On GeoGebra Automated Reasoning Tools

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Joint work with

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Zoltán Kovács (The Private University College of Education of the Diocese of Linz (Austria))

J. Rafael Sendra (UAH)

M.Pilar Vélez (U Nebrija)

Carlos Villarino (UAH)

.... +++

MTM2017-88796-P

What is GeoGebra?

- Hohenwarter, M. (2002). “Ein Softwaresystem für dynamische Geometrie und Algebra der Ebene”. Master’s thesis. Salzburg University.

- GeoGebra is dynamic mathematics software for all levels of education that brings together geometry, algebra, spreadsheets, graphing, statistics and calculus in one easy-to-use package.

- In 2013, Bernard Parisse's **Giac** was integrated into GeoGebra's CAS view.

geogebra.org

About GeoGebra – GeoGebra

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What is GeoGebra?

GeoGebra is dynamic mathematics software for all levels of education that brings together geometry, algebra, spreadsheets, graphing, statistics and calculus in one easy-to-use package. GeoGebra is a rapidly expanding community of millions of users located in just about every country. GeoGebra has become the leading provider of dynamic mathematics software, supporting science, technology, engineering and mathematics (STEM) education and innovations in teaching and learning worldwide.

Quick Facts

- Geometry, Algebra and Spreadsheet are connected and fully dynamic
- Easy-to-use interface, yet many powerful features
- Authoring tool to create interactive learning resources as web pages
- Available in many languages for our millions of users around the world
- Open source software freely available for non-commercial users

Awards

- Archimedes 2016: MNU Award in category Mathematics (Hamburg, Germany)
- Microsoft Partner of the Year Award 2015: Finalist, Public Sector: Education (Redmond, WA, USA)
- MERLOT Classics Award 2013: Multimedia Educational Resource for Learning and Online Teaching (Las Vegas, Nevada, USA)
- NTLC Award 2010: National Technology Leadership Award (Washington D.C., USA)
- Tech Award 2009: Laureat in the Education Category (San Jose, California, USA)
- BETT Award 2009: Finalist in London for British Educational Technology Award
- SourceForge.net Community Choice Awards 2008: Finalist, Best Project for Educators
- AECT Distinguished Development Award 2008: Association for Educational Communications and Technology (Orlando, USA)
- Learnie Award 2006: Austrian Educational Software Award (Vienna, Austria)
- eTwinning Award 2006: 1st prize for "Crop Circles Challenge" with GeoGebra (Linz, Austria)
- Les Trophées du Libre 2005: International Free Software Award, category Education (Soisson, France)
- Comenius 2004: German Educational Media Award (Berlin, Germany)
- Learnie Award 2005: Austrian Educational Software Award (Vienna, Austria)
- digita 2004: German Educational Software Award (Cologne, Germany)
- Learnie Award 2003: Austrian Educational Software Award (Vienna, Austria)
- EASA 2002: European Academic Software Award (Ronneby, Sweden)

ɔut GeoGebra
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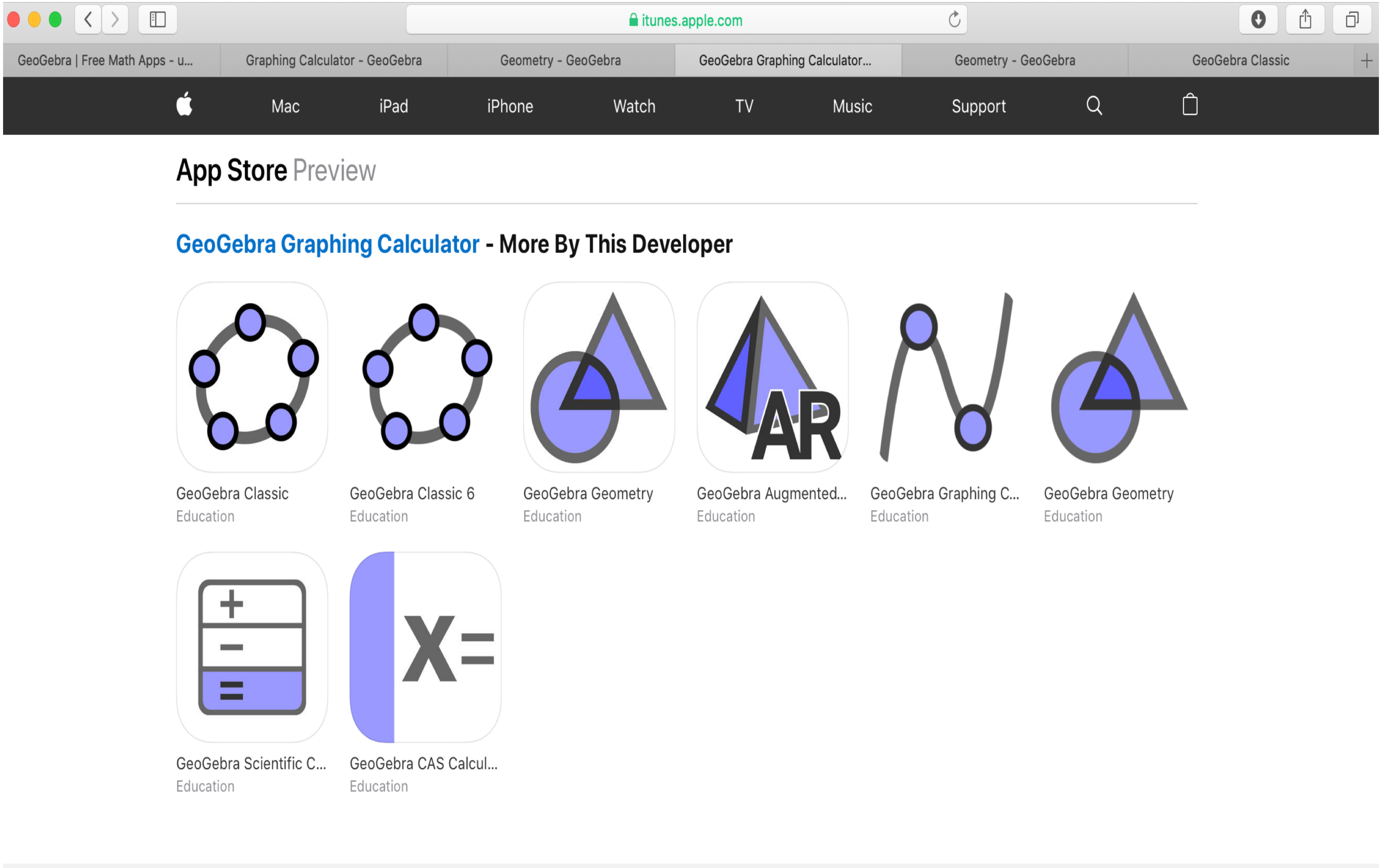
Language: English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GeoGebra | Free Math Apps - u...
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Geometry - GeoGebra
GeoGebra Graphing Calculator...
Geometry - GeoGebra
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Music
Support
App Store Preview
GeoGebra Graphing Calculator - More By This Developer
GeoGebra Classic
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GeoGebra Classic 6
Education
GeoGebra Geometry
Education
GeoGebra Augmented...
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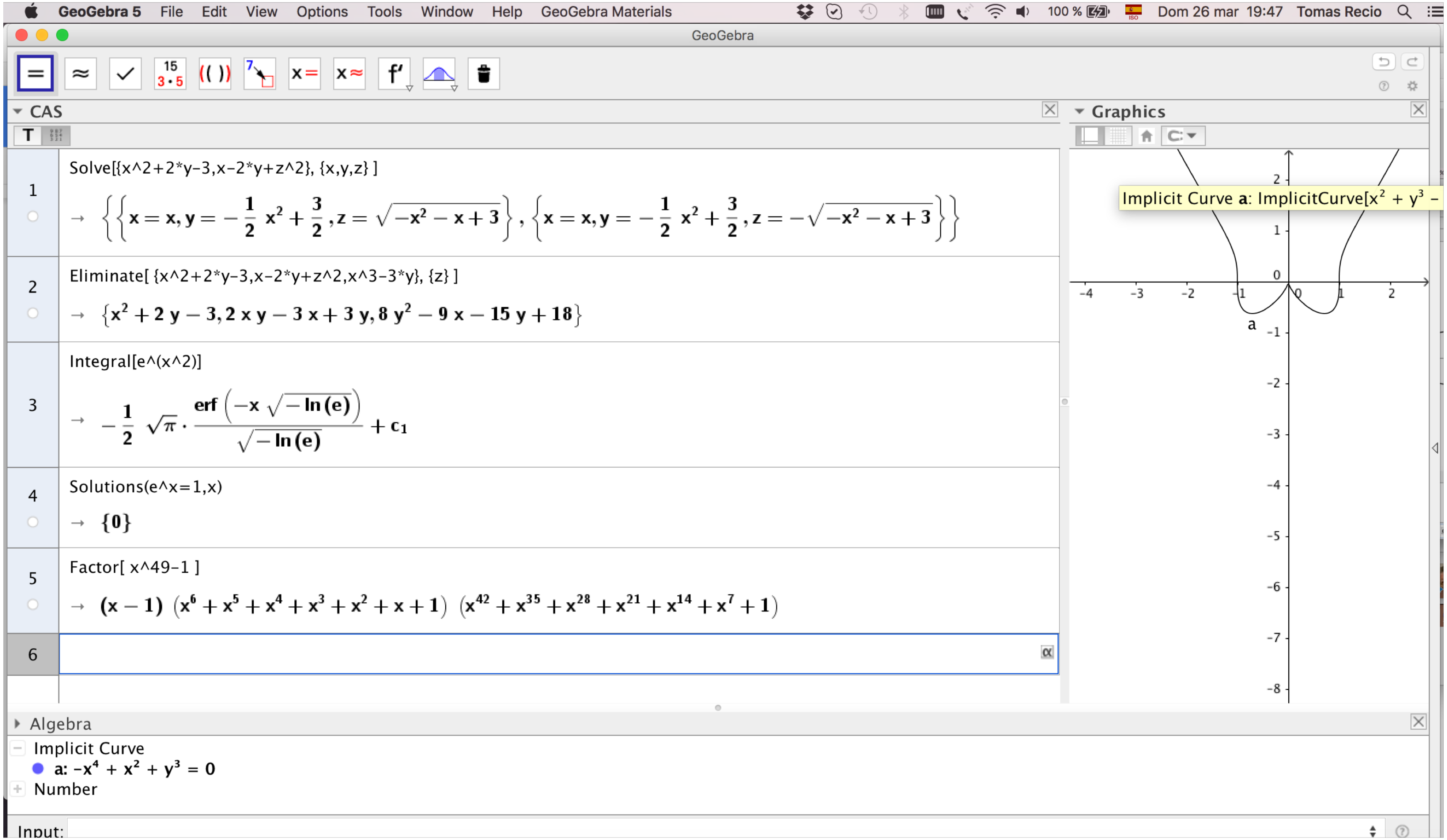
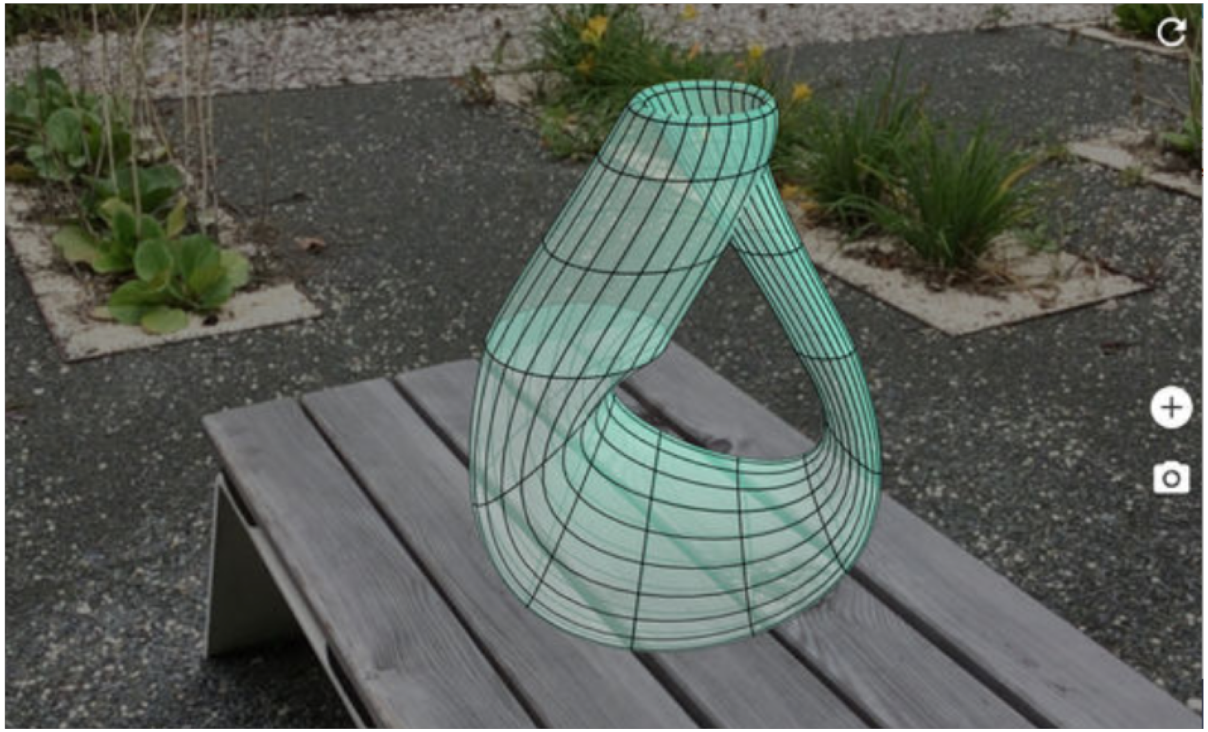
Education

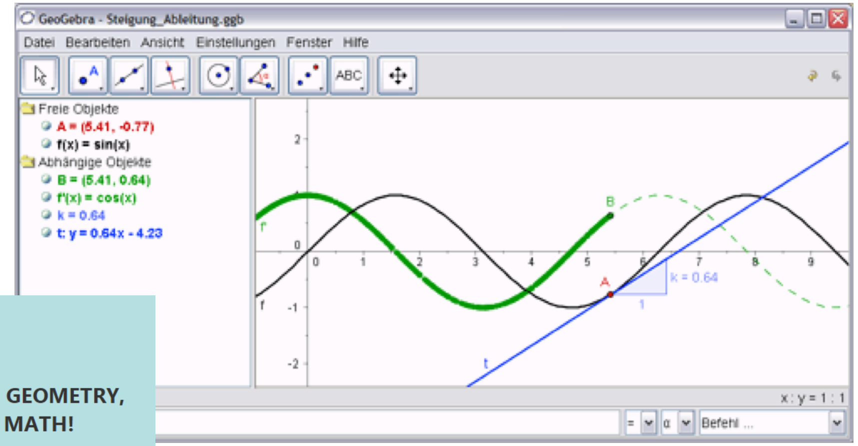

GeoGebra 5 File Edit View Options Tools Window Help GeoGebra Materials
100 % Dom 26 mar 19:47 Tomas Recio
GeoGebra
CAS
Graphics
Solve[{x^2+2*y-3,x-2*y+z^2}, {x,y,z}]
→ {{x = x, y = −1/2 x² + 3/2, z = √(−x² − x + 3)}, {x = x, y = −1/2 x² + 3/2, z = −√(−x² − x + 3)}}
Eliminate[{x^2+2*y-3,x-2*y+z^2,x^3-3*y}, {z}]
→ {x² + 2 y − 3, 2 x y − 3 x + 3 y, 8 y² − 9 x − 15 y + 18}
Integral[e^(x^2)]
→ −1/2 √π · erf(−x √(−ln(e))) / √(−ln(e)) + c1
Solutions(e^x=1,x)
→ {0}
Factor[x^49-1]
→ (x − 1) (x⁶ + x⁵ + x⁴ + x³ + x² + x + 1) (x⁴² + x³⁵ + x²⁸ + x²¹ + x¹⁴ + x⁷ + 1)
Implicit Curve a: ImplicitCurve[x² + y³ −
Algebra
Implicit Curve
a: −x⁴ + x² + y³ = 0
Number
Input:

Augmented reality

https://www.facebook.com/geogebra/videos/10155662725038232/



GeoGebra
Dynamic Mathematics for Everyone



GEOGEBRA

THE GRAPHING CALCULATOR FOR FUNCTIONS, GEOMETRY, ALGEBRA, CALCULUS, STATISTICS AND 3D MATH!

