1 Lecture 1: The systolic inequality and its relatives.

The *systole* of a space M, denoted by $sys(M)$ is the length of the shortest non-contractible closed curve in M. The more general systolic inequality claims the following:

Theorem 1.1 (Gromov 1983)**.** *There exists a constant $C_d > 0$ such that every Riemannian essential manifold M satisfies:*

$$sys(M)^d \leq C_d vol(M)$$

1.1 What is a manifold?

Before a definition remember that manifolds often arise as fibers $f^{-1}(\vec{v})$ of smooth functions $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$. For example consider f to be given by some polynomial, $n = 1$ and $k = 1$, to make things easier let us assume that the polynomial is of degree two so

$$p(x, y) = ax^2 + bxy + cy^2 + dx + fy + e$$

In this case one can completely understand what the fibers look like. It is essentially the same as for each α looking at the solutions of the equation:

$$\alpha = ax^2 + bxy + cy^2 + dx + fy$$

For instance if $a = c$ and $b = d = f = 0$ then fibers are all the circles centered at the origin. For other coefficients one can completely understand the solutions, each of the fibers is a conic (an ellipse, a hyperbola a parabola or a union of two lines) geometrically this corresponds to the intersection of a plane with a cone in three dimensions and it is completely understood which case is which depending on the coefficients.

Assume that f is smooth and $f^{-1}(t)$ is a curve and $(x_0, y_0) = t$. The equation of the tangent line to the curve at (x_0, y_0) is:

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0$$

For example if $f(x,y) = x^2 + y^2$, $t = 1$ and $(x_0, y_0) = (1, 0)$, then $\frac{\partial f}{\partial x}(x_0, y_0) = 2x|_1 = 2$, $\frac{\partial f}{\partial y}(x_0, y_0) = 2y|_0 = 0$, so the tangent line has equation:

$$2(x-1) = 0$$

Similarly if $f\colon \mathbb{R}^3 \to \mathbb{R}$ then the tangent plane at a point (x_0, y_0, z_0) such that $f(x_0, y_0, z_0) = t$ on the surface $f^{-1}(t)$ is given by

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Notice however that these equations for the tangent planes might fail. If all the partial derivatives are zero, then this equation does not define a hyperplane at all. In that case we say that the point is critical.

Definition 1.2. *A map $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ is* **smooth** *if all partial derivatives exist and are continuous.*

Definition 1.3. *Given a smooth function $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$, we say that a point $\vec{x} \in \mathbb{R}^{n+k}$ is* **critical** *if the matrix of partial derivatives:*

$$(Jf)_{i,j}(\vec{x}) = \frac{\partial f_i}{\partial x_j}(\vec{x})$$

has rank smaller than k. In this case we say that $f(\vec{x})$ is a critical value. Otherwise we say that the point $\vec{x}$ is **regular**. *We say that $\vec{y} \in \mathbb{R}^k$ is a* **regular value** *if* all *the elements of $f^{-1}(\vec{y})$ are regular.*

Example: Show that $(0,0)$ is a critical point of $f(x,y) = x^2 - y^2$. Interpret this geometrically. Show that 1 is a regular value.

The implicit function theorem tells us that when the Jacobian, i.e. the matrix of partial derivatives has full rank, then there is a local diffeomorphism:

Theorem 1.4. *If $F\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ is smooth, $M := \{(x,y) \in \mathbb{R}^{n+k} : F(x,y) = 0\}$ with $\frac{\partial F}{\partial y}(x_0, y_0)$ of full rank, then there exists $\epsilon > 0$ and an open set $U \subset \mathbb{R}^n$ around x_0 and a homeomorphism $g\colon U \to \mathbb{R}^k$ such $F(x, g(x))$.*

Denote $\mathbb{R}^{n+k}_+ := \{z \in \mathbb{R}^{n+k} : z_{n+k} \leq 0\}$. A linear map $\mathbb{R}^n \to \mathbb{R}^m$. Denote by $B^n := \{x \in \mathbb{R}^n : |x|_2 < 1\}$, and by $B^n_+ = \{x \in \mathbb{R}^n : |x|_2 < 1, x_n \geq 0\}$

Definition 1.5. *A map $f\colon A \subset \mathbb{R}^{n+k} \to \mathbb{R}^k$ is* **smooth** *if there exists an open set $U \supset A$ and a map $F\colon U \to \mathbb{R}^k$, such that $F = f$ in A, and all the partial derivatives of F exist and are continuous. A map is called a* **diffeomorphism** *if it is smooth with smooth inverse.*

Theorem 1.6. *If $F\colon \mathbb{R}^{n+k}_+ \to \mathbb{R}^k$ is smooth, $M := \{z \in \mathbb{R}^{n+k} : F(z) = 0\}$ with Jacobian $J_{z_0}(F)$ of maximal rank, then there exists $g\colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ that parametrizes M in a neighborhood around z_0, i.e. there exists an open set $U \subset M \subset \mathbb{R}^{n+k}$ and a* diffeomorphism *$g\colon B^n \to U$.*

Definition 1.7. *A* **manifold** *$M \subset \mathbb{R}^{n+k}$ is a subset such that for every point $x \in M$ there exists an open neighborhood $U \subset M$, $x \in U$ which is diffeomorphic to the open ball B^n.*

Definition 1.8. *A* **manifold with boundary** *is a subset of $\mathbb{R}^{n+k}$ is a subset such that for every point $x \in M$ there exists an open neighborhood $U \subset M$ which is diffeomorphic to the open ball B^n or to the open semiball B^n_+.*

Lemma 1.9. *Assume that $f\colon M \to N$ is a smooth map between manifolds of the same dimension, assume that $y \in N$ is a regular value of f. Then there exists an open neighborhood U around y, such that $\#f^{-1}(y') = \#f^{-1}(y)$ for every $y' \in U$.*

Proof. This is a direct consequence of the inverse function theorem. (Exercise: write down this formally). □

Abstract manifolds One can define a manifold M, which doesn't have to be a subset $\mathbb{R}^{n+k}$. It is a paracompact Hausdorf topological space which is locally diffeomorphic to euclidean space. The catch of this definition is then talking about the tangent space. If M is a manifold embedded in $\mathbb{R}^{n+k}$ then for each $p \in M$ there exists an open set $U \subset M$, and $V \subset \mathbb{R}^n$. Such that $\phi\colon V \to U$ is a parametrization. This allow us to describe the tangent space at p. It is an affine flat of dimension n. Then for a map between two manifolds $f\colon M \to N$, the derivative at p sends the tangent space at p, denoted T_p to the tangent space at $T_{f(p)}$. One of the difficulties of differentiable geometry is describing what is the tangent space when M is not necessarily embedded in $\mathbb{R}^{n+k}$. If one wants to work seriously on the topic one should definitely get used to it t realize it is quite easy. For this course we skip it. Whitney showed that every manifold of dimension n (not necessarily embedded), can be in fact embedded in dimension $2n$. For this reason we feel justified to not discuss abstract manifolds in detail. As a substitute we will describe a combinatorial cousin of Whitney's result in the next section and point out how does one go around proving Whitney result informally. The important point now is that we can do not need the definition of abstract manifold. A second technical remark is that depending on the context the word manifold might refer topological manifold, rather than smooth manifold as in the previous definition. A topological manifold is a topological space which is paracompact Hausdorff that is locally homeomorphic to $\mathbb{R}^d$ for some d.

