Homework 1:Due Friday 23th at midnight.

Systolic Geometry Course

October 15, 2020

1. Write a small introductory letter. What are your interests? What is your background in general? What is your background in geometry and topology?

2. What are the cardinalities of the fibers of the map $z \to z^3$ on $\{z \in \mathbb{C} : |z| = 1\}$?

3. Is 0 a regular value of $p(x, y) = x^2 - y^2$? What about 1? Draw the curves $p(x, y) = 0$, $p(x, y) = 1$.

4. Show for every point $p \in \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + y^2 = 1\}$ there exists an open neighborhood U, with $p \in U$ and a diffeomorphim $\mathring{\mathbb{D}}^2 \to U$ where $\mathring{\mathbb{D}}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

5. For every d construct a map from the sphere $\mathbb{S}^2$ to itself such that almost every value y has d elements in the fiber: $\#f^{-1}(y) = d$.

6. Prove that $2d + 2$ random points are affinely independent. Prove that if $2d + 2$ points are affinely independent then any partition into two $(d + 1)$-tuples span affine spaces that do not intersect.

7. Show that the complete graph on 5 vertices is not planar.

8. Why is the proof of the Jordan theorem we gave not valid for general simple closed curves.

9. The Brouwer fixed point theorem claims that every continuous map $f : [0, 1]^2 \to [0, 1]^2$ has a fixed point, i.e. a point $(x, y) \in [0, 1]^2$ such that $f(x, y) = (x, y)$.

   - Using the intermediate value theorem show that every map $f : [0, 1] \to [0, 1]$ has a fixed point.
   - Evaluate whether the following argument is correct:
     Write $f = (f_1, f_2)$. For y let $h_y(x) = f_1(x, y) : [0, 1] \to [0, 1]$. Apply the one dimensional fixed point theorem to h_y to obtain a value $c(y)$ such that $f_1(c(y), y) = h_y(c(y)) = c(y)$. Now consider y as a variable and consider the function $g(y) := f_2(c(y), y)$. Since g is a function from $[0, 1]$ to $[0, 1]$ by the one dimensional fixed point theorem it has a fixed point. This is a y_0, such that $g(y_0) = y_0$. Unwinding we have shon the existence of a point $c(y_0), y_0)$ such that: $f(c(y_0), y_0) = (f_1(c(y_0), y_0), f_2(c(y_0), y_0)) = (c(y_0), y_0)$.
     Explain whether you find this proof convincing. Can you make a picture that explains the situation?