DYNAMIC MATHEMATICS FOR LEARNING AND TEACHING

- Free/Open source software
- (40) Millions of users worldwide, +1 million resources
- Intense collaboration research project /GeoGebra for the development of automated augmented reality, derivation, proof, discovery tools.
- Geogebra as an EPO (Ente Promotor Observador)

GeoGebra **ART**: Automated reasoning tools

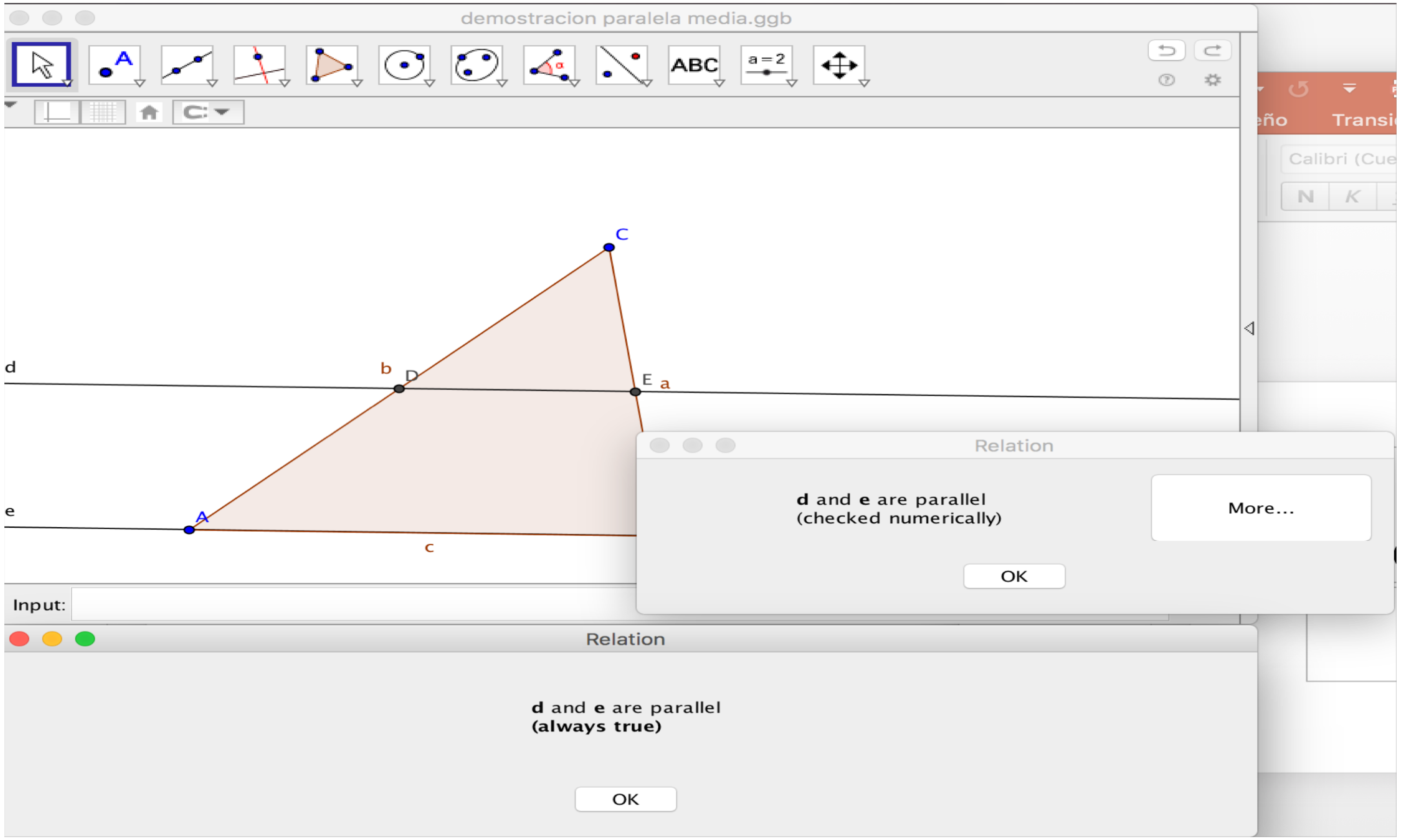
- Automated derivation
- Automated proving
- Automated Discovery
- Locus: mover-tracer, boolean, envelopes, etc.

- Botana, F.; Hohenwarter, M.; Janicic, J.; Kovács, Z.; Petrovic, I.; Recio, T.; Weitzhofer, S.: Automated Theorem Proving in GeoGebra: Current Achievements. Journal of Automated Reasoning, June 2015, Volume 55, Issue 1, pp 39-59

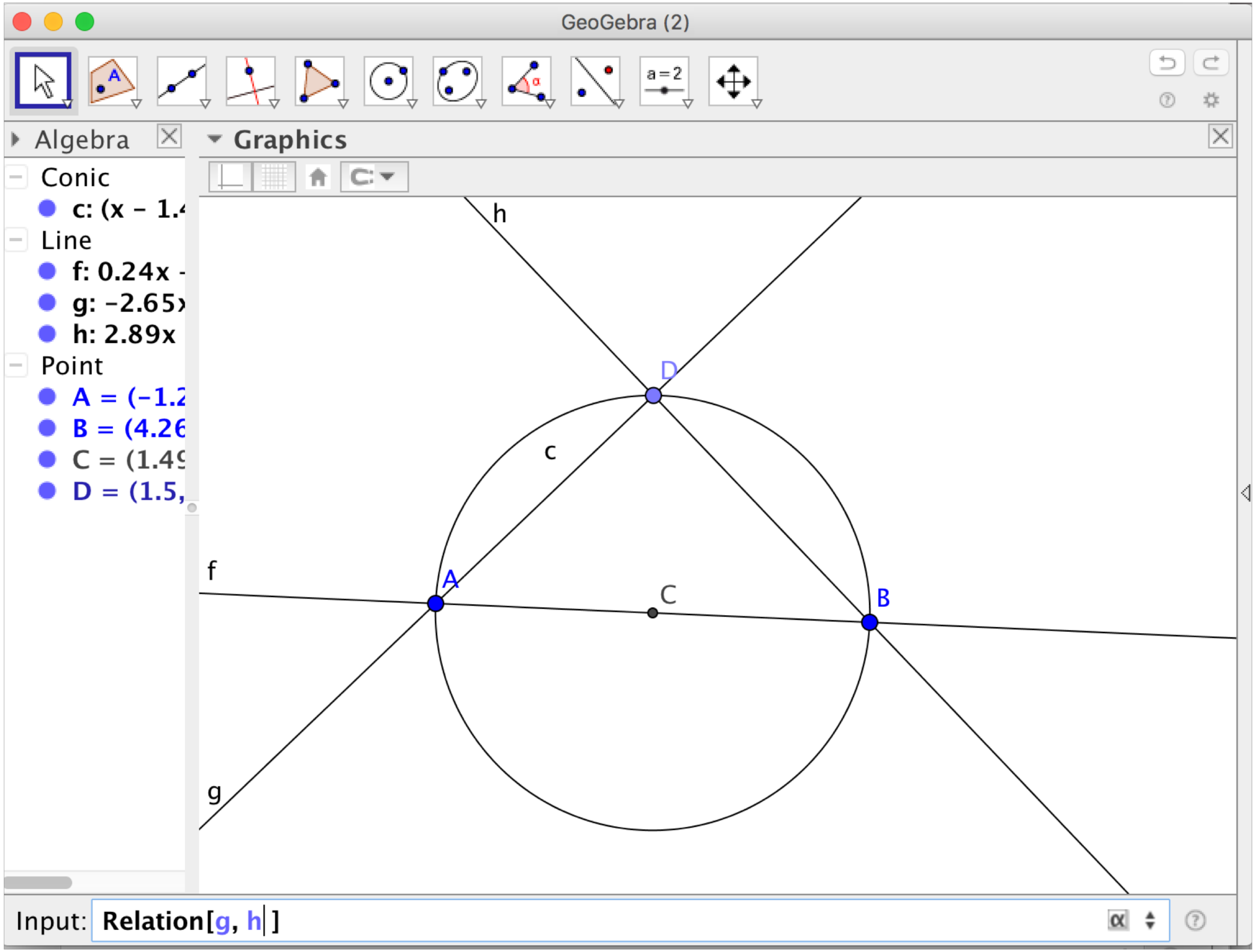
- Abánades, M.; Botana, F.; Kovács, Z.; Recio, T.; Sólyom-Gecse,C.: Development of automatic reasoning tools in GeoGebra. Software Demo Award at ISSAC 2016. ACM Communications in Computer Algebra. Volume 50 Issue 3, September 2016. Pages: 85-88

- Kovács, Z.; Recio, T.; Vélez, M.P. : Using Automated Reasoning Tools in GeoGebra in the Teaching and Learning of Proving in Geometry. International Journal of Technology in Mathematics Education. Vol. 25, no. 2. pp. 33-50. 2018.

Verify: numerically and formally

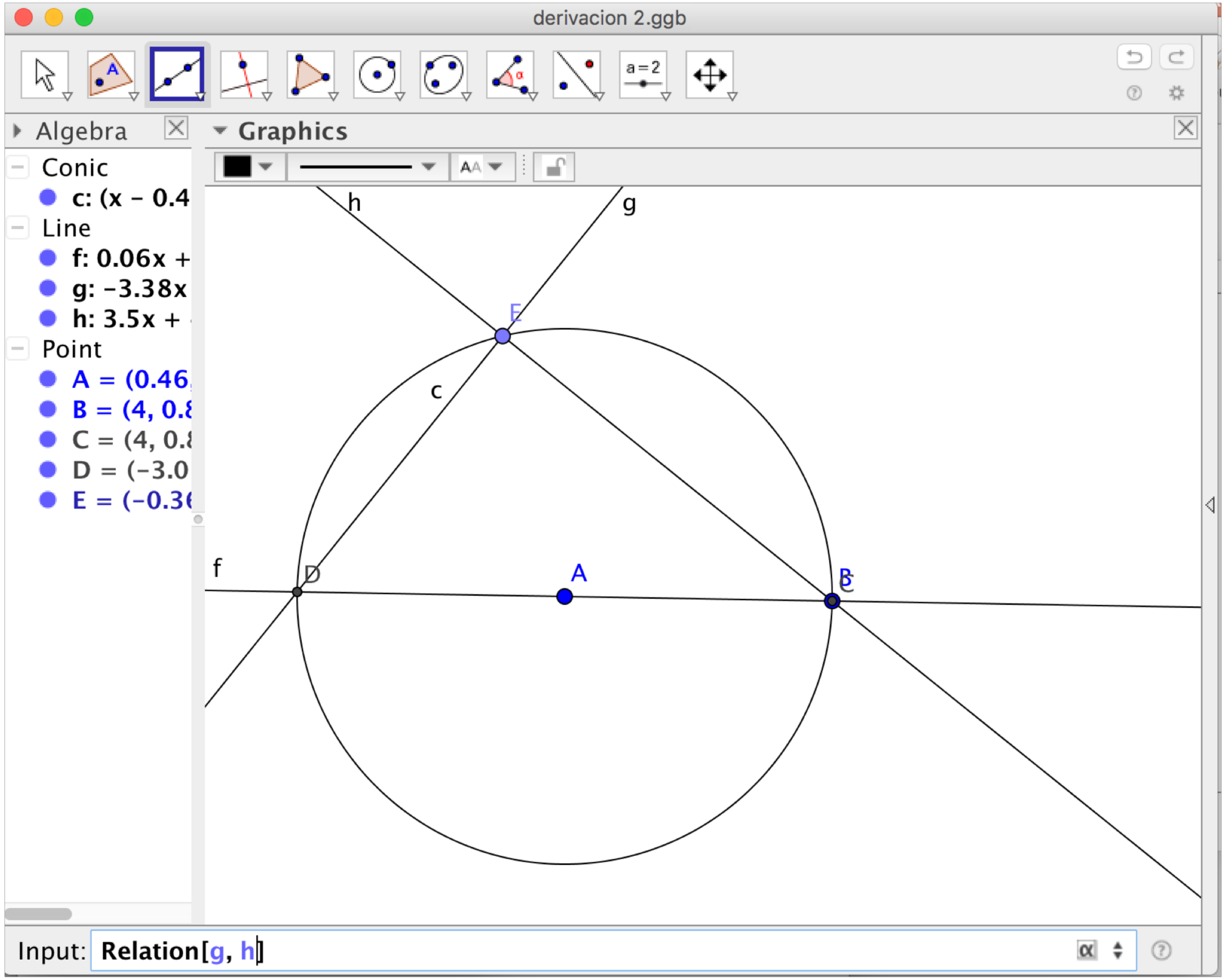


- DERIVATION



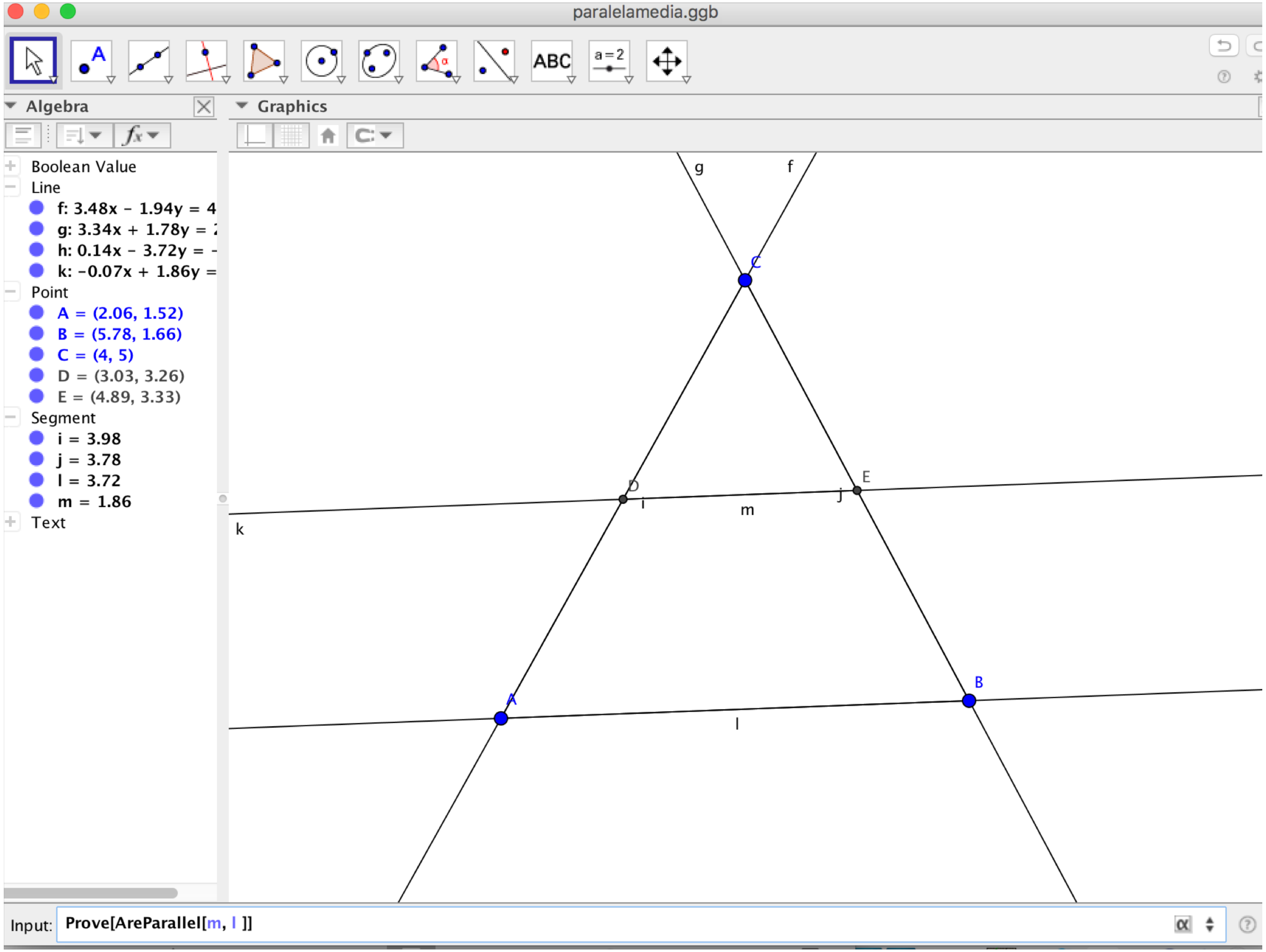
GeoGebra (2)
Algebra
Graphics
Conic
c: (x − 1.
Line
f: 0.24x
g: −2.65
h: 2.89x
Point
A = (−1.2
B = (4.26
C = (1.49
D = (1.5,
h
D
c
f
A
C
B
g
Input: Relation[g, h]