We go back to smooth manifolds. In the following it is assumed that M and N are embedded in $\mathbb{R}^{n+k}$ and $\mathbb{R}^{m+k}$ and each has dimension n and m, as before maps are smooth means that there

Example: What are the sizes of the fiber of $z \to z^3$ on $\{z \in \mathbb{C} : |z| = 1\}$? Exercise: For every d construct a map from the sphere $\mathbb{S}^2$ to itself such that almost every value y has d elements in the fiber: $\#f^{-1}(y) = d$.

The fundamental theorem of algebra Let us give a first application of these ideas.

Theorem 1.10. *Every non constant complex polynomial has root on the complex numbers.*

The complex numbers are $z = a+ib$ with the rules $(a+ib)+(c+id) = a + c + i(b + d)$ and $(a + ib)(c + id) = ac - bd + i(ad + bc)$. We can give $a + ib$ the coordinate (a, b) and represent them in the plane. Polynomials are expressions of the form $p(z) = \sum_{i=0}^{n} c_i z^i$.

Proof. To each polynomial we can assign a function from the plane to the plane. This function which we still denote by p is differentiable (in fact it is complex differentiable which is a stronger notion). For the proof to work we need a compact space. We add one point to $\mathbb{R}^2$ at infinity, what we obtain is diffeomorphic to $\mathbb{S}^2$. Specifically let $\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, and consider the stereographic projection to the plane $x_3 = 0$. That is we draw the line from the north point to a point in the sphere and we send it to the point where this line intersects the plane. This is a diffeomorphism from the sphere minus the north pole and the plane, denote it by s_N (s for stereographic). There is a second homeomorphisms for the south pole s_S. Notice that these two maps are enough to show that the sphere is a two manifold (one needs to check differentiability). Define a map $\hat{p} \colon \mathbb{S}^2 \to \mathbb{S}^2$, by

$$\hat{p} = s_N^{-1} p s_N \colon \mathbb{S}^2 - N \to \mathbb{S}^2 - N.$$

For every polynomial the norm of $\lim_{|z|\to\infty} p(z)$ goes to infinity, so $\lim_{s_N^{-1}(z)\to N} \hat{p}(z) = N$, so define $\hat{p}(N) := N$. One can show that this map is differentiable even at the nord pole.
Finally the complex derivative $p'(z)$ is another polynomial. Whenever $p'(z) \neq 0$, z is a regular point of p and hence $s_N^{-1}(z)$ is a regular point of $\hat{p}$. Now p' has only a finite number of zeros. Indeed if z_0 is a zero of a polynomial p then $p(z) = (z - z_0)q(z)$ (to see this assume it by induction and apply Euclid's algorithm to p and $(z - z_0)$ to obtain $p = (z - z_0)q + r$. With r of smaller degree than q, substituting $z = z_0$ we obtain that $r(z_0) = 0$, but then $(z - z_0)$ divides r, hence the degree of r must be 0.) So the number of zeros of p' is at most $(d-1)$. The complement of the zero set is connected, so outside this set $\#\hat{p}^{-1} = \#p^{-1}$ must be constant. This fiber cannot have cardinality 0

everywhere, so it cannot have cardinality 0 anywhere. In other words, the map must be surjective, and in particular $s_N^{-1}(0) = (0, 0, -1)$ is in its image. $\square$

Skipped details: We have not been very cautious about certain details of the last proof to emphasize the topological essence of the argument. See if you can find where we have been sloopy and try to fill in these details.

1.2 Simplicial Complexes

Triangulations appeared as a tool early in the development of topology. From a pure combinatorics perspective simplicial complexes are very satisfying. A simplicial complex is a hereditary family of subsets of a set. We will use $[n] = \{1, 2, 3 \dots n\}$ for the set and X for the family of subsets. Hereditary means that if $x \subset y$ and $y \in X$, then $x \in X$. The relation to topology comes from considering the geometric realization of X. This is a topological space which can be defined as follows: for every element $i \in [n]$, consider the vector $\vec{v} \in \mathbb{R}^n$ with $\vec{v}_i = 1$ and $\vec{v}_j = 0$ for all $j \neq i$. Now for each set $x \in X$, consider $\operatorname{conv}(\{\vec{v}_i\}_{i \in x})$. The union of these *simplicies* is a set inheriting a topology from $\mathbb{R}^n$. For example if all the sets in X have cardinality 1 or 2, then we obtain a graph. The dimension of a simplicial complex is the maximum among the cardinalities of its elements minus 1. For example $X = \{1, 2, 3, 4, (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (234), (134), (124), (123)\}$ has dimension two and by construction it is contained in $\mathbb{R}^4$. But in fact we can visualize it in dimension 3. It is the boundary of a tetrahedron. Topologically a tetrahedron is equivalent to $\mathbb{S}^2$. More precisely there exists a continuous bijective map, with continuous inverse $h\colon ||X|| \to \mathbb{S}^2$.

Let us say that a drawing of a simplicial complex in some space M is a map $f\colon ||X|| \to M$ if the map is injective then it is an embedding. Here we are using $||X||$ to refer to the geometric realization. That is, X is a family of finite subsets of a ground set of vertices indexed by the numbers $[n]$, and $||X||$ is a topological space defined by introducing one simplex for each set in X. We emphasize that once that we have used the construction above we forget about it and just think about it as a topological space. In the future we will forget about the notation, and just write $f\colon X \to M$, here X is assumed to be the geometric realization (probably should be called the topological realization). The definition of a simplicial complex X is one plus the maximal cardinality of a subset $x \in X$. The reason is obvious having the geometric realization in mind.

Proposition 1.11. *Every simplicial complex of dimension d can be topologically embedded in dimension $2d+1$.*

For example every graph can be embedded in $\mathbb{R}^3$. The following embedding works in general: take points $\{p_1, p_2, \dots p_n\}$ in $\mathbb{R}^{2d+1}$ at random, and for each set $x \in X$, consider $\mathrm{conv}_{i\in x}(p_i)$. Since the any subset of $2d+2$ points are affinely independent, than no two such affine d-planes spanned by them intersect in dimension $2d+1$ (give a proof of this fact using linear algebra).
This argument uses general position, which is a very powerful tool. On the other hand general position and probabilistic constructions are not always constructive. Let us give a second construction that can be easily implemented. Consider the Veronesse map $\nu\colon \mathbb{R} \to \mathbb{R}^{2d+1}$:

$$t \to (t, t^2, t^3 \dots t^{2d+1}),$$

the image of this map is called the moment curve. This relates bijectively polynomials in one dimension to hyperplanes in $2d+1$ dimensions. Specifically the polynomial $P = a_1 t + a_2 t^2 + \ldots + a_{2d+1} t^{2d+1} - b$ will correspond to the linear space $H_{\vec{a},b} := \{\vec{v} : \langle \vec{v}, \vec{a} \rangle = b\}$, where $\vec{a} = (a_1, a_2 \dots a_{2d+1})$ so that the zeros of P are in bijection via the Veronesse embedding with the intersections of $H_{\vec{a},b}$ with the moment curve.
Now we embed the vertex set of the simplicial complex sending $i \to \nu(i)$ and taking the convex hull of the points corresponding to each face. Notice that no $2d+2$ points on the moment curve lie on the same hyperplane. Indeed this will correspond to a polynomial of degree $2d+1$ with $2d+2$ distinct roots which is absurd. This means that every subset of the $2d+2$ points is affinely independent, so two d-dimensional affine spaces in $2d+1$ have empty intersection.

Sperner and Brouwer Let T be a triangulation of a triangle. This is a 2-dimensional simplicial complex homeomorphic to a disk. Choose three vertices v_0, v_1, v_2 on the boundary and call a 3-coloring of the vertices an s-coloring if for every $i \in \{0, 1, 2\}$, v_i is colored i and the path on the boundary from v_i to v_{i+1} does not use the color $i-1$ (indices are understood modulo i).

Proposition 1.12. *For any s-coloring T there exists an heterochromatic triangle.*

The content of this theorem is combinatorial-topological, but in the classical statement T is an equilateral euclidean triangle and each face of the triangulation is an euclidean triangle. More formally there exists a map from the geometric realization of the simplicial complex to an equilateral triangle in the plane, that is an affine map on each simplex.

We prove the geometric statement first and then a second proof for the more general case stated here. Actually it turns out that in two dimensions they are equivalent. As an exercise explain what does this mean. How would you show that they are not equivalent in dimension d?

Proof. For every face $\sigma \in T^2$, let σ_t be a triangle which linearly interpolates between $\sigma = x_0, x_1, x_2$ and $v_{\chi(x_0)}, v_{\chi(x_1)}, v_{\chi(x_2)}$. The function $P(t) = \sum_{\sigma \in T^2} area(\sigma_t))$ is a polynomial. On the other hand for t small enough the union is still a triangulation of T, so this function must be equal to the area of T for small t. Hence the polynomial is constant! So $P(1) = \sum_{\sigma \in T^2} area(\sigma_t)) = area(T)$. On the other hand observe that for every simplex σ which is not heterochromatic $area(\sigma_1) = 0$. □

Here is another proof that doesn't use geometry.

Proof. Consider the intersection graph of the simplicies and add a vertex for the exterior face which is connected to all the faces on the boundary. Consider the subgraph spanned by edges that have one end of color 0 and one end of color 1. Now heterochromatic triangles are in bijection with vertices of degree 1. Monochromatic triangles are in bijection with isolated vertices and triangles with two 0s and one 1 or two 1s and one 0 have degree 2. There is an odd number of edges from the outside face into the triangulation so one of there must exist a vertex of odd degree in the interior. □

Exercise: Generalize these proofs to higher dimensions.

Theorem 1.13. *Every map from a topological closed disk to itself has a fixed point.*

A topological closed disk is one is a set that is homeomorphic to the euclidean disk. Exercise: Show that a disk is homeomorphic to a simplex.

Proof. Let D be the topological disk, $f\colon D \to D$ the function. Let $h\colon D \to T$ be a homeomorphism. Showing that f has a fixed point is the same as showing that hfh^{-1} has a fixed point. So we can assume that $T = D$. Consider a very fine triangulation of T, that is a triangulation in which every simplex has small diameter (can you show that there exists such a thing?). Use f to define an s-coloring using the barycentric coordinates if $x = \sum_{i=0}^{3} \lambda_i v_i$ is some vertex of the triangulation where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, and $f(x) = \sum_{i=0}^{3} \alpha_i v_i$,$\alpha_i \geq 0$ and $\sum \alpha_i = 1$, then if $\lambda_0 < \alpha_0$ color x with 0, otherwise $\lambda_1 < \alpha_1$ color x with 1 and if neither of the previous ones occurs then color it 2. This is an s-coloring. We look at the heterochromatic triangle and restrict f to it, □

Classically one shows that:

Claim 1.14. *There is no continuous map from the disk to its boundary that fixes the boundary*

and derives the previous theorem from this. Actually this claim is equivalent to the fixed point theorem above, using the conjugation argument we can assume that we are on the round disk D. First if g maps D to ∂D fixing the boundary, we can compose it with the antipodal map $x \to -x$ to obtain a map without fixed points. On the other hand if there exists a map without fixed points f, then we can define $g(x)$ to be the point on the boundary intersected by the ray starting at $f(x)$ and passing through x. One shows that if g is continuous whenever it is defined. Clearly $g(x) = x$ for the points in the boundary, and the only way that g is not defined is when f has a fixed point.

We are going to generalize these statements later on.

Planar graphs Every graph embeds in $\mathbb{R}^3$, but there are some graphs that embed in $\mathbb{R}^2$ and others that do not. Now if a graph embeds in $\mathbb{R}^2$ (show that) equivalently it embeds in $\mathbb{S}^2$. Then the number of edges (subsets of size 2) is at most three times the number of vertices. To see this we need to show a foundational result in the field:

Theorem 1.15. *(Euler's formula) For any planar graph*

$$V - E + F = 2$$

Here V is the number of vertices, E the number of edges and F is the number of faces. If G is a graph, and $f\colon ||G|| \to \mathbb{S}^2$ is an embedding, a face is a connected component of $\mathbb{S}^2 \setminus f(||G||)$. A priori the number of faces depends on the embedding but this formula implies that it doesn't.
Excercise: Construct two embeddings of the same graph, $f_1, f_2\colon ||G|| \to \mathbb{R}^2$ such that there does not exist a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $hf_1 = f_2$. (We used the geometric realization notation: $||.||$, for the last time, from now on we skip it).
The proof of this theorem now rests in the following lemma.

Theorem 1.16 (Jordan)**.** *The complement of a simple closed curve C in the plane has two connected components, one bounded and one unbounded.*

This theorem is hard to proof. A proper write up of Jordan's proof by Hales takes 15 pages. It was long thought that Jordan's proof was incorrect. According to Hales the only thing that Jordan skipped was the easy case in which the curve is assumed to be polygonal. Presumably Jordan thought that this case was obvious, and is the one that we care about here: we prove the theorem under the extra assumption that the simple closed curve is polygonal, that is, it is a concatenation of segments.

Theorem 1.17 (Jordan)**.** *The complement of a simple closed polygonal curve C in the plane has two connected components, one bounded and one unbounded.*

Proof. For each point y in the complement of C consider an piecewise linear ray that departs from y to infinity. For each point x of intersection between the ray and the curve, consider a metric ball $B(x, \epsilon)$, where $\epsilon > 0$ is small enough that it does not contain any vertex of the polygonal curve, except possibly x itself and it intersects C in one connected arc. By the choice of ϵ, C divides $B(x, \epsilon)$ in two regions and we count the intersection as a proper intersection if the ray touches both regions. Now to each point we assign a number in $\{0, 1\}$ depending on the parity of the number of proper intersections of any ray. The crucial observation is that this number does not depend on the ray which can be shown immediately moving around the ray, also for any y we can choose $0 < \epsilon < d(y, C)$, and every point in $B(y, \epsilon)$ has the same number. Choosing points close to the curve on different sides shows that the parities are different. Now if some two points y, y' have the same parity they must be in the same connected component, or in other words, if they have different parities they must be in different components. Indeed otherwise we can connect the points with a ray in a component and continue to infinity. Similarly if two points are in the same connected component then they must have the same parity. $\square$

In fact something stronger is true, one of the components is homeomorphic to a disk. Indeed it is bounded and the union with C is closed, so the union with C is a polygonal compact set. We can subdivide it into convex triangles. Now observe that the intersection graph of the convex triangles is a tree, as otherwise it would have two connected components. We can now embed the tree into the disk in such a way that each vertex has a convex assigned to it so that the intersection graph of these convexes is the same as of the convex decomposition of the polygon.