- DERIVATION



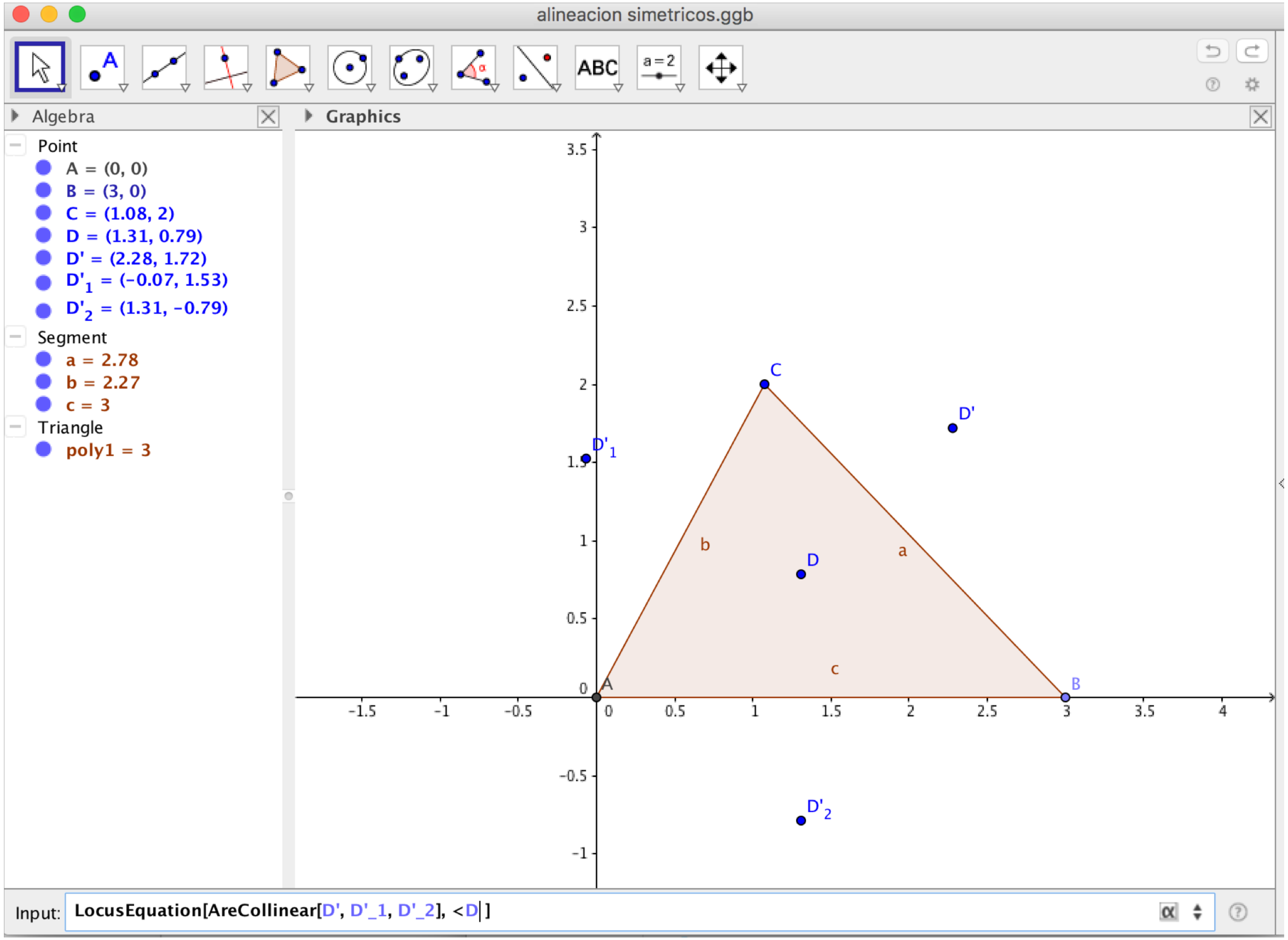
derivacion 2.ggb
Algebra
Graphics
Conic
c: (x − 0.4
Line
f: 0.06x +
g: −3.38x
h: 3.5x +
Point
A = (0.46,
B = (4, 0.8
C = (4, 0.8
D = (−3.0
E = (−0.36
h
g
E
c
f
D
A
B
C
Input: Relation[g, h]

- PROVE



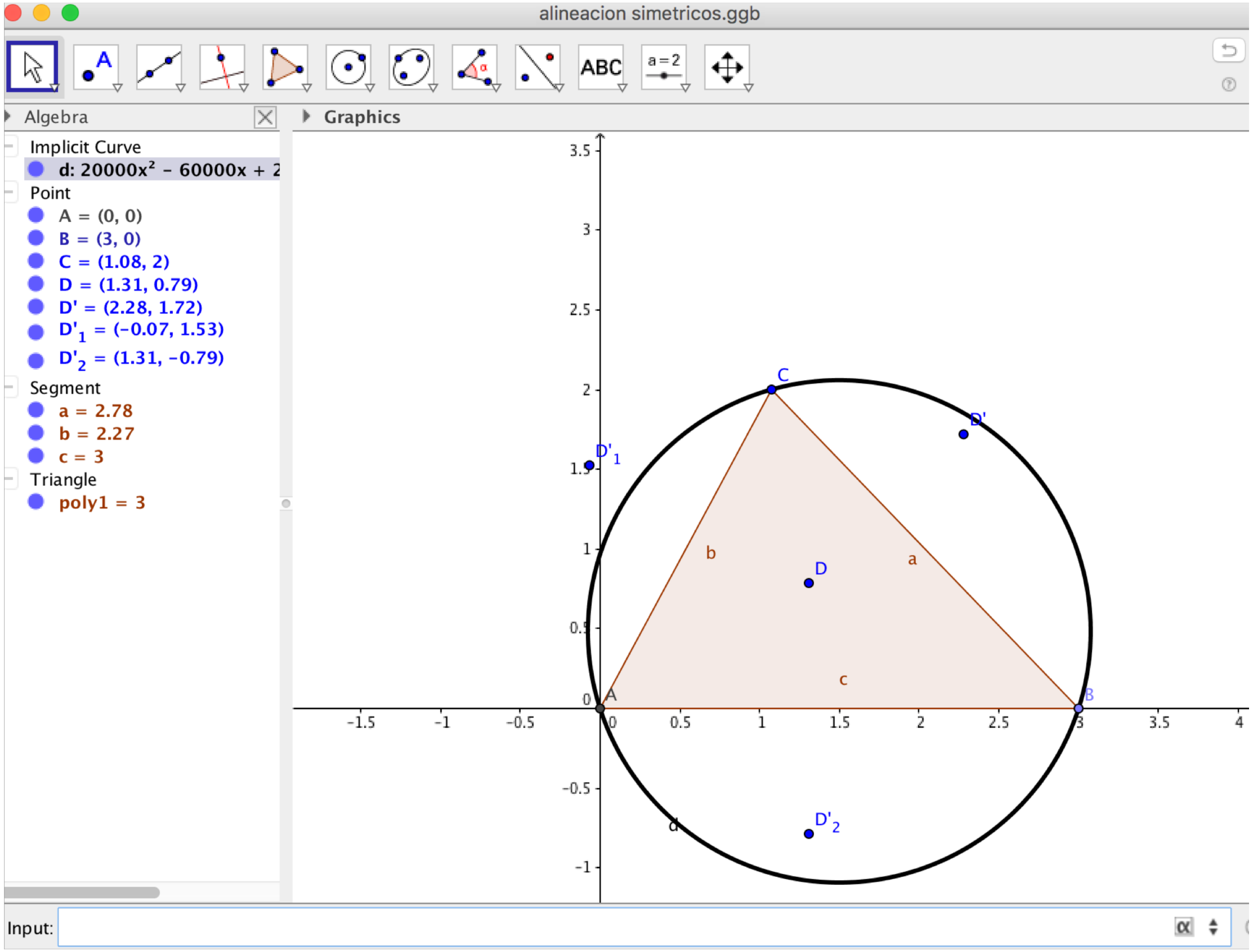
paralelamedia.ggb
Algebra
Graphics
Boolean Value
Line
f: 3.48x − 1.94y = 4
g: 3.34x + 1.78y =
h: 0.14x − 3.72y = -
k: −0.07x + 1.86y =
Point
A = (2.06, 1.52)
B = (5.78, 1.66)
C = (4, 5)
D = (3.03, 3.26)
E = (4.89, 3.33)
Segment
i = 3.98
j = 3.78
l = 3.72
m = 1.86
Text
Input: Prove[AreParallel[m, l]]

- DISCOVERY



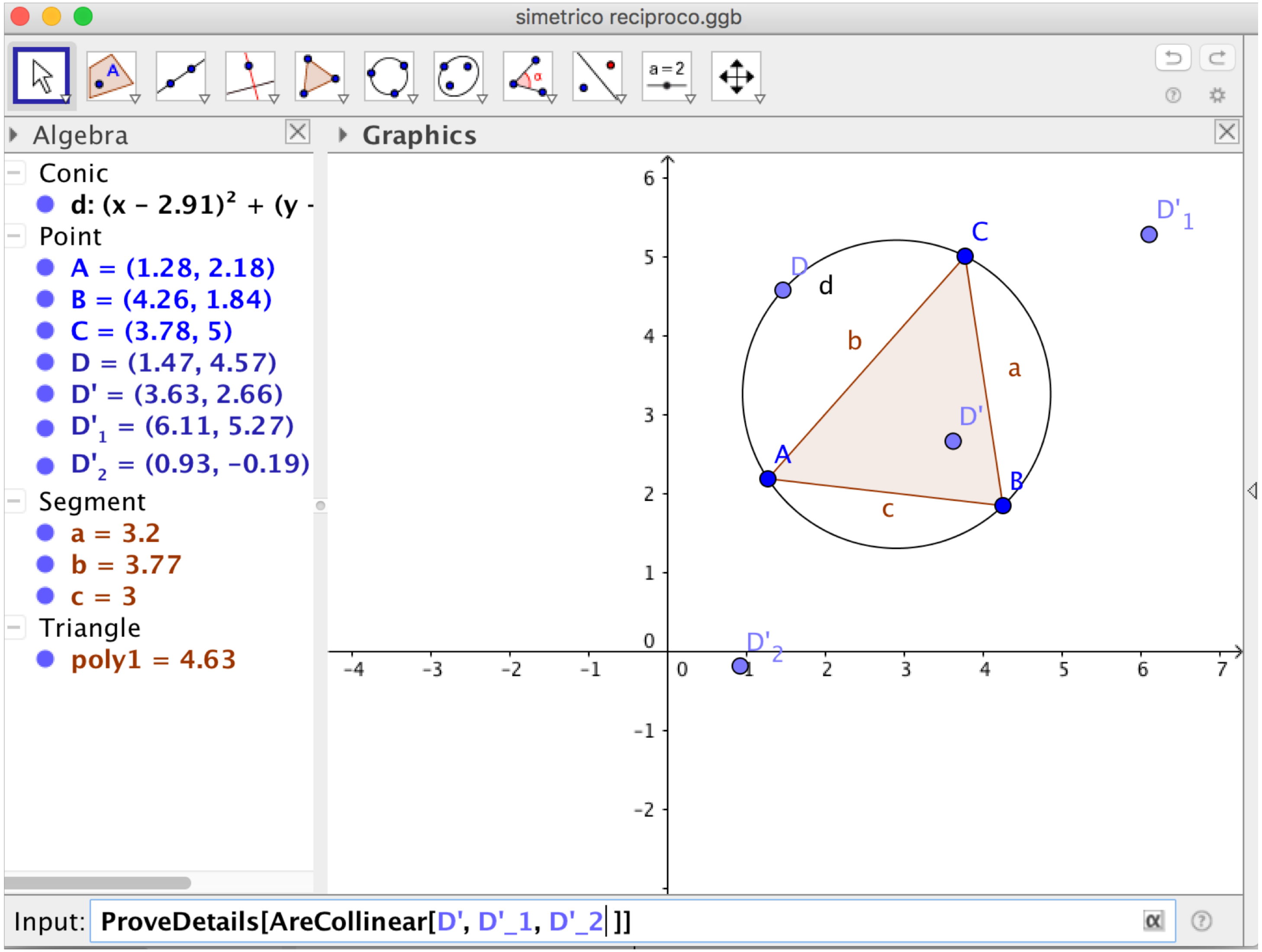
alineacion simetricos.ggb
Algebra
Graphics
Point
A = (0, 0)
B = (3, 0)
C = (1.08, 2)
D = (1.31, 0.79)
D' = (2.28, 1.72)
D'_1 = (−0.07, 1.53)
D'_2 = (1.31, −0.79)
Segment
a = 2.78
b = 2.27
c = 3
Triangle
poly1 = 3
Input: LocusEquation[AreCollinear[D', D'_1, D'_2], <D]

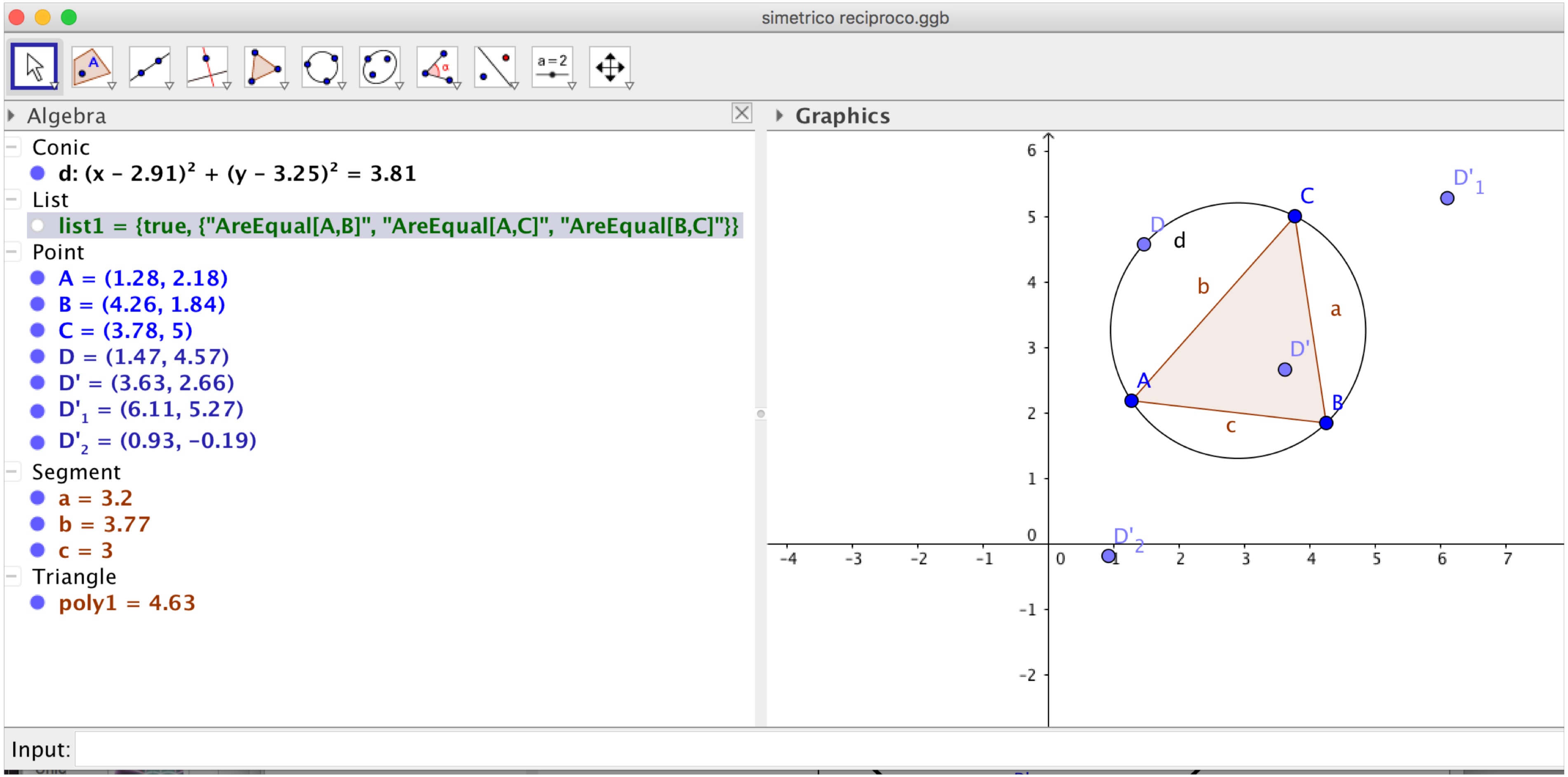
- DISCOVERY



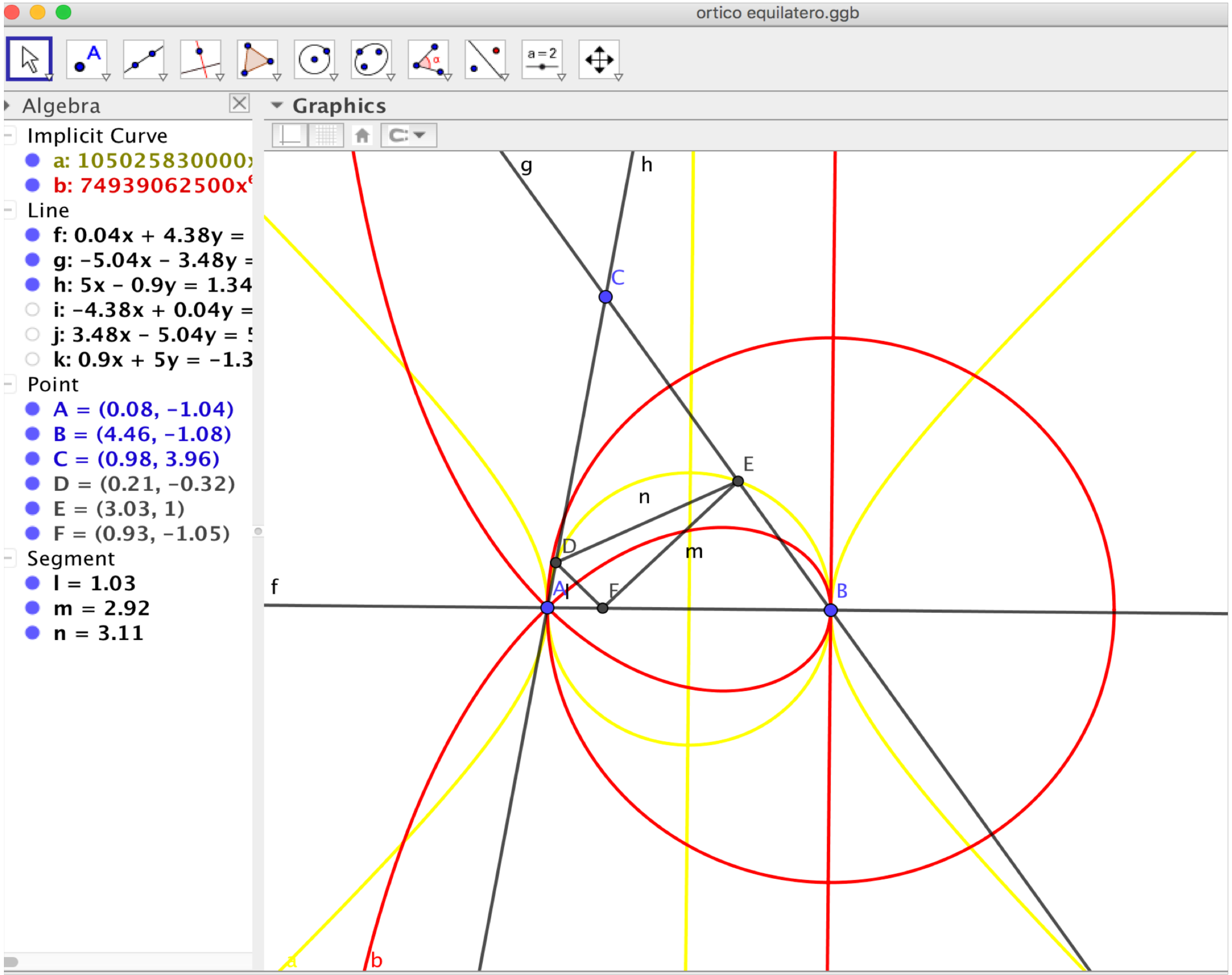
alineacion simetricos.ggb
ABC
a=2
Algebra
Graphics
Implicit Curve
d: 20000x² − 60000x + 2
Point
A = (0, 0)
B = (3, 0)
C = (1.08, 2)
D = (1.31, 0.79)
D' = (2.28, 1.72)
D'1 = (−0.07, 1.53)
D'2 = (1.31, −0.79)
Segment
a = 2.78
b = 2.27
c = 3
Triangle
poly1 = 3
Input:

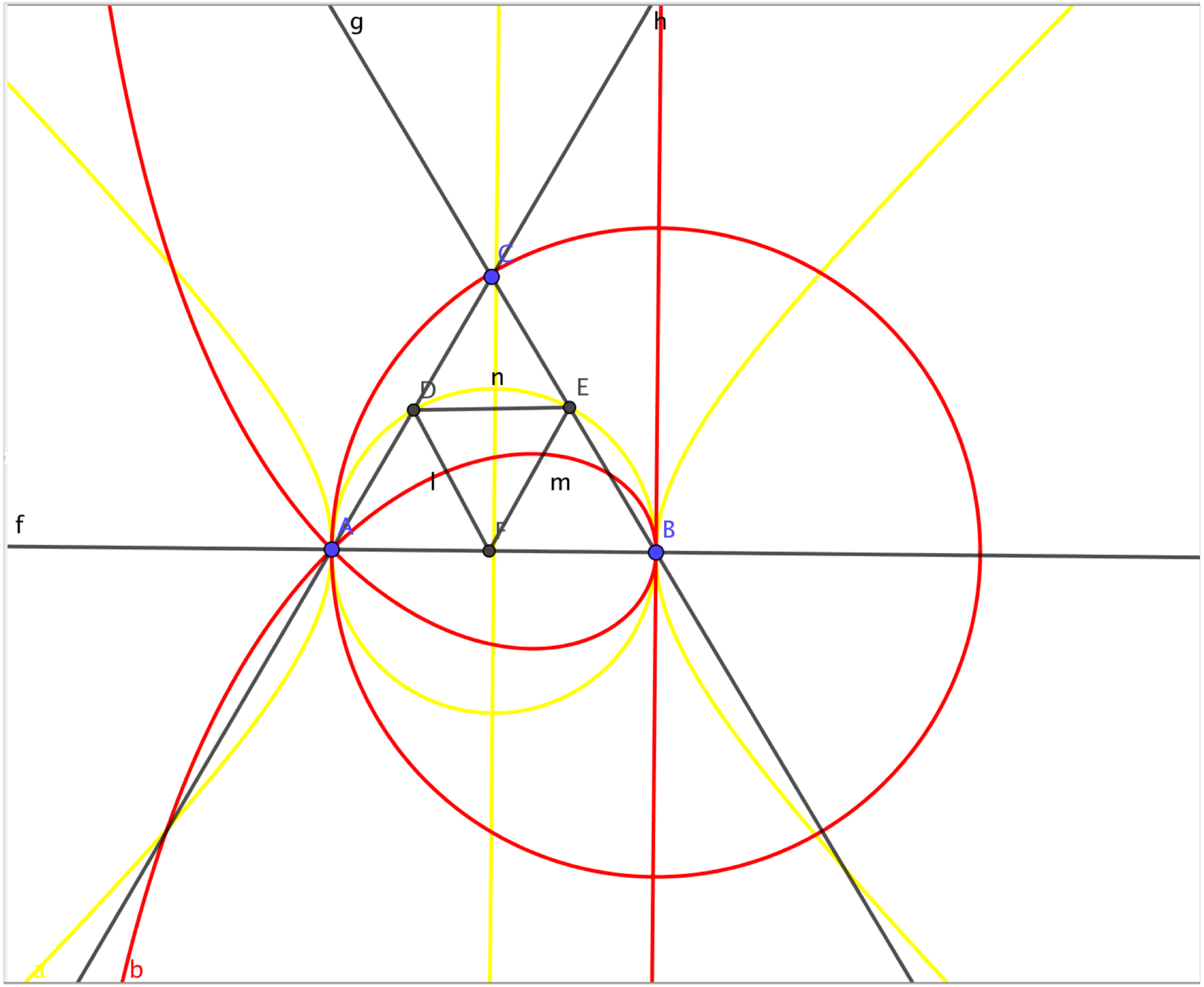
- DISCOVERY

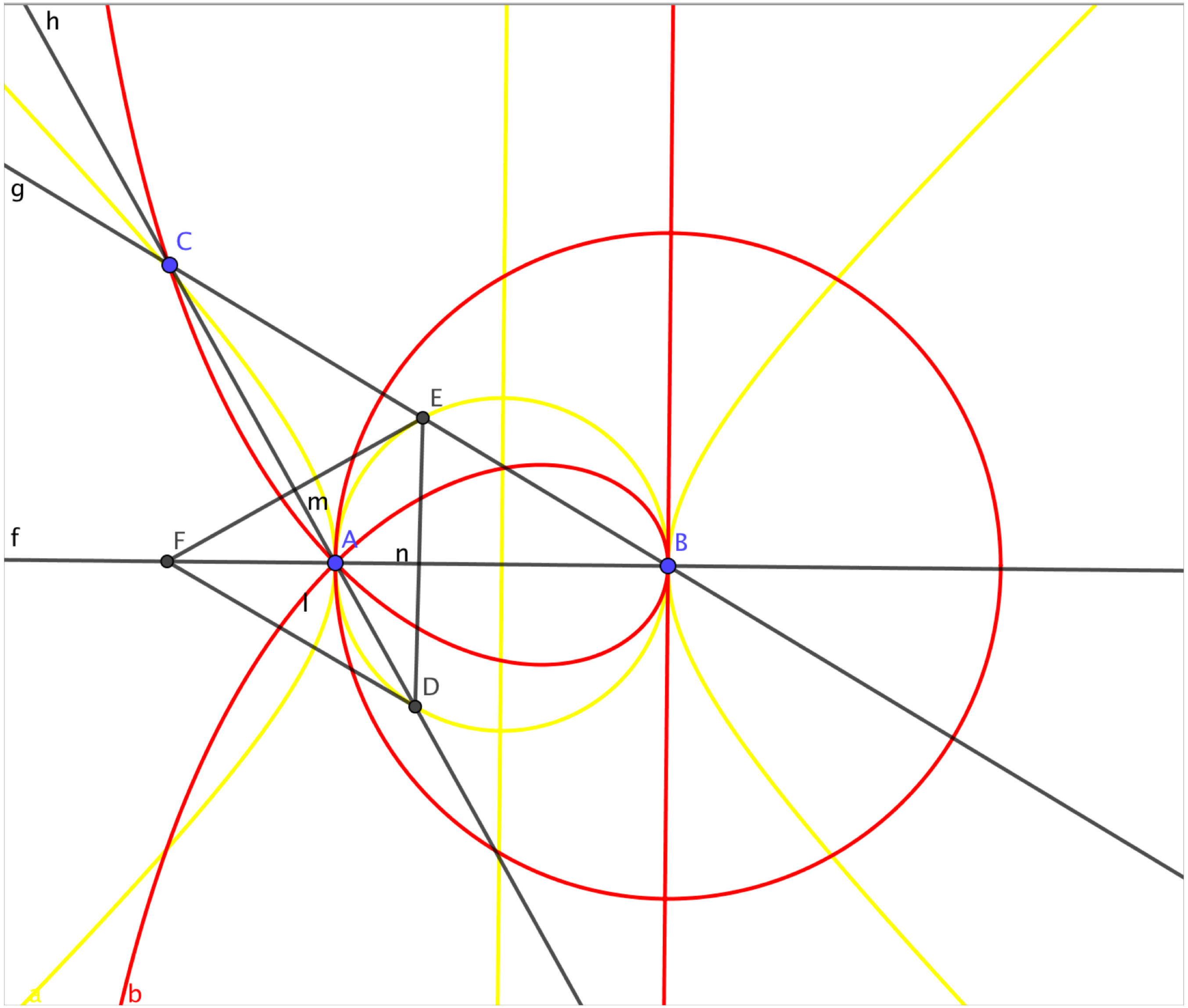

simetrico reciproco.ggb
Algebra
Graphics
Conic
d: (x − 2.91)² + (y -
Point
A = (1.28, 2.18)
B = (4.26, 1.84)
C = (3.78, 5)
D = (1.47, 4.57)
D' = (3.63, 2.66)
D'₁ = (6.11, 5.27)
D'₂ = (0.93, −0.19)
Segment
a = 3.2
b = 3.77
c = 3
Triangle
poly1 = 4.63
Input: ProveDetails[AreCollinear[D', D'_1, D'_2]]



simetrico reciproco.ggb
Algebra
Graphics
Conic
d: (x – 2.91)² + (y – 3.25)² = 3.81
List
list1 = {true, {"AreEqual[A,B]", "AreEqual[A,C]", "AreEqual[B,C]"}}
Point
A = (1.28, 2.18)
B = (4.26, 1.84)
C = (3.78, 5)
D = (1.47, 4.57)
D' = (3.63, 2.66)
D'1 = (6.11, 5.27)
D'2 = (0.93, –0.19)
Segment
a = 3.2
b = 3.77
c = 3
Triangle
poly1 = 4.63
Input:


ortico equilatero.ggb
Algebra
Graphics
Implicit Curve
a: 105025830000x
b: 74939062500x
Line
f: 0.04x + 4.38y =
g: −5.04x − 3.48y =
h: 5x − 0.9y = 1.34
i: −4.38x + 0.04y =
j: 3.48x − 5.04y =
k: 0.9x + 5y = −1.3
Point
A = (0.08, −1.04)
B = (4.46, −1.08)
C = (0.98, 3.96)
D = (0.21, −0.32)
E = (3.03, 1)
F = (0.93, −1.05)
Segment
l = 1.03
m = 2.92
n = 3.11
g
h
C
E
n
D
m
f
A
F
B
a
b


g
h
C
n
D
E
l
m
f
A
F
B
a
b


h
g
C
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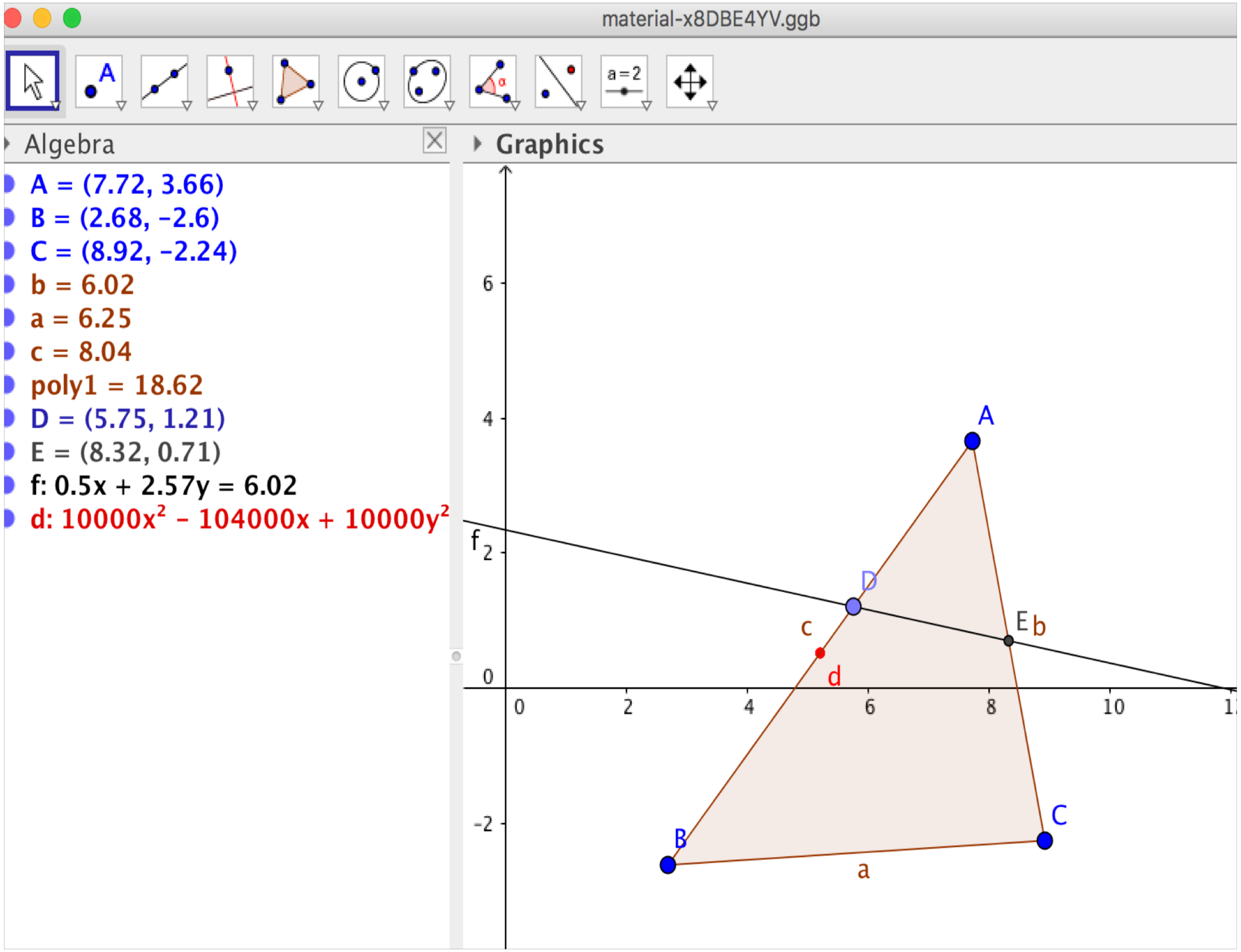
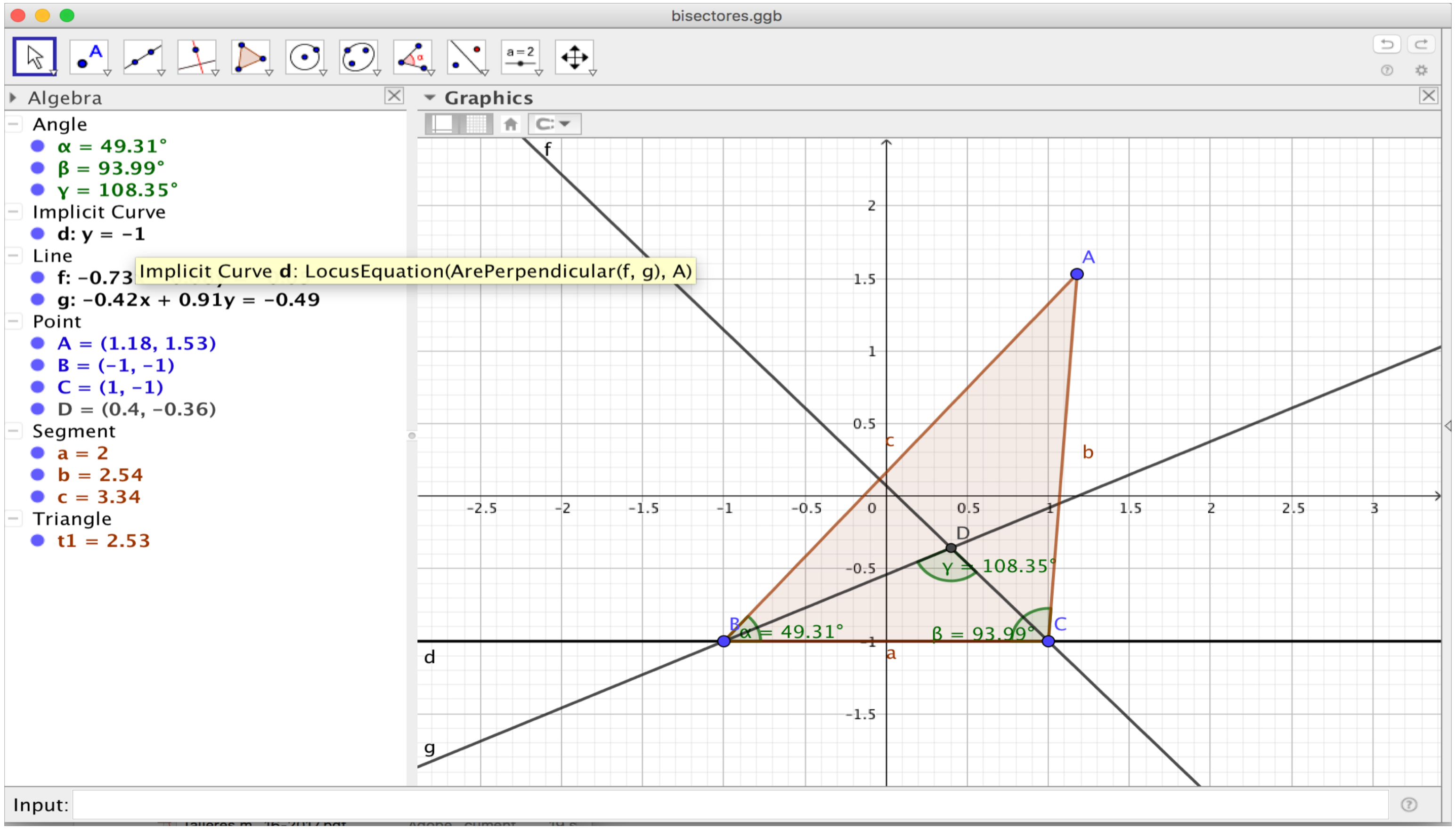
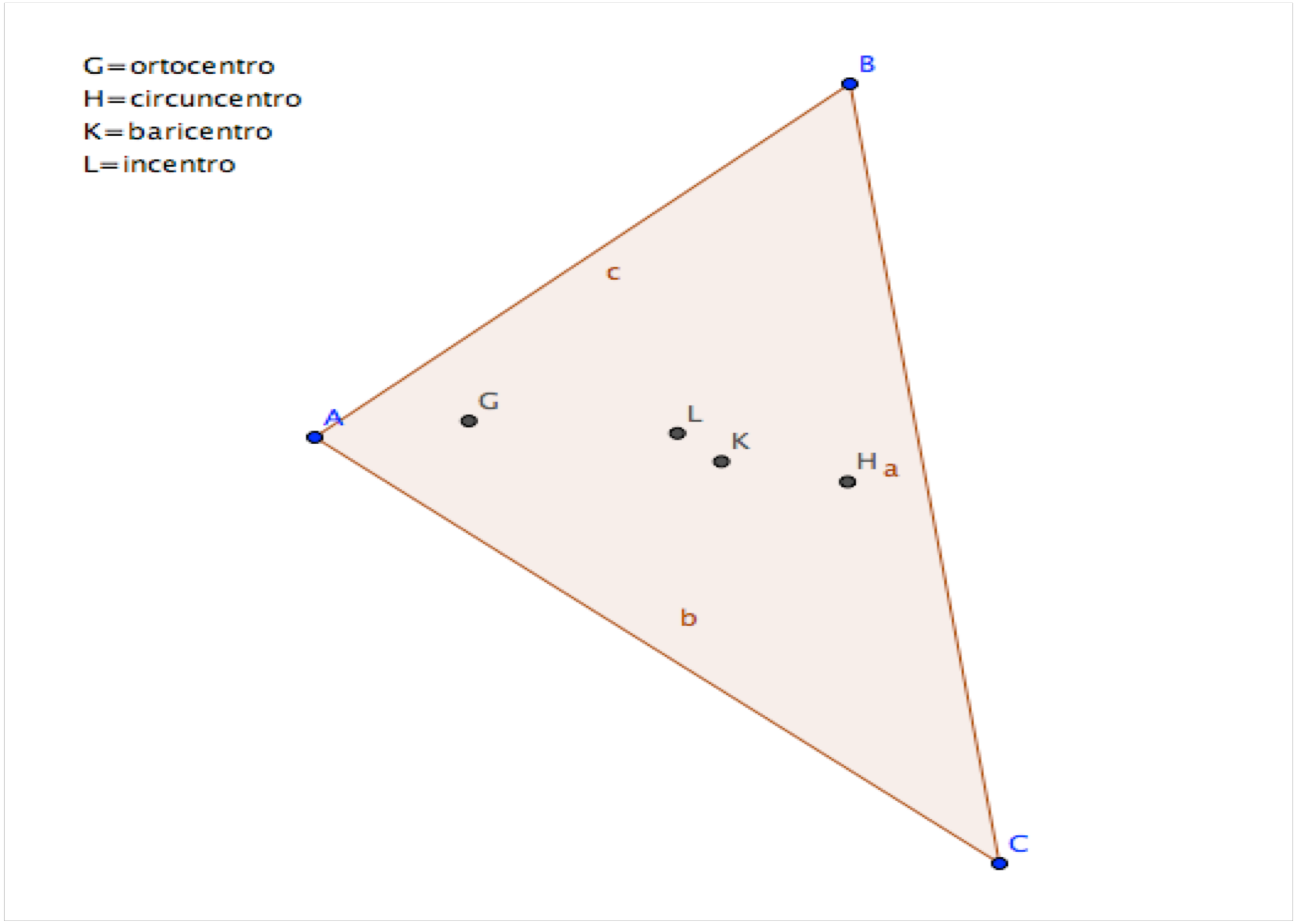
l
D
a
b