PL-manifolds Whitney showed that every smooth manifold is homeomorphic to a simplicial complex. Such a simplicial complex is called piecewise linear manifold or PL manifold. This is in fact a third family of manifolds other than topological and smooth.

The ***link*** of a simplex σ is the set of all the simplicies τ disjoint from σ such that $\sigma \cup \tau$ is a simplex. The ***star*** of a simplex σ is the set all the simplicies containing σ. Notice that in the case in which σ is a vertex, then the star is the cone of the link over the vertex. Now a PL-manifold is a simplicial complex such that the link of every simplex is a PL-sphere. a 0-sphere consists of two disjoint vertices. For example a tetrahedra is a PL-sphere. Milnor amazed us showing that there are PL manifolds that cannot be smoothed! In two and three dimensions this is not the case.
When the topology of the manifold gets more complicated the number of simplicies (subsets) grows, more importantly certain operations on simplicial complexes are not simplicial complexes. Topologists prefer CW-complexes, or Σ-complexes or simplicial sets. In this course we are mainly interested in surfaces and graphs so the generalizations that we need are easy to describe.

Graphs A graph or multigraph is a pair $([n], E)$, each edge has two endpoints, two edges might have the same endpoints and one edge might have twice the same endpoint. The reason we introduce this more general concept is that we want to take the dual of a graph.

For the proof recall some terms: a cycle is a sequence of edges $e_0, e_1 \dots e_{k-1}$ such that the e_i and e_{i+1} share a common vertex (subindices are modulo k), that a tree is a connected graph without cycles, and that is easy to see, by induction that a tree satisfies $V(T) - E(T) = 1$. Let us also define the ***dual*** graph of G denoted by G^*: in fact G^* depends on the embedding $f\colon G \to \mathbb{R}^2$. The vertex set of G^* is the set of faces of $f(G)$. Two such vertices of G^* are connected by an edge if they are both incident to a common edge. Notice that G^* comes equipped with an embedding into $\mathbb{R}^2$ in which each pair of vertices of the dual are connected through a Jordan arc contained in the union of the corresponding faces. The following follows directly from the Jordan curve theorem.

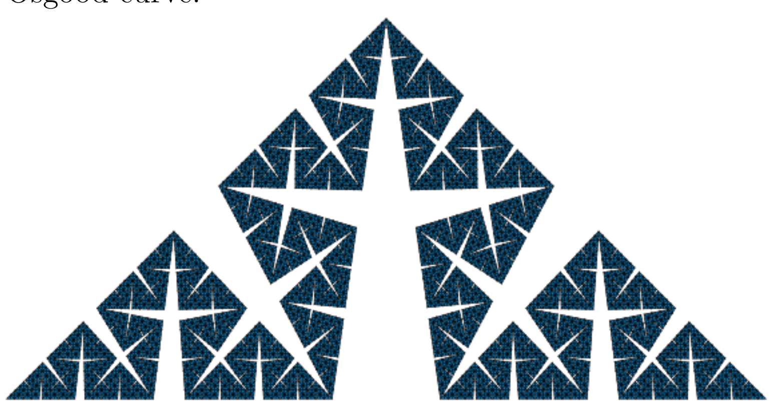
Corollary 1.18. *A set of edges of G support a cycle if and only if erasing the dual edges separates G^*.*

Let us give a proof of Euler's formula.

Proof. Assume the result by induction on the number of edges. If there is a vertex of degree one, erase it and erase the incident edge. If there are no vertices of degree 1, there must be a cycle (why?). If there is a cycle there must be a simple cycle. Erase any edge on that cycle, the number of faces goes down by one and the number of edges goes down by one. $\square$

Proof. Consider a spanning tree T of G, and let T^* be the edges dual to $E(G) - E(T)$. By the previous statement T^* spans a tree. Indeed it has no disconnected componnents and if it had cycles than T will not be spanning. Now $E(T) + E(T^*) = E(G)$, but $E(T) - V(T) = 1$ and $E(T^*) - V(T^*) = E(T^*) - F(G) = 1$, so $E(G) = V(G) - 1 + F(G) - 1$, and the result follows. $\square$

To understand the type of difficulties around the Jordan curve one might consider the following algorithmic question: Given a polygonal jordan curve of n points and a point not on the curve determine if the point is in the inside or the oustide of the Jordan curve. Its alos worth noting that there are jordan curves with positive area, called the Osgood curve:



2 Smooth manifolds

2.1 A curve has zero or two boundary points.

After this detour into the discrete and piecewise linear let us go back to the smooth world. Our point of departure was the implicit function theorem, with slightly more language let us revisit it. Recall that a value is regular if each of its inverse images is differentiable and the differential has the larget possible rank.

Theorem 2.1. *Let $f\colon M \to N$ be a smooth map between manifolds. Then for any regular value y, $f^{-1}(y)$ is a smooth manifold of dimension $m-n$. Moreover if M is a manifold with boundary and y is a regular value when restricted to the boundary then $f^{-1}(y)$ is a manifold with boundary and $\partial f^{-1}(y) = \partial M \cap f^{-1}(y)$.*

proof idea. We compose the parametrization of M with the map f and with the coordinate chart of N. We obtain a regular value of a map from $\mathbb{R}^m \to \mathbb{R}^n$ and the implicit function theorem implies that the inverse image is a manifold. In the case of a boundary point one has to be careful about what does it mean to be regular. It means that the tangent of the boundary intersecst generically to the tangent of $f^{-1}y$, which yields the result. □

Theorem 2.2. *Any connected compact smooth* 1*-manifold is either diffeomorphic to* $[0,1]$ *or to* $\mathbb{S}^1$*.*

Refer to Milnor little book for a proper proof.

Theorem 2.3. *There is no smooth map from a manifold to its boundary that fixes the boundary*

Proof. Let $f\colon M \to \partial M$ be the supposed map. Assume that y is a regular value of f, then $f^{-1}(y)$ is a 1-manifold. By the previous theorem this is a union of topological segments and closed curves. The important part: **each component of $f^{-1}(y)$ has 0 or 2 points on the boundary. There are an even number of boundary points**. This contradicts the fact that f was supposed to be the identity on the boundary. $\square$

Theorem 2.4. *Any continuous function $f\colon \overline{B^n} \to \overline{B^n}$ has a fixed point.*

In fact this statement is equivalent to the previous theorem in the case of $M = \overline{B^n}$ and f smooth. We prove it first for smooth functions and then for continuous ones.

Here are two tools that we used:

Theorem 2.5 (Sard-Brown)**.** *If U is an open set and $f\colon U \to \mathbb{R}^k$ is a smooth map, then the image of the critical points has k-Lebesgue measure* 0*. If $f\colon M \to N$ is smooth the set of regular values is dense in N.*

Again Milnor is a good reference, the idea is that

Here is a third useful technical tool which follows from the Weierstrass theorem (for example):

Proposition 2.6. *For any a continuous map between compact smooth manifolds $f\colon M \to N$ every $\epsilon > 0$ and every point $y \in N$ there exists a smooth function f_ϵ such that $|d(f_\epsilon, f)| < \epsilon$ and y is a regular point of f_ϵ.*

We keep postponing technical issues because we have not defined a distance between functions. We might assume that M and N are in fact metric spaces which have the same topology as the one coming from their embedding in euclidean space. In fact M and N come with a natural metric. For any pair of points in N consider all the paths between them, measure their euclidean length and define their distance as the minimum among these lengths. Once we have a metric in N we can compare functions f, f_ϵ which are valued in N by considering $\sup_{x\in M} d_N(f(x), f_\epsilon(x))$

With this theorem at hand we prove the Brouwer fixed point theorem again for continuous maps. Indeed for each ϵ, we obtain x_ϵ such that $f_\epsilon(x_\epsilon) = x_\epsilon$. Taking a convergent subsequence of the x_ϵ when $\epsilon = 1/n$ we obtain $\lim_{n_k\to\infty} x_{1/n_k} = x$ which satisfies $f(x) = x$.