material-x8DBE4YV.ggb
Algebra
A = (7.72, 3.66)
B = (2.68, -2.6)
C = (8.92, -2.24)
b = 6.02
a = 6.25
c = 8.04
poly1 = 18.62
D = (5.75, 1.21)
E = (8.32, 0.71)
f: 0.5x + 2.57y = 6.02
d: 10000x² − 104000x + 10000y²
Graphics

Discover H for T, even when it is impossible!!

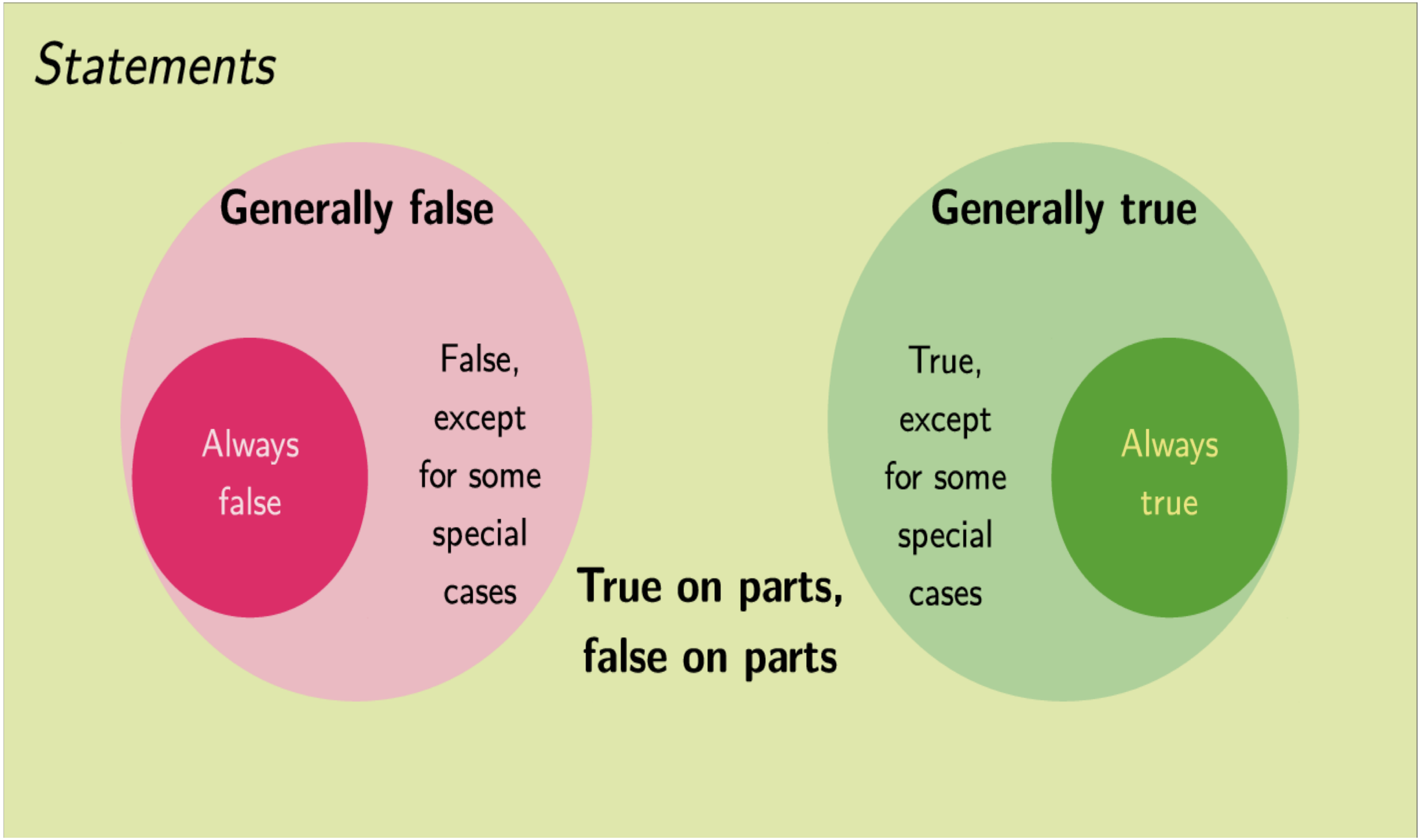



G=ortocentro
H=circuncentro
K=baricentro
L=incentro
B
c
A
G
L
K
H
a
b
C

Recio, T. and Vélez, M.P.: Automatic Discovery of Theorems in Elementary Geometry. Journal of Automated Reasoning 23, 63-82 (1999)

- H => T is generally true if the thesis T vanishes on all non-degenerate K-components of the hypotheses variety V(H).

- H =>T is generally false if the thesis T vanishes on none of the non-degenerate K-components of the hypotheses variety V(H).


Statements
Generally false
Always false
False, except for some special cases
Generally true
True, except for some special cases
Always true
True on parts, false on parts

Behind the scene

	Elimination of (H,T)	Elimination of (H, T*z-1)
not gen.true and not gen. false	(0)	(0)
generally true (and, thus, not generally false)	(0)	Not(0)
generally false (and, thus, not generally true)	Not(0)	(0)

The importance of being zero. C. Villarino, R. Sendra, T.R. ACM. Proceedings ISSAC 2018. ISBN 978-1-4503-5550-6/18/07. pp. 327-333

Breiding, P.; Kališnik Verovšek, S.; **Sturmfels, B.;** Weinstein M.: Learning algebraic varieties from samples. Revista Matematica Complutense, **31** (2018) 3, p. 545-593

- We establish a procedure for deciding, **with a finite number of tests**, given a polynomial ideal (of hypotheses and [negated] theses), whether the result of eliminating in the ideal some variables, yields the zero ideal or not.

Proof by exhaustion, also known as proof by cases… is a method …in which the statement to be proved is split into a finite number of cases and each case is checked to see if the proposition in question holds (Wikipedia)

Example: sum of first n natural numbers S(n).

Assume S(n) polynomial, degree at most 2.

n=0, n=1, n=2, n(n+1)/2

(C. McBride, Calculemus 2012, notes by JHD)

Intersection of three heights

- Given a triangle (0,0), (1,0), (r, s), each height equation is linear in (x, y) with coeffs. linear in (r, s)

- The intersection of the three heights is the vanishing of a 3x3 determinant, with linear entries in (r,s).

- Given a cubic curve in (r,s), it is identically zero iff it passes through a certain number of suitable distributed points in the (r, s) plane.

General procedure:

- We assume that we only know some very limited data: number of variables and an upper bound for the geometric degree (in the sense of Heintz) gdeg(V(I)).

- And we want to accomplish the zero test just by means of an oracle that allows us to check, given a point in K^r, whether this point is or not in V(I).

The elimination is not zero

iff

its zero set is contained in a hypersurface of degree bounded by D

iff

the projection is contained in a hypersurface of degree bounded by D

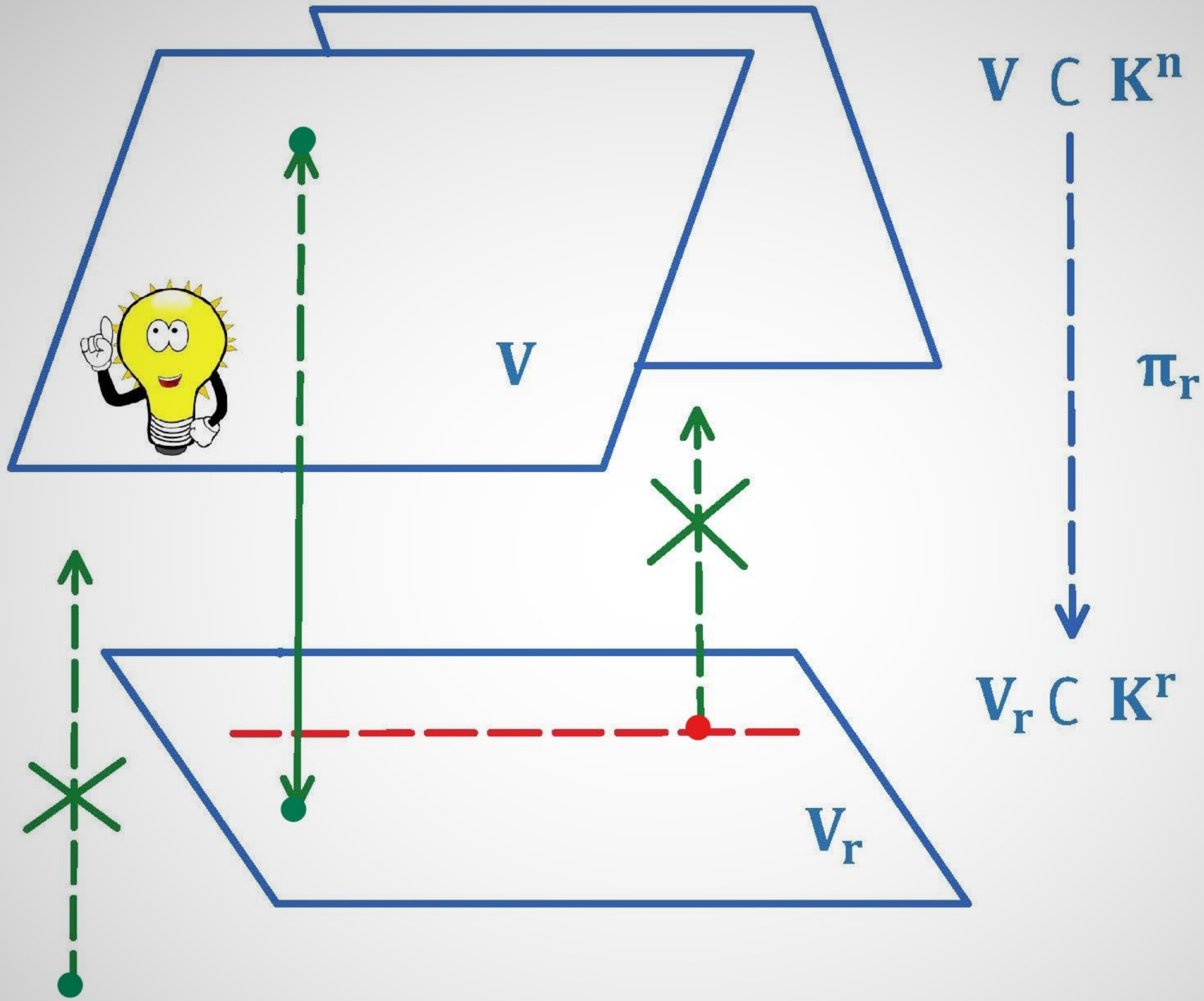
iff

the statement is generally true.

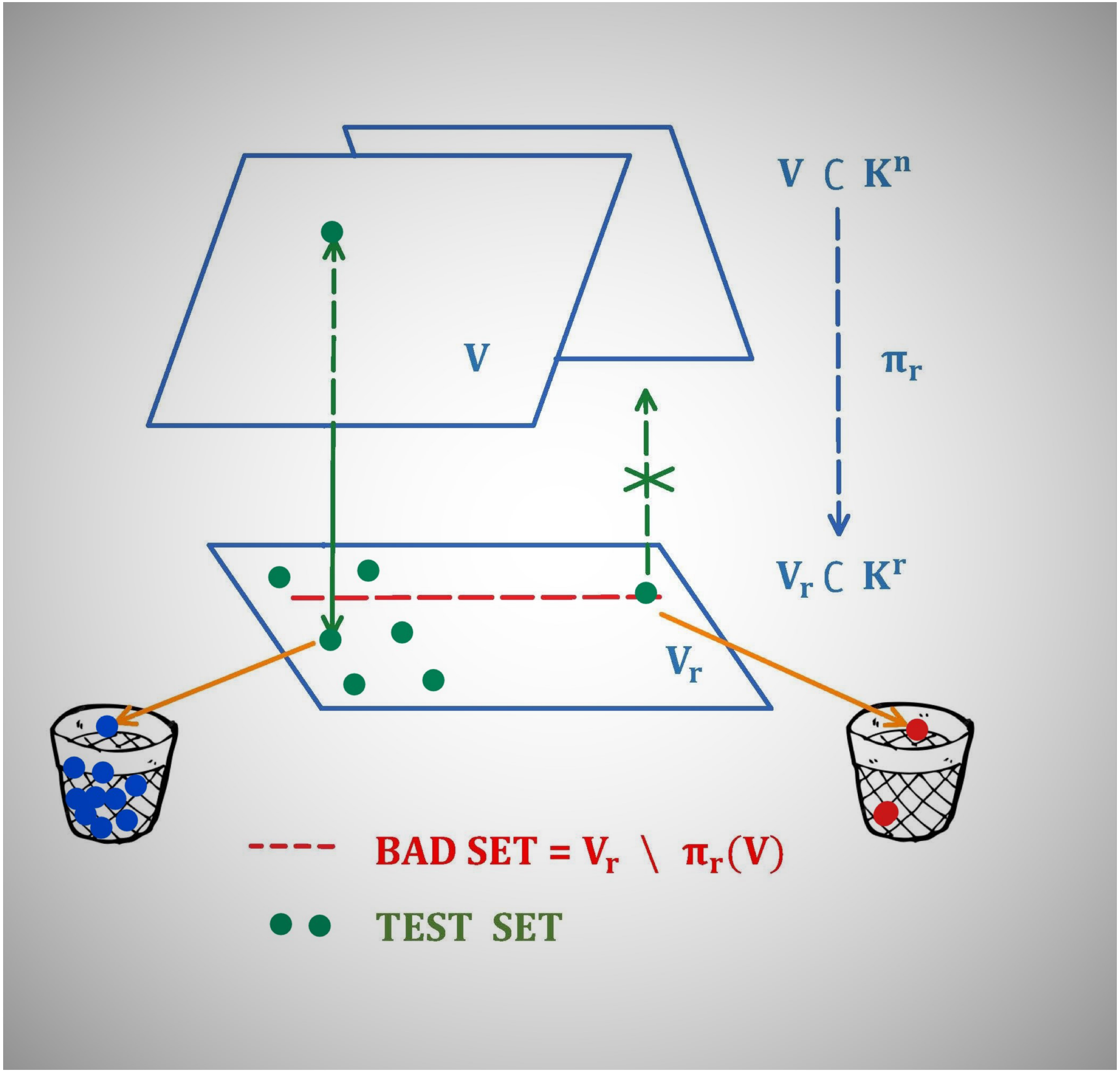
Similarly, for the generally false case.

Definition (TEST SET): A finite subset A of K^r is a TEST SET for the varieties of geometric degree less or equal than d with d > 0, (shortly a (d; r)-test set), if no proper variety W of K^r of gdeg(W) ≤ d contains A.

Supp(d; r) is a (d; r)-test set of minimum cardinality.

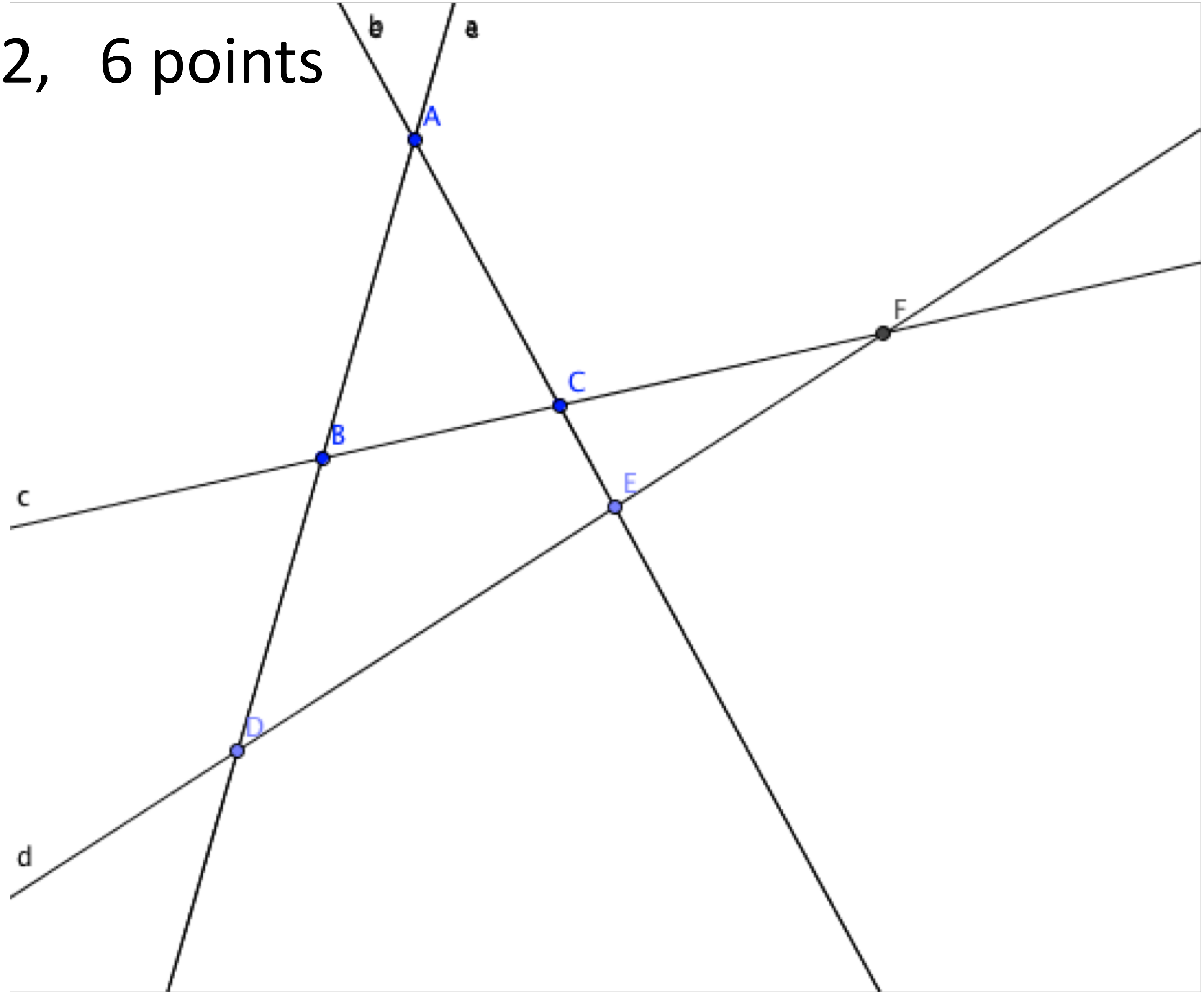


---- BAD SET = $V_r \setminus \pi_r(V)$

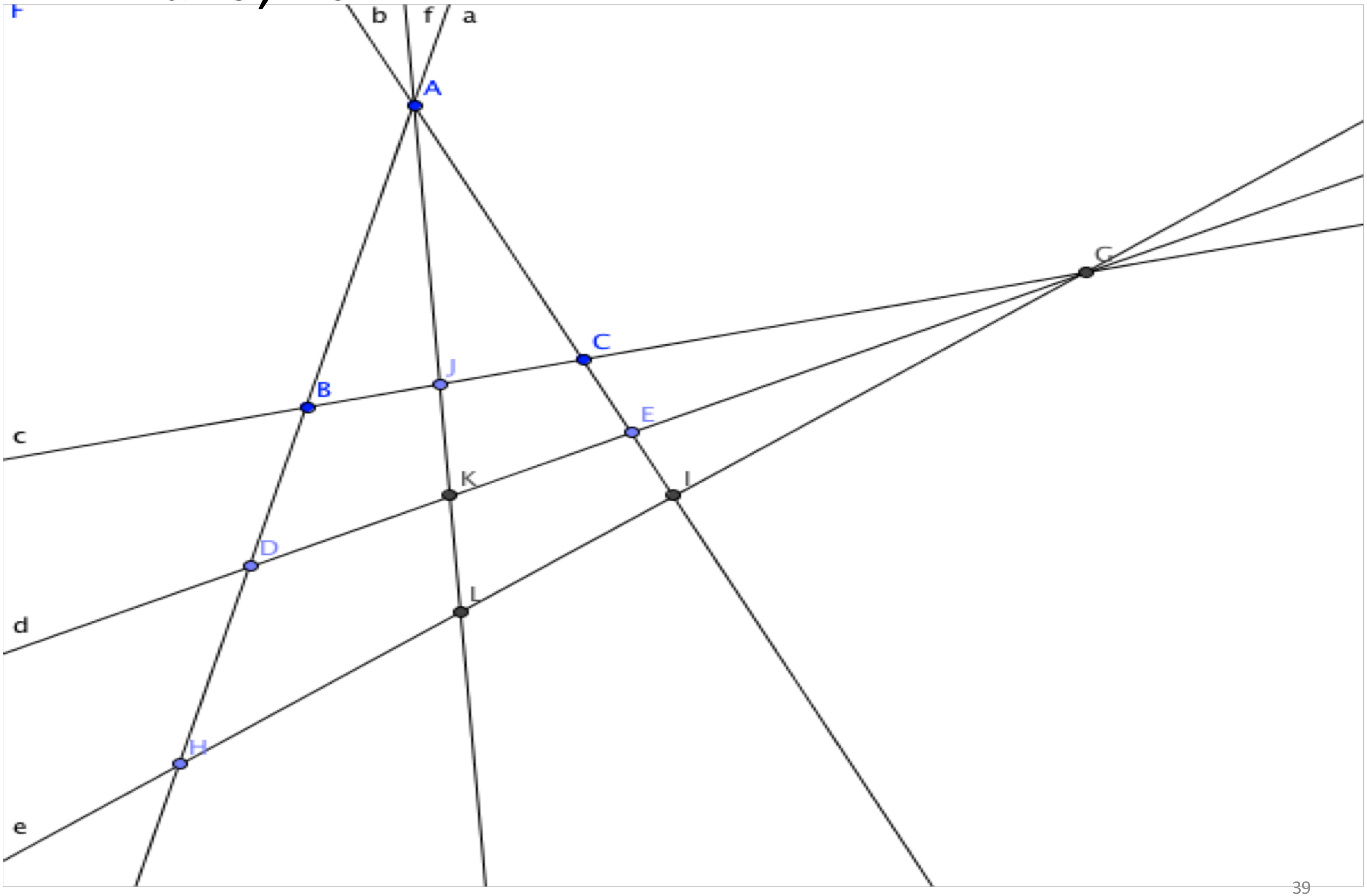


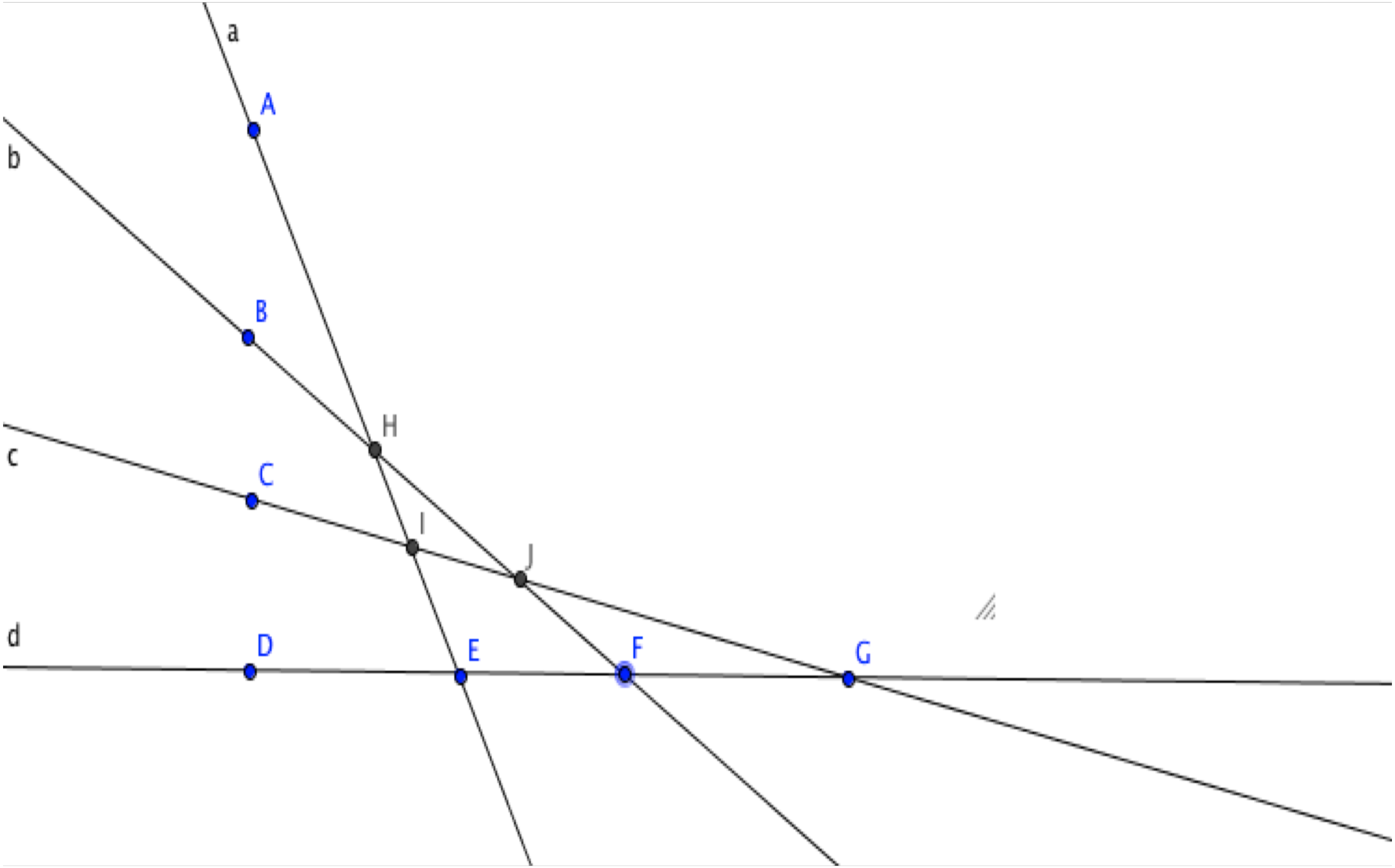
$V \subset K^n$
π_r
$V_r \subset K^r$
V
V_r
BAD SET = $V_r \setminus \pi_r(V)$
TEST SET

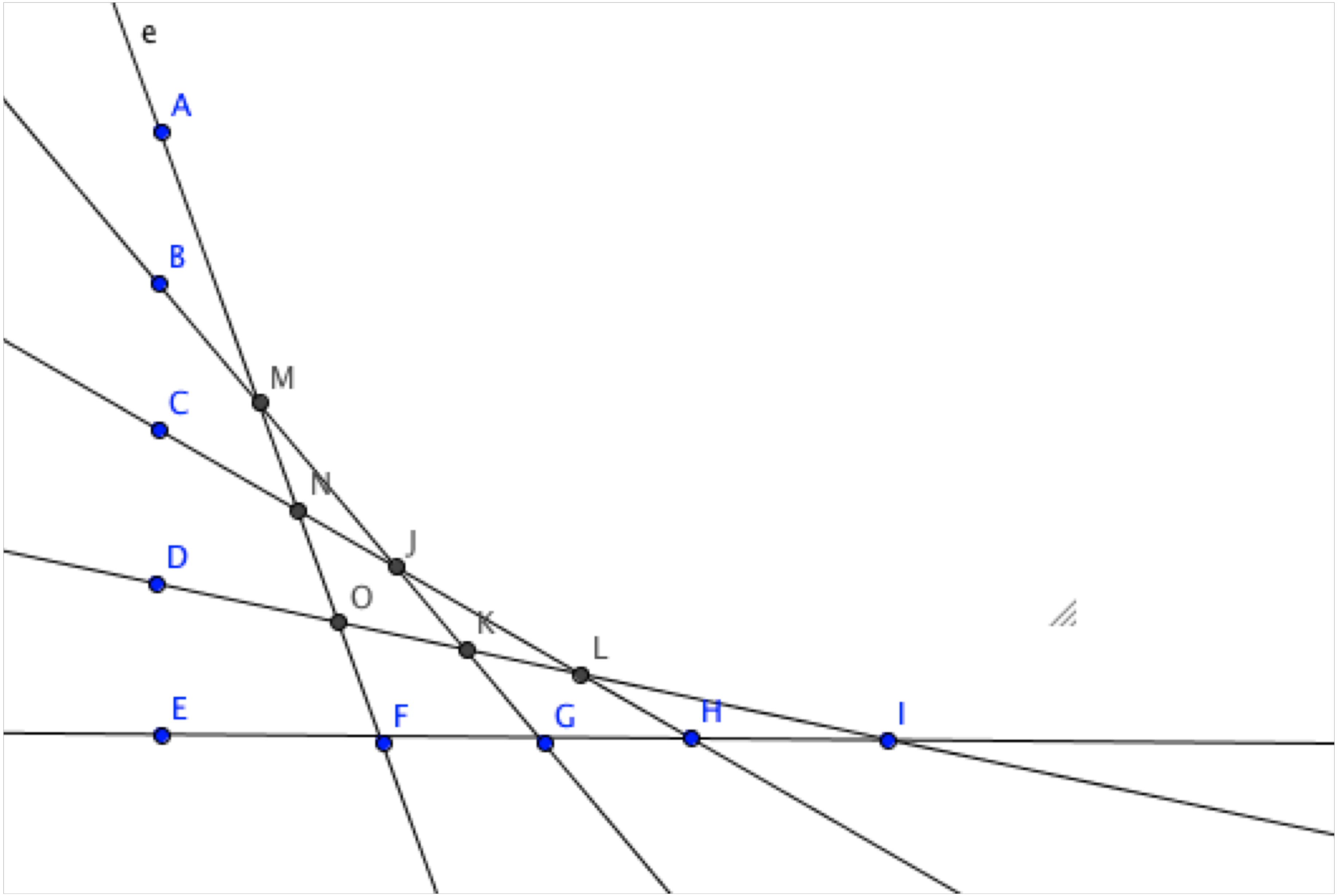
d=2, 6 points



d=3, 10




a
b
c
d
A
B
C
D
E
F
G
H
I
J


e
A
B
M
C
N
J
D
O
K
L
E
F
G
H
I

- Definition. Let d;,r, and N = #(Supp(d; r)). We say that a finite set A, with #(A)≥N, is a (d; r)-**disjunctive test set** if any subset of A of cardinal N is a (d; r)-test set.