Now let us recall a crucial concept.

Definition 2.7. *Two smooth (or continuous) maps $f, g\colon M \to N$ are* **homotopy equivalent** *if there exists a smooth (or continuous) map $h\colon M \times I \to N$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. The map h is a* **homotopy** *between the maps f and g.*

The homotopy h is like a movie that starts in f and finishes in g. From here one can derive the more complicated notion of homotopy equivalence between spaces. One of the main ideas of algebraic topology is that by quotiening by homotopy the extremely wiggly world of topology becomes rigid.

Definition 2.8. *A closed curve $\gamma\colon \mathbb{S}^1 \to M$ is said to be* **contractible** *if it is homotopic to a constant map.*

Observe that a curve is contractible if and only if we can extend the map $\gamma\colon \mathbb{S}^1 \to M$ to a map $f\colon D^2 \to M$.

Theorem 2.9. *For any two homotopically equivalent smooth functions $f, g\colon M \to N$ between manifolds of the same dimension, for any regular value y of the homotopy h,*

$$\# f^{-1}(y) = \# g^{-1}(y) \mod 2$$

Proof. Assuming that y is a regular value for h. The fiber $h^{-1}(y)$ is a one manifold with boundary points on $M \times 0 \cup M \times 1$. Since $\# f^{-1}(y) = \# h^{-1}(y) \cap (M \times 0)$ and $\# g{-}1(y) = \# h^{-1}(y) \cap (M \times 1)$ if a connected $h^{-1}(y)$ intersects $M \times 0$ in one point then it must also intersect $M \times 1$. The connected components of $h^{-1}(y)$ that don't intersect $M \times 1$ in one point do not modify the parity of neither boundary component. If y is not a regular value of h then we need that it is contained in a small neighborhood around a regular value where we can apply the inverse function theorem. $\square$

Corollary 2.10. *Let $f\colon M \to N$ be a smooth function between compact connected manifolds of equal dimension. For any two regular values y, y' of f,*

$$\#f^{-1}(y) = \#f^{-1}(y') \mod 2$$

This number is called the degree mod two of the map. Try to give a definition of the degree of a map.

Proof. Assume that we have a map $h\colon N \times I \to N$ which is a homotopy between the identity and a map that sends y to y', and which moreover for each $t \in I$, $h(.,t)$ is a homeomorphism. Define $f' := f \circ h$ is homotopic to f, so $f'^{-1}(y') = (f \circ h)^{-1}(y') = f^{-1}(y)$. On the other hand $\#f^{-1}(y) = \#f'^{-1}(y) \mod 2$. Now to construct the map h, observe that it is enough to do it in an open ball, as we can then cover N by open balls and carry any point to any other point. Specifically we need to show that there is an isotopy (an homotopy in which every time is a homemomorphism) from the open ball to itself sending any point to any other. For this we can assume that one of the points is the origin and compose the maps. The last statement accepts a proof by picture $\square$

Let $D^n = \overline{B^n}$ and S^{n-1} be the sphere of radius 1, and rS^{n-1} be the sphere of radius r.

Corollary 2.11. *Let $f\colon D^n \to D^n$ be a map such that $f|_\partial D^n\colon S^{n-1} \to S^{n-1}$ has degree 1 mod 0, then f is surjective.*

Lemma 2.12. *If a map $g\colon S^{n-1} \to S^{n-1}$ can be extended to a map $D^n \to S^{n-1}$ then g has zero degree.*

Proof. Indeed, we just need to interpret the extension as a homotopy between g a constant map. Parametrize D^n as $\cup_{r\in[0,1]} rS^{n-1}$. We obtain a map $G\colon S^{n-1} \times [0,1] \to D^n \to S^{n-1}$ by composing the closure of this parametrization with g. The map G is the homotopy bewteen g and a constant map. □

Proof. Assume that f is not surjective let $x \in D^n$ be a point that is not covered and consider a function h that takes each $x' \neq x$ to the boundary point on the extension of the segment between x and x'. Now the function hf is an extension of a map of degree 1 to all of D^n, this is a contradiction. □

Essential and aspherical manifolds As homework try to show that $\mathbb{S}^1 \times \mathbb{S}^2$ does not satisfy the systolic inequality. That is an example of a non-essential manifold. To define essential first we define aspherical, which is a slightly more particular concept (i.e. aspherical manifolds are essential).

Definition 2.13. *We say that a manifold is* **aspherical** *if any map from any sphere S^{n-1} with $n > 1$ can be extended to a map of the disk D^n.*

The previous corollary shows that spheres are not aspherical. Try to show that surfaces other than the projective plane and the sphere are aspherical, and the torus on any dimension is aspherical.

Theorem 2.14 (Borsuk-Ulam)**.** *For every continous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ there exists a pair of antipodal points $x, -x$ such that $f(x) = f(-x)$.*

Proof. Equivalently the map $g(x) := f(x) - f(-x)$ has a zero. This statement is obvious for the projection π that forgets the last coordinate: the north and south pole are mapped to $\vec{0}$, moreover this is the only pair of antipodal points which maps to 0. Now consider the homotopy equivalence $h(x,t) = tg(x) + (1-t)\pi(x)$. Notice that it is enough to show that g has some zero when g is smooth and 0 is a regular value as we can approximate g by a smooth map for which $\vec{0}$ is a regular value. Now look at $g^{-1}(\vec{0})$. It is a one manifold which is antipodally invariant at each t. Similarly to our previous argument the number of pairs of antipodal points which are mapped to zero by g must be odd. $\square$

Corollary 2.15 (Ham-Sandwich theorem)**.** *Let $\mu_0, \mu_1, \dots \mu_{d-1}$ me measures in $\mathbb{R}^n$ then there exists a hyperplane that bisects them simultanously.*

Proof. Parametrize oriented hyperplanes by points on a sphere and define an antipodal map, then by the BU theorem there exists a zero of this map. $\square$

3 Geometry

A Reimannian metric on a manifold M is one that arises when we embed M in some $\mathbb{R}^{n+k}$ and consider the length structure given by paths restricted to M measured by the Euclidean metric. Like in the definition of smooth manifold there exists an intrinsic definition. Thanks to a theorem of John Nash (which is much harder than that of Whitney) every Riemannian manifold embedds in some Eucldiean space so that the induced length metric is the original one. We can use the surface area of Euclidean space, alternatively, a map that does not increase distances is called ***non-expanding*** or *Lipschitz*. We can define the volume of a euclidean cube to be the product of the length of its sides, and for a general manifold, volume is a positive measure (meaning it is additive) which satisfies that if $M \to N$ is non-contracting then $vol(M) \geq vol(N)$.