- The motivation of this notion is the following. Assume that A is disjunctive and #(A) ≥ 2N - 1 and B subset of A, then either B or A \ B is a (d; r)-test set.

Procedure

- Calculate an upper bound d for the degree of the zero set of hypotheses and negated thesis.
- Create a (d; r)-**disjunctive test set** of 2N-1 points A with and N = #(Supp(d; r)), on the affine space of independent variables and test for each point of this collection if it is contained in the projection.

- B=do not lift up

- Indeed, if #(B) $\geq$ N, the statement holds by the definition of disjunctive test set and bound on deg. of bad set.

- Else, #(A \ B) $\geq$ N, and thus A\ B is a (d; r)-test set. and

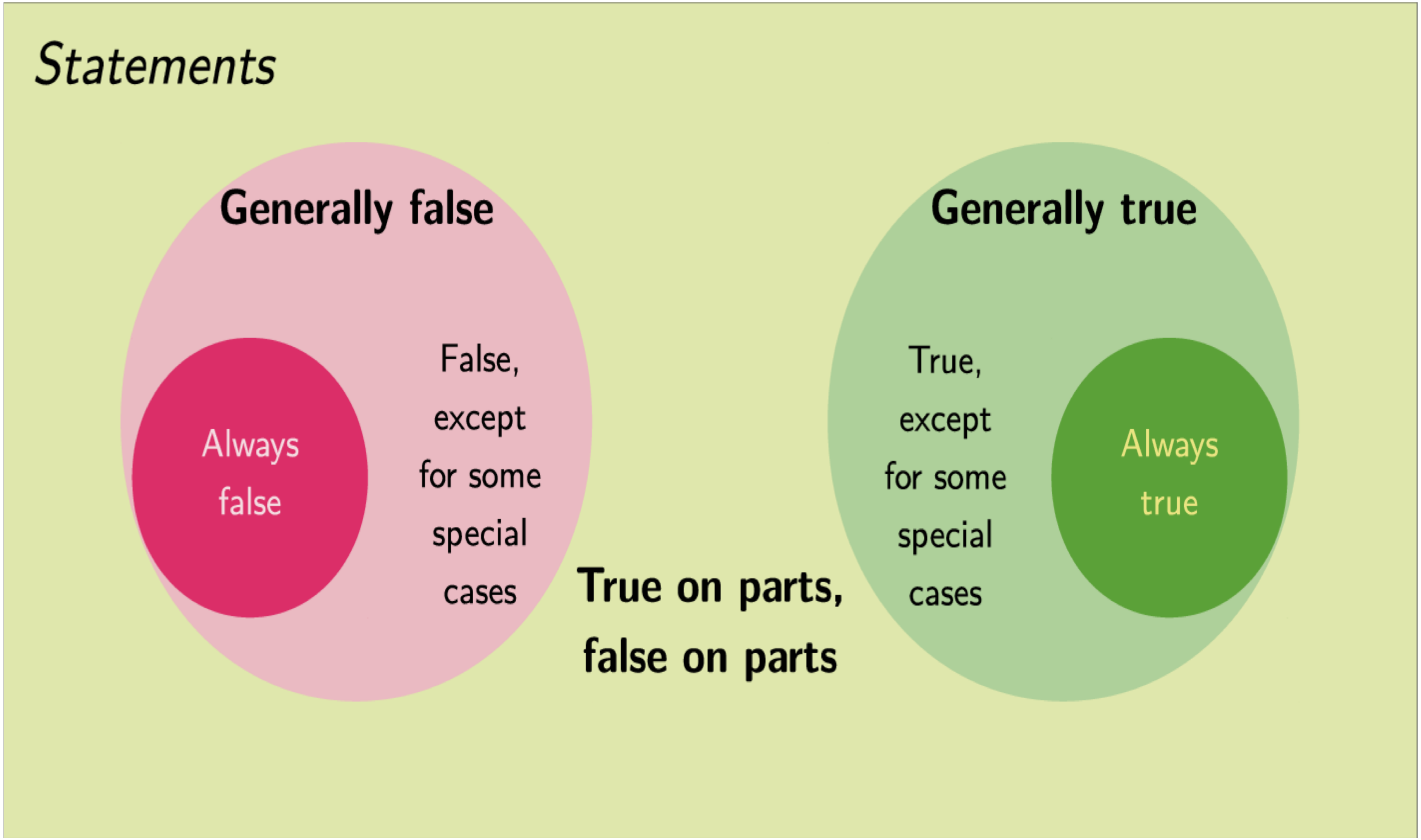
- If so, then the projection is the whole space and the elimination ideal is zero.

- Else, the projection is obviously not the whole space and thus the elimination is not the zero ideal.

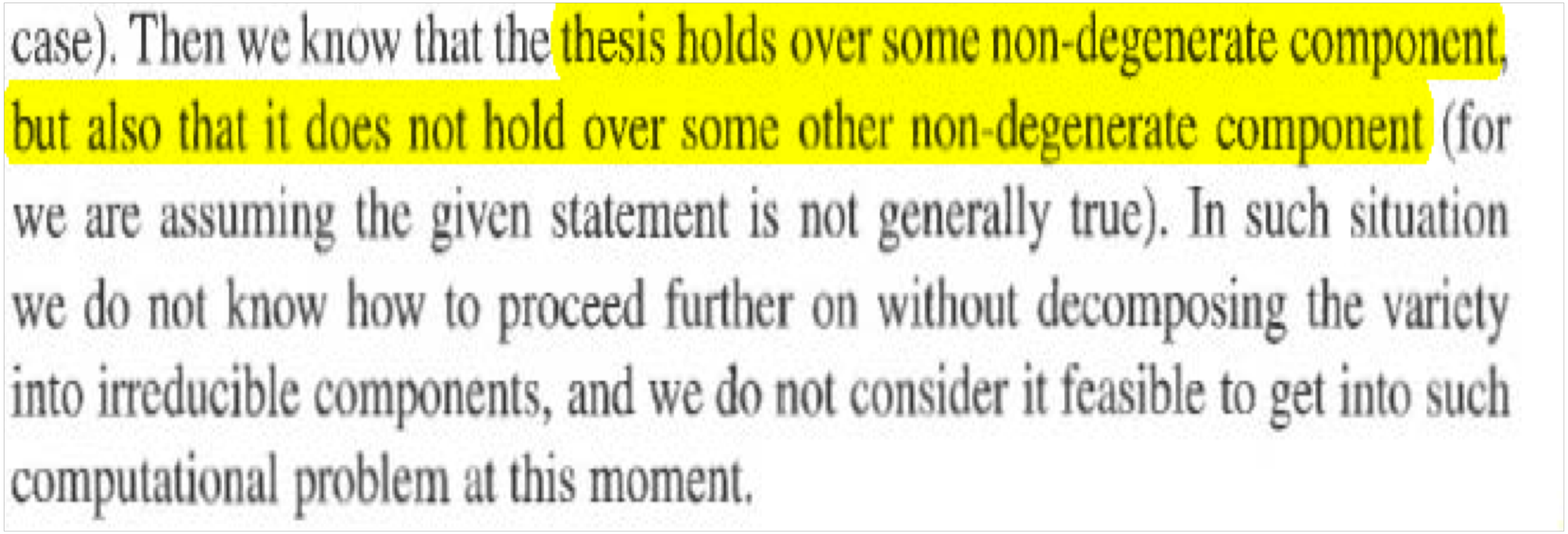
- Precision for checking numerically each instance
- Possibility to do it symbolically in GeoGebra
- Yet, “minor” episthemological obstacle. What if our paper / pencil proof is wrong? What if our computer has a bug?
- At least, different kind of obstacle from “probabilistic” checking.

- https://prover-test.geogebra.org/job/GeoGebra-provertest/870/artifact/test/scripts/benchmark/prover/html/all.html
- http://test.geogebra.org/~kovzol/data/Prove-20150219b/
- http://prover-test.geogebra.org/~kovzol/prover-20190206/README

Test file	Engine 1		Engine 2		Engine 2, Giac		Engine 3a		Engine 3b		Auto	
	Result	Speed	Result	Speed	Result	Speed	Result	Speed	Result	Speed	Result	Speed
lines-parallel.ggb	false	6	false	29		261	false	71	false	72	false	4
points-collinear.ggb	false	3	false	25		245	false	65	false	72	false	6
points-equal.ggb	false	5	false	76		242	false	58	false	71	false	6
bisector-midpoint.ggb	true	6	true	84	true	268	true	75	true	77	true	4
centroid-median-ratio1.ggb		3	true	39	true	274	true	495	true	136	true	251
centroid-median-ratio2.ggb		1		15		242		71	true	83	true	365
Ceva1.ggb		2		55		346		62	true	92	true	399
Ceva2.ggb		2		19		240		68	true	94	true	333
Ceva3.ggb		2		16		239		74	true	87	true	340
circumcenter1.ggb	true	9	true	34	true	266	true	200	true	115	true	7
circumcenter2.ggb	true	5	true	33	true	268	true	113	true	94	true	5
circumcenter3.ggb	true	8	true	29	true	250	true	148	true	103	true	4
circumcenter4.ggb	true	8	true	32	true	273	true	136	true	102	true	6
circumcenter5.ggb		2	true	38	true	253	true	107	true	96	true	244
circumcenter6.ggb	true	12	true	101	true	268	true	131	true	106	true	16
construct-perpendicular-line.ggb		2		19		251		80		84		323
construct-tangent-to-circle-point.ggb		5		7		252		79		88		323
def-line-perpline-perpline.ggb	true	6	true	33	true	270	true	67	true	70	true	4


Statements
Generally false
Always false
False, except for some special cases
True on parts, false on parts
Generally true
True, except for some special cases
Always true

- T. Recio and M.P. Velez (1999). Automatic Discovery of Theorems in Elementary Geometry. Journal of Automated Reasoning 23, 63-82

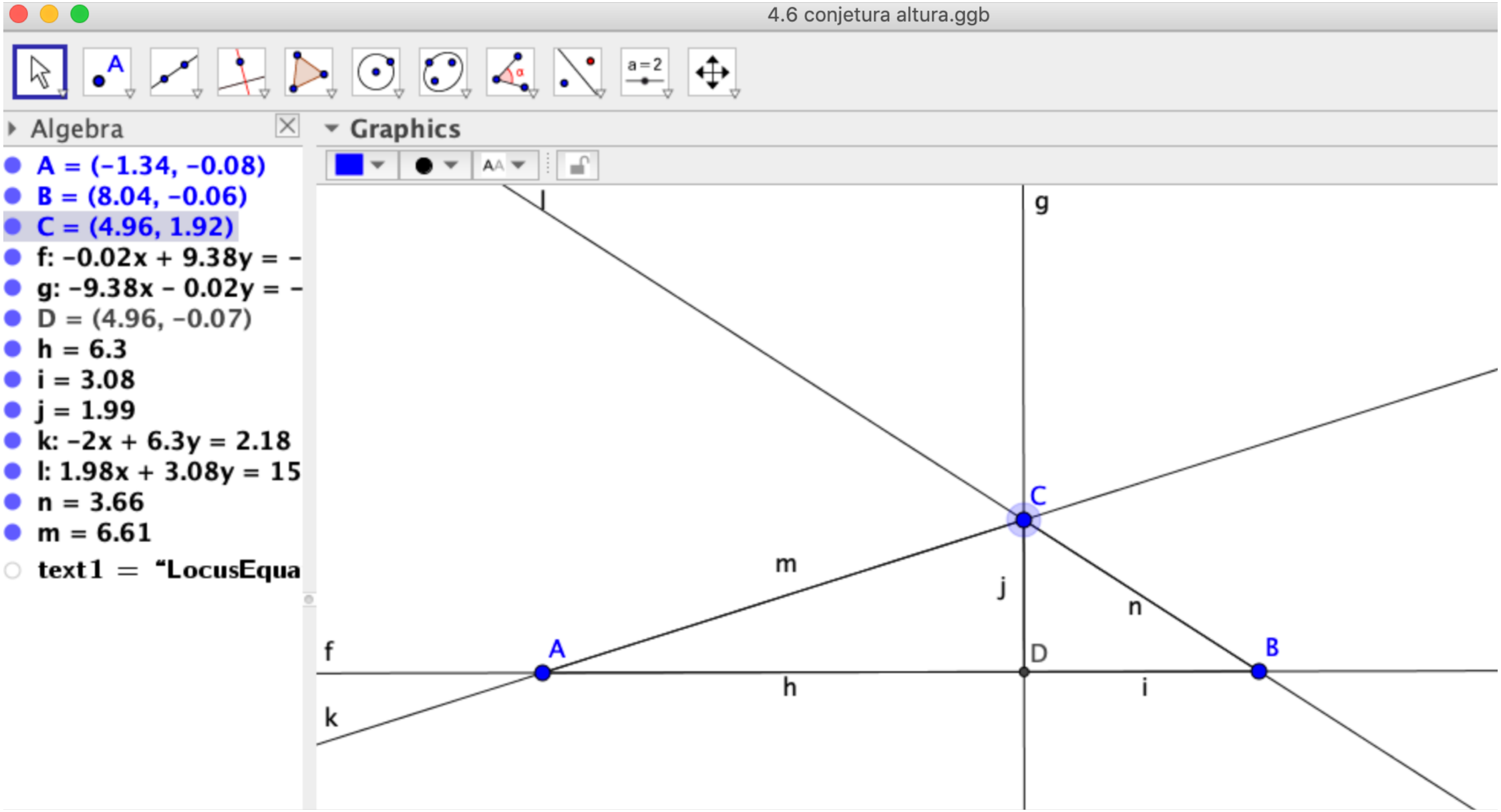

case). Then we know that the thesis holds over some non-degenerate component, but also that it does not hold over some other non-degenerate component (for we are assuming the given statement is not generally true). In such situation we do not know how to proceed further on without decomposing the variety into irreducible components, and we do not consider it feasible to get into such computational problem at this moment.

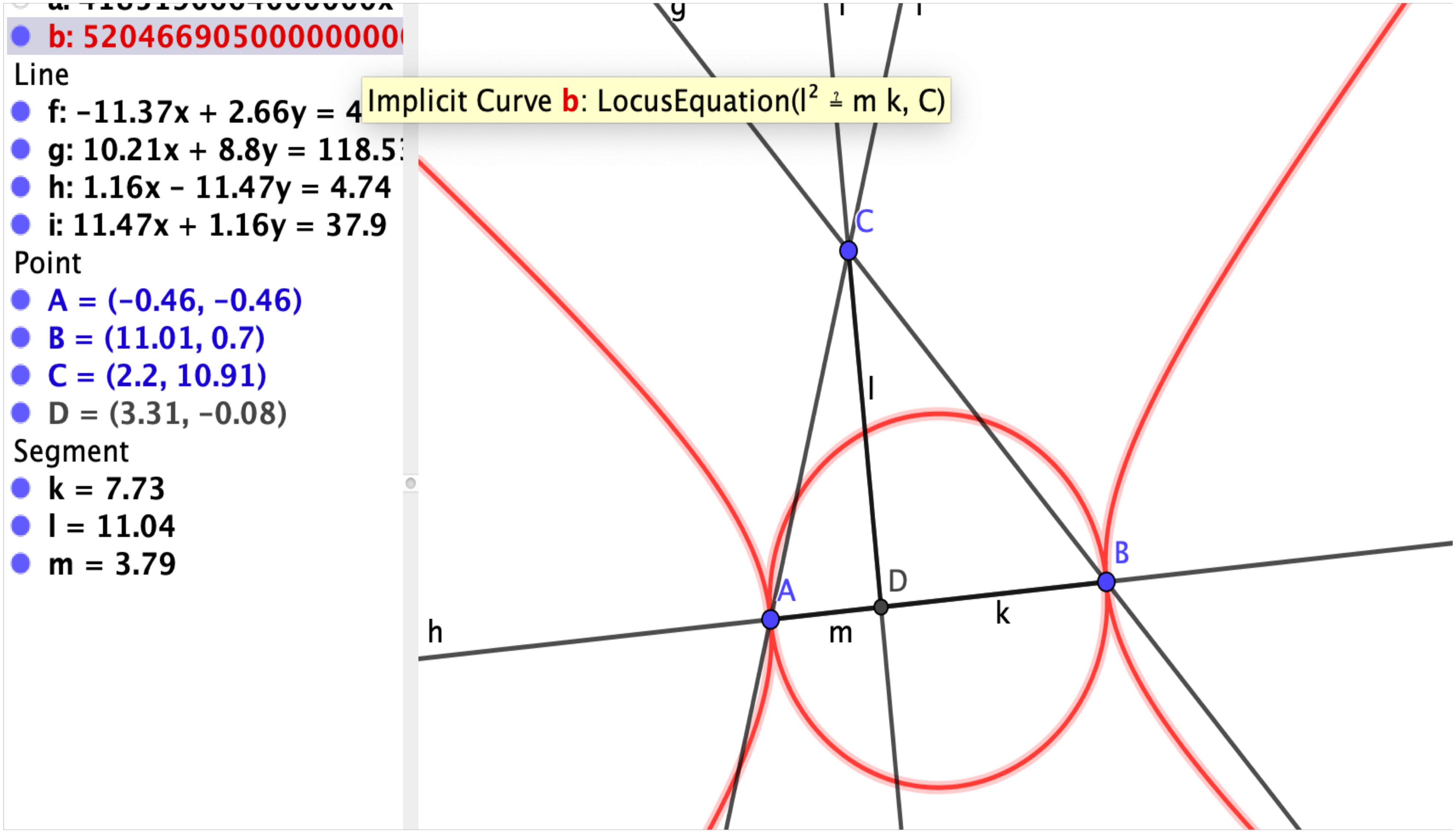
- J. Zhou, D. Wang and Y. Sun (2017). Automated reducible geometric theorem proving and discovery by Gröbner basis method. Journal of Automated Reasoning 59(3), 331-344

The geometric statement is called *true* if f vanishes on every point of V. The geometric statement is called *generally true* if f vanishes on all non-degenerate components of V, i.e. f vanishes on $V_1 \cup \cdots \cup V_p$. The geometric statement is called *generally true on components* if f vanishes on some but not all non-degenerate components of V. Otherwise, the geometric statement is called *generally false*.