In the intrinsic point of view, we can content ouserlves with defining what a Riemannian metric in one chart is since both length and volume are additive. On a disk B^n, we have a positive definite form g_x at each point x, and the dependence on the point x of the form g_x is continuous or smooth. Remember that a positive definite form g takes two vectors v, u and gives back a number, for vectors $u = v$ $g(u, u) \geq 0$ with equality if and only if $u = 0$. One way to represent g in a vector space with a basis is via a symmetric matrix with positive eignvalues A, and $g(u, v) = uAv^* = \langle Au, v \rangle$. Euclidean space corresponds to the matrix $A = I$. To each such matrix we can assign its unit ball, the set of v such that $\langle Av, v \rangle = 1$. This is an ellipsoid, an ellipse in the plane.

To measure the length of a vector we see how much the ellipse needs to be scaled to be on the boundary. Equivalently $|u|_g = \sqrt{g(u,u)}$ hence

$$length(\gamma) = \int \sqrt{|\gamma'(t)|_g} dt$$

, and here g might depend on $\gamma(t)$. As for the volume we integrate

$$\int \sqrt{det(g(x))} dvol(x),$$

where $x \in D^n$ and vol is the Lebesgue measure.

3.1 Kuratowksi embedding

Now we explain one major ingredient of the proof of the systolic inequality. We motivated the definition of manifold looking at zeros of smooth functions. We motivated Reimannian manifolds as length metrics induced from manifolds embedded in $\mathbb{R}^n$, we mentioned the theorems of Nash. It is important to remark this theorem gives local information on the geodesics on the manifold but no global information Said differently: if M is a Nash embedding in $\mathbb{R}^n$ then the distances $d_{\mathbb{R}^n}(x, y)$ and $d_M(x, y)$ might have nothing to do with each other. One might wonder if by going even into higher dimension one might get an isometric embedding. That is one for which $d_{\mathbb{R}^n}(x, y) = d_M(x, y)$, for every pair of points $x, y \in M$. As it turns out this is impossible, essentially because M might be curved. More concretely let $0, 1, 2, 3$ and define the distances $d(i, i + j) = j$, where both i and $i + j$ are considered modulo 4. Is not difficult to construct a manifold that contains a copy of this metric space, for example a circle of the right scale contains it, but this cannot be embedded isometrically in any Euclidean space (you'll prove this in the homework).

Normed spaces A norm is a nice metric on a vector space. More precesely it is a function $|.|: \mathbb{R}^n \to \mathbb{R}^+$ that satisfies $|0| = 0$, $|\lambda v| = |\lambda||v|$, (where λ is a number and $|\lambda|$ is the absolute value) and $|u+v| \leq |u|+|v|$. The most famous norms are the l_p norms: $|x|_p \to (\sum |x_i|^p)^{1/p}$, when $p \to \infty$ this tends to the function $x \to max_i|x_i|$ and is denoted $(\mathbb{R}^n, l_\infty)$. The other main examples that play roles in computer science are l_1 and l_2. The relevance of l_1 is intuitively clear in a combinatorial context, l_2 is of course Euclidean space. The l_p norms appear in several areas of analysis, particularly in harmonic analysis. The spaces of functions L_p have some similarities with $(\mathbb{R}^\infty, l_p)$. In finite dimensions norms are naturally related to their unit balls. It turns out that norms are in bijection with centrally symmetric convex bodies. Homework: Draw the unit ball of $(\mathbb{R}^3, l_\infty)$, and of $(\mathbb{R}^3, l_1)$. Here is the finite version of the Kuratowski embedding theorem:

Theorem 3.1. *For any finite metric space* (X, d)*, denote* $X = \{x_1, x_2, \ldots x_n\}$*,* *then the map* $K(x) = (d(x, x_1), d(x, x_2), d(x, x_3) \dots d(x, x_n))$*, is an isometric embedding into* $(\mathbb{R}^n, l_\infty)$

Proof. Indeed $d(x_i, x_j) \geq |d(x_i, x_k) - d(x_j, x_i)|$ for all k by the triangle inequality, so $d(x_i, x_j) \geq |K(x_i) - K(x_j)$—. On the other hand $K(x_i) - K(x_j)| \geq |K(x_i)_j - K(x_j)_j| = |d(x_i, x_j) - d(x_j, x_j)| = d(x_i, x_j)$. □

Now if M is a simplicial complex or a manifold with a Riemannian metric.

Theorem 3.2. *If M is a simplicial complex with a Riemannian metric, then the map $x \to (d(x, .) \colon M \to \mathbb{R})$ is an isometric embedding into the space L_∞ of essentially bounded functions.*

The proof is essentially the same but L_∞ is more intimidating.

3.2 The isoperimetric inequality

The classic isoperimetric inequality says that for all the decent sets of euclidean space that have the same perimeter the circle bounds the largest area. This generalizes to higher dimensions, of all the decent sets of given surface area the euclidean balls encloses the largest area. An equivalent way to state this is that among all sets of a given volume the unit ball has the least surface area. (Prove that these are equivalent). This theorem is classical, and has many proofs. The classical proofs either use symmetrization, as we make the set more and more symmetric the area gets smaller, and after enough symmetrization we reach a ball. A second important proof uses the Brunn-Minkowski inequality.

The sparsest cut problem There are many variants of this inequality, many of which are important in computer science. For example the sparsest cut problem: given a graph G find the partition $V = A \cup B$ such that the number of edges between A and B quotiented by their product is minimized, i.e. $min\frac{|E(A,B)|}{|A||B|}$.

It turns out that the best way to approximate this quantity is via a bilipshitz embedding into l_1. Similar to the Kuratowski embedding we define $K(x)$ as the weighted distance to a random set S. So the entries of $K(x)$ are indexed by subsets of X and $K(x)_S = p(S)d(x, S)$, where $p(S)$ is the probabilty of S for some distribution.

Higher co-dimension isoperimetric inequality In geometric measure theory, the isoperimetric inequality was generalized in an interesting fashion. To get started, suppose that we have a circle embedded in three dimensions. The Plateau problem consists of finding the surface of smallest area that bounds this circle. Now we might wonder if we can bound the area of the surface in terms of the length of the circle. A generalized isoperimetric inequality (introduced by Federer and Fleming) generalizes this phenomena in the following way:

Theorem 3.3 (Federer-Fleming 60s)**.** *Suppose that M is an n-cycle in Euclidean space $\mathbb{R}^{n+m}$. Then there exists C such that $\partial C = M$ obeying the following estimates:*

1. $vol(C) \leq c_{n+m} vol(M)^{\frac{n+1}{n}}$
2. *C is contained in a neighborhood of M of size $c'_{m+n} vol(M)^{\frac{1}{n}}$*

A crucial idea behind this proof which carries out later is to push M from a random point to the standard cubulation of euclidean space inductively on the dimension.

Later Simon-Bombieri and Almgreen refined this to obtain the following:

Theorem 3.4. *Suppose that M is an n-cycle in Euclidean space $\mathbb{R}^{n+m}$ with $vol(M) = vol(S^n)$. Then there exists N such that $\partial C = M$ obeying the following estimate: $vol(C) \leq vol(D^{n+1})$*

More generally and coming back to our topic let us introduce a new concept. Given a manifold $M \subset R^{n+m}$ we define a ***filling*** of M as $C \subset \mathbb{R}^{n+m}$ such that $\partial C = M$. Now the ***filling volume*** of a manifold is the minimal volume of a C that fills M. The ***filling radius*** of M is the smallest r such that the neighborhood denoted $M_{+r} := \{x : d(x, y) < r, y \in M\}$ contains a C such that $\partial C = M$. Here if it is not specified otherwise then it is assumed that M is embedded in $L^{\infty}(M)$ via the Kuratoswki embedding.

The filling radius measures how thick a Riemannian manifold is. For example, the filling radius of the cylinder $S^1 \times R$ is $\pi/3$, but the filling radius of R^2 is infinite.The filling radius of an ellipse is equal to its smallest principal axis. The filling radius of the cylinder $S^1 \times R \subset R^3$ is the radius of S^1. Now we can give two theorems that together imply the systolic inequality.

Theorem 3.5. *If M is a closed aspherical manifold then*

$$sys(M, g) \leq 6FillRad(M, g)$$

Theorem 3.6. *For any closed manifold M,*

$$FillRad(M, g) \leq C_n Volume(M, g)^{1/n}$$

.

There is a huge detail under the rug that won't be discussed precisely. Some manifolds are not the boundaries in the topological sense, so C is not quite a set, it is a chain in the sense of homology. We do not have enough time to present this properly, we might use some other way to avoid filling radius and filing area. However we prove an important theorem using them in the case $n = 1$. The following theorem is important both in metric geometry, in topological graph theory and in scientific computing.

3.3 Separator theorem

Theorem 3.7. *Let M be a Riemannian 2-sphere (S^2, g) or an embedded graph (S^2, G) with vertex set V. Then there exists a simple loop $\gamma \subset S^2$ such that if we denote $A \cup B = S \setminus \gamma =$ the connected components of its complement. Then*

1. $min(area(A), area(B)) \geq area(M)/4$
2. $length(\gamma) \leq 2\sqrt{2area(S^2)}$

In the discrete case, there are no edges between A and B, and

1. $\min(\#A, \#B) \geq \#V/4$
2. $\#(V \cap \gamma) \leq 2\sqrt{2\#V)}$

3.4 The filling area of the circle

Similar to the Kuratowski embeddings L^∞, l_∞ satisfy the following nice property:

Lemma 3.8. *Given $Y \subset X$ metric spaces, any Lipschitz function $f\colon Y \to L^\infty$ or $f\colon Y \to l^\infty$ can be extended to a Lipschitz function defined on all of X.*

We don't proof this but just emphasize that we could give an alternative definition of filling radius and filing area. In this section we are interested in the filling area of the simplest of case, the circle $S^1 \subset R^2$ with its induced length Riemannian metric. An isometric filling of S^1 is a surface with boundary M, such that ∂M is isometric to S^1. We abuse notation and set $S^1 = \partial M$ and we say that M is an isometric filling of S^1 if $d_{S^1}(x, y) \geq d_M(x, y)$ for all $x, y \in \partial M = S^1$. Similarly if C_{2n} is the cycle graph of $2n$ vertices we ask for the quadrangulated surface M such that $\partial M = C_{2n}$ and $d_{C_{2n}}(x, y) \geq d_M(x, y)$, for any $x, y \in \partial M = C_{2n}$. The filling area problem is to estimate the minimal area of such M. In the discrete case this is the minimal number of quadrangles. We denote by S^n_+ a hemisphere.

Theorem 3.9. *If M Riemannian is an isometric filling of S^1 then $area(M) \geq \frac{1}{2}area(S^2_+)$. If M a quadrangulation which is an isometric filling of C_{2n} then the number of quadrangles is at least $\frac{1}{4}n(n-1)$.*

It has long been conjectured that the filling area of the circle is the area of a round hemisphere. That is $area(M) \geq area(S^2_+)$ in the continuous setting and $\frac{1}{2}n(n-1)$ in the discrete one.

3.5 Systole-Filling radius inequality

Let us proof

Theorem 3.10. *If M is a closed aspherical manifold then*

$$sys(M,g) \leq 6FillRad(M,g)$$

Proof. Let C be a filling of M which we assume is triangulated and for contradiction we assume that it is contained in $N_R(M)$ for some $R < sys(M)/6$ in the space L^∞ (which we pretend to be $\mathbb{R}^N$ for some large N). We can assume that every edge in the triangulation is very short. The contradiction will come because we construct a map $f\colon C \to M$ that is the identity in M, but by definition $\partial C = M$ and this contradicts a theorem we have shown suing degree theory. We define $f\colon \partial C \to \partial C$ to be the identity. To extend this map we induct on the dimension of the cells of the tringulation. We begin defining f on the vertices of C. For each v a vertex of C, there exists a point on ∂C at distance less than R, pick any such point to define $f(v)$. Now we want to define the map on the edges which are very short, call an upper bound on this distance δ. Notice that if $e = (v,u)$ is an edge then $d(f(u), d(v)) \leq 2R + \delta$ by the triangle inequality we define $f(e)$ to be a shortets path between $f(v)$ and $f(u)$. Now we look at a triangle, it has perimeter $6R + 3\delta$. Since $R < sys(M)/6$, if δ is small engough then $6R + 3\delta < sys(M)$. This means that if σ is a face, and $f(\partial\sigma) \subset M$ is contractible. This means that we can fill it by some *triangle* to define $f(\sigma)$. Now we look at a tetrahedra $\tau \subset C$, since M is aspherical we can extend the map from the boundary $f(\partial\tau)$ to the interior, we continue going up in dimension. At the end of this process we have defined a map from $C \to \partial C$ that is the identity on the boundary. $\square$

Stephan Wenger has a very short proof of the second theorem we need:

$$FillRad(M,g) \leq C_n Volume(M,g)^{1/n}$$

. However short and mostly elementary it is not easy. The constant in this isoperimetric inequality is very big, so the constant it derives for the systolic inequality 6 times very big. In fact the examples of the torus and the projective plane show that the optimal constant should be around $\sqrt{n}$, Wegner's proof gives something around $27^n n!$. A much smaller constant was obtained recently by Nabutovsky, following ideas of Papasoglu and Guth, which is around n and is somehow simpler.

3.6 Nerves and partitions of unity

Given a family of sets $U_1, U_2 \dots U_k$ the ***nerve*** $\mathcal{N}[U_1, U_2, \dots U_k]$ is a simplicial complex that has one vertex for each set U_i and one simplex σ for any family of sets with non-empty intersection: $\cap_{i\in\sigma} U_i \neq 0$ iff $\sigma \in \mathcal{N}[U_1, U_2, \dots U_k]$.
The dimension of this covering is the the multiplicity of the nerve minus one. If $\{U_i\}$ is an open covering of a space X and there are functions $\phi_i \colon U_i \to \mathbb{R}$ such that for any point x, $\sum \phi_i(x) = 1$ then we call the pair of open cover and functions a ***partition of unity***. It comes handy in manifold theory to pass from statements in $\mathbb{R}^n$ to statements in a manifold. For example to show Sard's theorem or to show that the smooth approximation of a continuous function has enough regular points this is the technical tool to use. We will use it in slightly different fashion as our main object is a distance function, we are interested in metric rather than smooth objects.

Lemma 3.11. *If $\{U_i\}$ are open subsets the map $\phi(x) = \frac{d(.,X\setminus U_i)}{\sum_j d(.,X\setminus U_j)}$ is a partition of union, and $\psi : X \to \mathcal{N}$,*

$$\psi(x) := \sum_{i:x\in U_i} v_i \frac{d(.,X\setminus U_i)}{\sum_j d(.,X\setminus U_j)}$$

is a 1-Lipschitz map such that $\psi(U_i) \subset st(v_i)$.

3.7 Width and volume profile

The ***radius*** of a set U on a metric space is the smallest R such that there exists a point $x \in U$ such that $U \in \overline{B(x,R)}$. The ***i-width*** is the least upper bound $R > 0$ such that X can be covered with open sets $\{U_i\}$ of radius R each, such that the multiplicity of the covering is at most $i+1$. Clearly the i-widths are decreasing in i. For compact metric spaces, one can equivalently define the i-width as the least upper bound, of the maximal radius of a fiber of a map $f\colon X \to T$, where T is a simplicial complex of dimension i. As an exercise you can show the equivalence between these two definitions.

Volume profile For a given Riemannian manifold, or more generally Riemannian simplicial complex X the volume profile is the function $v_X(r) := sup\{vol(B(p,r)) : p \in X\}$, this function is not decreasing and for r big enough $v_X(r) = vol(X)$. Here is an easy observation that will work as the basis of the induction of the following theorem.

Lemma 3.12. *Let X be a Riemannian graph, if $v_X(R) < R$ for some $R > 0$ then $width_0 < R$.*

Our next goal are the following theorems:

Theorem 3.13. *Let X is an n-dimensional Riemannian polyhedron, if for some R*

$$R > n(v_X(R))^{\frac{1}{n}}$$

then

$$width_{n-1}(X) < R.$$

In particular $width_{n-1}(X) \leq nvol(X)^{\frac{1}{n}}$

Theorem 3.14. *Let X is an n-dimensional aspherical Riemannian manifold, then $sys(X) < 6width_{n-1}(X)$.*

Together they imply the systolic inequality. Let's begin from the second one, its proof is similar to the filing radius systole inequality.

Theorem 3.15. *Let X is an n-dimensional aspherical Riemannian manifold, then $sys(X) < 6width_{n-1}(X)$.*

Proof. Assume the opposite. Let N be the nerve equipped with $\psi\colon X \to N$, let us construct a simplexwise smooth map $f\colon N \to X$ and a homotopy h between the identity in X and $f\psi$. This is absurd because on one hand the identity has degree 1 and on the other, a smooth approximation of $f\psi$ has degree 0, because f is simplex-wise smooth and N is of dimension $n-1$, this means that it $f(N)$ has n-volume 0, so for the map f and hence for the map $f\psi$ almost every regular value has 0 fibers.
For every vertex $v_i \in N$, choose some point $p_i \in U_i \subset X$, $f(v_i) = p_i$. For each edge $e = v_i, v_j$ in N choose the shortest path between p_i and p_j, so $f(e) = [p_i, p_j])$. Notice that $d(p_i, p_j) < 2width_{n-1}(X)$ so triangles in N^1 map to triangles with perimeter $< sys(X)$ we can fill them and then use asphericity to fill f going up in dimensions.
Now assume that we have a very fine triangulation on X such that every edge is contained in one of the sets of the open cover. Extend this triangulation to a triangulation of all of $X \times [0, 1]$. We have the map h defined on the top and on the bottom. We extend the map to the 0-skeleton, and built the map inductively on the dimension. □

3.8 Almost minimal separating complexes

Let us come back to the width volume inequality. We have shown that for a graph if $v_X(R) < R$ for some $R > 0$ then $width_1 < R$. This is the base of an induction proof, claiming taht if $(v_X(R))^{\frac{1}{n}} < \frac{R}{n}$ then $width_{n-1}(X) < R$. The main idea of the proof is similar to the proof of the separator theorem. Let's call S an R-***separating*** simplicial complex if $S \subset X$ has dimension $(n-1)$ and $X \setminus S$ consists of connected components of radius less than R.

Lemma 3.16. *Let X be a Riemainnian simplicial complex, for any $R > 0$ and any $\epsilon > 0$, there exists an R-separating complex $Q \subset X$, such that for any $r_0 < r_r < R$:*

$$v_Q(r_0) < \frac{1}{r_1 - r_0} v_X(r_1)$$

As before we need to use Sard's theorem to transform the distance function into a smooth 1-Lipschitz function of each simplex. As in the proof of the systolic inequality for surfaces we need the co-area formula. We need it for Riemannian complexes but it easily follows from the formula on each simplex, namley we need that if $f : X \to \mathbb{R}$ is a simplex-wise smooth Lipschitz function then $\int_{r_0}^{r_1} vol_{n-1}(f^{-1}(t))dt \geq vol_n(f^{-1}([r_0, r_1])$.

Proof. We apply Sard theorem to the distance from a fixed point x obtain a function $dist'(p, .)$ which is close to the distance function and for which the "spheres" $S'(p, c) := \{x \in X : dist'(p, x) = c\}$ are unions of smooth $(n-1)$-manifolds with boundary which after possibly further subdividing into smaller simplicies becomes a Riemannian complex. We can assume that for every $c \in [r_0 + \delta, r_1 - \delta]$,

$$vol_{n-1}(S'(p, c)) \leq \frac{1}{r_1 - r_0 + \delta} vol_n(B(p, r_1)) < \frac{1}{r_1 - r_0} v_X(r_1) + \epsilon/2$$

Now suppose that Q is the R-separating complex with minimal volume upto a $\epsilon/2$ error. Note that for any p and any $c < R$ cutting out $B'(p, c) \cap Q$ and substituing it by $S'(p, c)$ yields a new R-separating complex, so by minimality:

$$vol_{n-1}[Q \cap B'(p, r_0)] - \epsilon/2 \leq vol_{n-1}(S'(p, c))$$

so,

$$vol_{n-1}[Q \cap B'(p, r_0)] \frac{1}{r_1 - r_0} v_X(r_1) + \epsilon$$

Since distances in Q are not smaller than the distances in X so $B_Q(p, r_0) \subset Q \cap B'(p, r_1)$. Since p was arbitrary: $v_Q(r_0) \leq \frac{1}{r_1 - r_0} v_X(r_1) + \epsilon$ as desired. $\square$

The previous lemma stated an upper bound on the area of a near minimal R-separating complex of X. The next lemma relates the width of an R-separating complex Q with the width of the complex X.

Lemma 3.17. *Let Q be an R-separating complex of X, if $width_{n-2}(Q) \leq R$ then $width_{n-1}(X) \leq R$*

Proof. We begin with the open cover $U_1, U_1, \ldots U_k$ of Q and we thicken the sets without increasing the multiplicity of any intersection we add the connected components $X - Q$. $\square$

We are ready to put the ingredient together, remember what we want to show

Theorem 3.18. *Let X is an n-dimensional Riemannian polyhedron, if for some R, $R > n(v_X(R))^{\frac{1}{n}}$ then $width_{n-1}(X) < R$. In particular $width_{n-1}(X) \leq nvol(X)^{\frac{1}{n}}$*

Proof. The induction step: By assumption, $R > nv_X(R)^{\frac{1}{n}}$ for some R. Let Q be a minimal separating complex, we can take $r = \frac{n-1}{n}R$, and use minimality to obtain:

$$v_Q(r) < \frac{1}{R-r}v_X(R) + \epsilon = \frac{n}{R}v_X(R) + \epsilon$$

We can assume that ϵ was chosen small enough so that,

$$\frac{n}{R}v_X(R) + \epsilon < \frac{n}{R}(\frac{R}{n})^n = (\frac{R}{n})^{n-1} = (\frac{r}{n-1})^{n-1}$$

We have shown that $v_Q(r) < (\frac{r}{n-1})^{n-1}$ which by the induction hypothesis implies that $width_{n-2}(Q) \leq r < R$. But Q was R-separating, so by the previous lemma $width_{n-1}(X) < R$. $\square$