Kóvacs, Z.; Recio, T.; Vélez, M.P.: Detecting truth, just on parts.

Revista Matemática Complutense. To appear. https://doi.org/10.1007/s13163-018-0286-1


4.6 conjetura altura.ggb
a = 2
Algebra
Graphics
A = (−1.34, −0.08)
B = (8.04, −0.06)
C = (4.96, 1.92)
f: −0.02x + 9.38y = −
g: −9.38x − 0.02y = −
D = (4.96, −0.07)
h = 6.3
i = 3.08
j = 1.99
k: −2x + 6.3y = 2.18
l: 1.98x + 3.08y = 15
n = 3.66
m = 6.61
text1 = "LocusEqua
l
g
C
m
j
n
f
A
D
B
h
i
k


b: 5204669050000000
Line
f: −11.37x + 2.66y = 4
g: 10.21x + 8.8y = 118.5
h: 1.16x − 11.47y = 4.74
i: 11.47x + 1.16y = 37.9
Point
A = (−0.46, −0.46)
B = (11.01, 0.7)
C = (2.2, 10.91)
D = (3.31, −0.08)
Segment
k = 7.73
l = 11.04
m = 3.79
Implicit Curve b: LocusEquation(l² ≟ m k, C)
g
C
l
A
D
B
h
m
k

```
> Altura:=<l^2-y^2, m^2-x^2,k^2-(1-x)^2,-x^4+y^4+2*x^3-x^2,(l^2-k*m)*t-1>;EliminationIdeal(Altura,{x});EliminationIdeal(Altura,{y});
```

$$Altura := \langle k^2 - (1-x)^2, \left(-k\,m + l^2\right) t - 1, l^2 - y^2, m^2 - x^2, -x^4 + y^4 + 2\,x^3 - x^2 \rangle$$

$$\langle 0 \rangle$$

$$\langle 0 \rangle \tag{8}$$

```
> EliminationIdeal(<l^2-y^2, m^2-x^2,k^2-(1-x)^2,-x^4+y^4+2*x^3-x^2,(l^2-k*m)>,{x});
```

$$\langle 0 \rangle \tag{9}$$

```
> EliminationIdeal(<l^2-y^2, m^2-x^2,k^2-(1-x)^2,-x^4+y^4+2*x^3-x^2,(l^2-k*m)>,{y});
```

$$\langle 0 \rangle \tag{10}$$

```
> L:={PrimaryDecomposition(<l^2-y^2, m^2-x^2,k^2-(1-x)^2,-x^4+
  y^4+2*x^3-x^2>)};
```

$$L := \{\langle l-y, m-x, k-1+x, -x^2+y^2+x\rangle, \langle l-y, m-x, k-1+x, x^2+y^2-x\rangle, \langle l-y, m-x, k+1-x, -x^2+y^2+x\rangle, \langle l-y, m-x, k+1-x, x^2+y^2-x\rangle, \langle l-y, m+x, k-1+x, -x^2+y^2+x\rangle, \langle l-y, m+x, k-1+x, x^2+y^2-x\rangle, \langle l-y, m+x, k+1-x, -x^2+y^2+x\rangle, \langle l-y, m+x, k+1-x, x^2+y^2-x\rangle, \langle l+y, m-x, k-1+x, -x^2+y^2+x\rangle, \langle l+y, m-x, k-1+x, x^2+y^2-x\rangle, \langle l+y, m-x, k+1-x, -x^2+y^2+x\rangle, \langle l+y, m-x, k+1-x, x^2+y^2-x\rangle, \langle l+y, m+x, k-1+x, -x^2+y^2+x\rangle, \langle l+y, m+x, k-1+x, x^2+y^2-x\rangle, \langle l+y, m+x, k+1-x, -x^2+y^2+x\rangle, \langle l+y, m+x, k+1-x, x^2+y^2-x\rangle\} \tag{11}$$

```
> nops(L);
```

$$16 \tag{12}$$

```
> for i from 1 to 16 do EliminationIdeal(op(i,L)+<(l^2-k*m)*t-1>,{x}),EliminationIdeal(op(i,L)+<
  (l^2-k*m)>,{x}) end do;
```

$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle x^2 - x\rangle$$

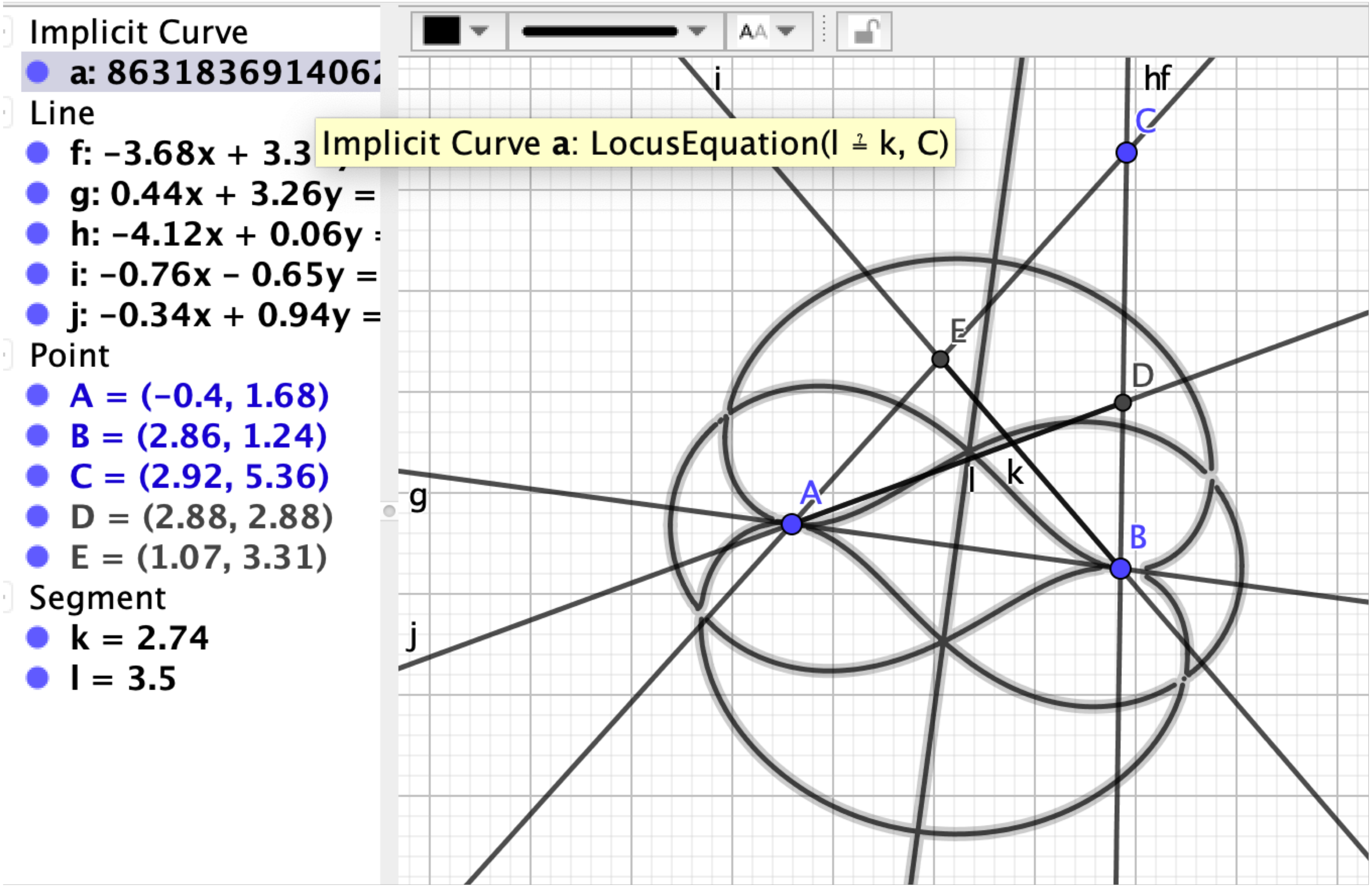
$$\langle 1\rangle, \langle 0\rangle$$

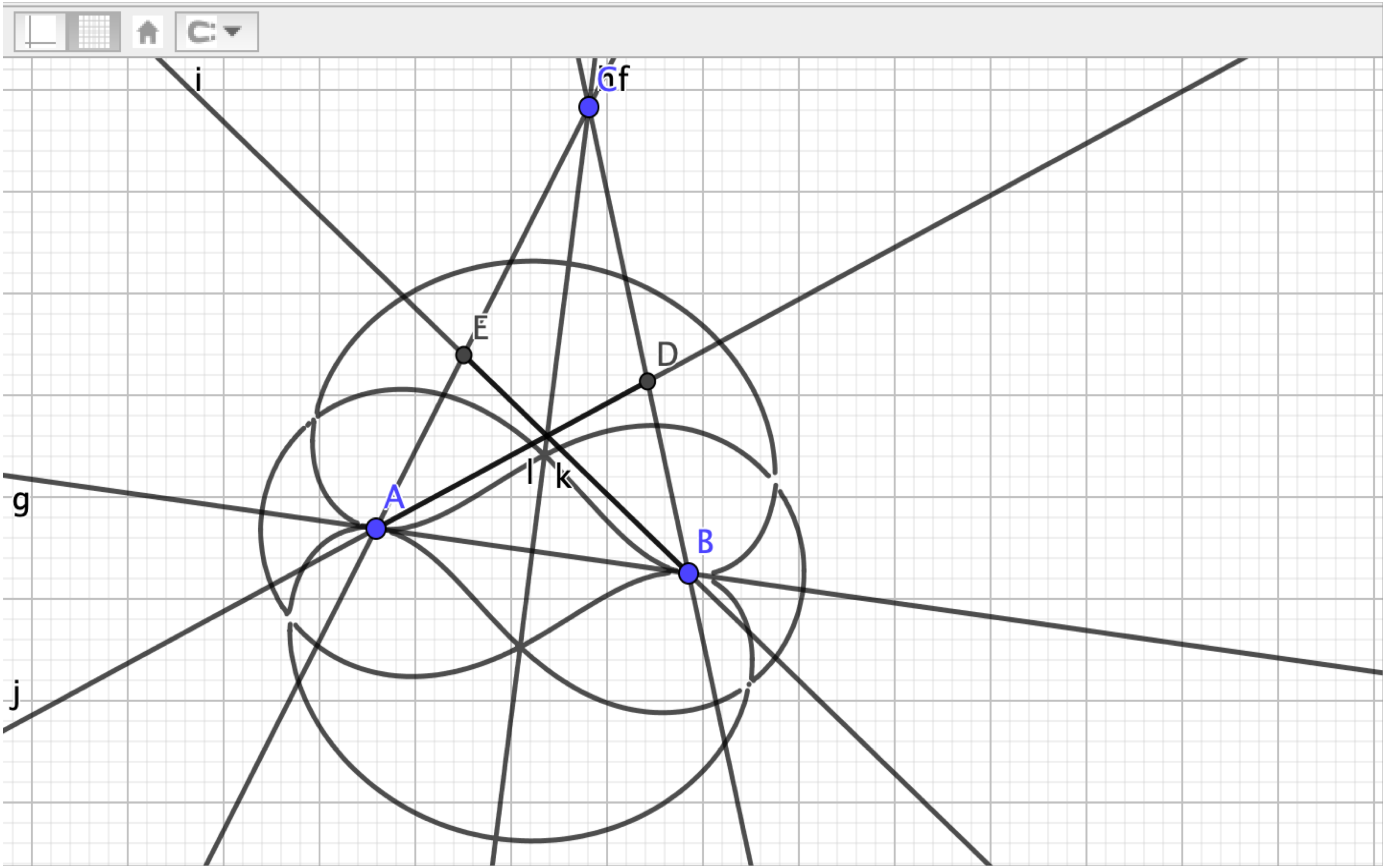
$$\langle 0\rangle, \langle x^2 - x\rangle$$

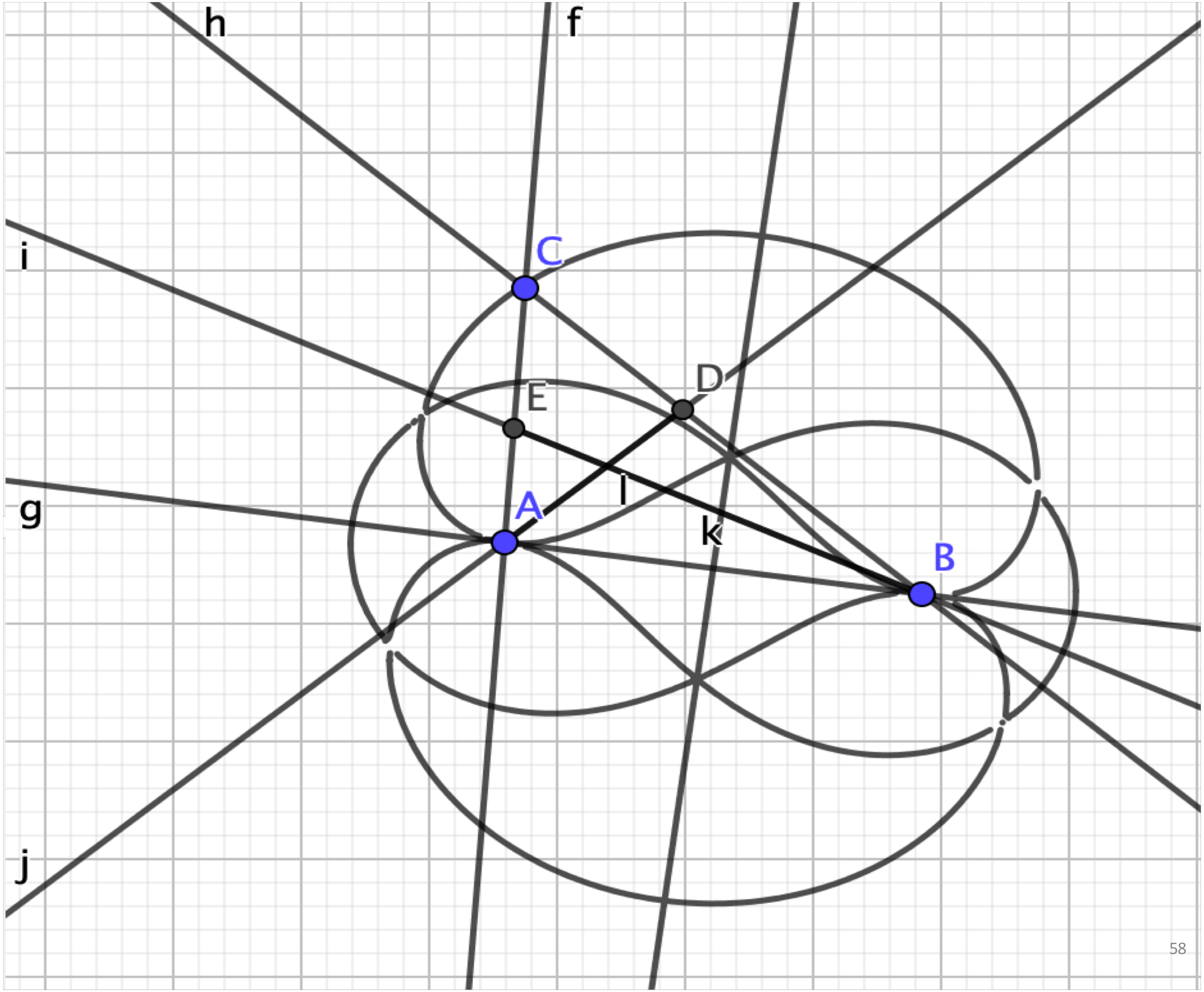
$$\langle 0\rangle, \langle x^2 - x\rangle$$

$$\langle 1\rangle, \langle 0\rangle$$

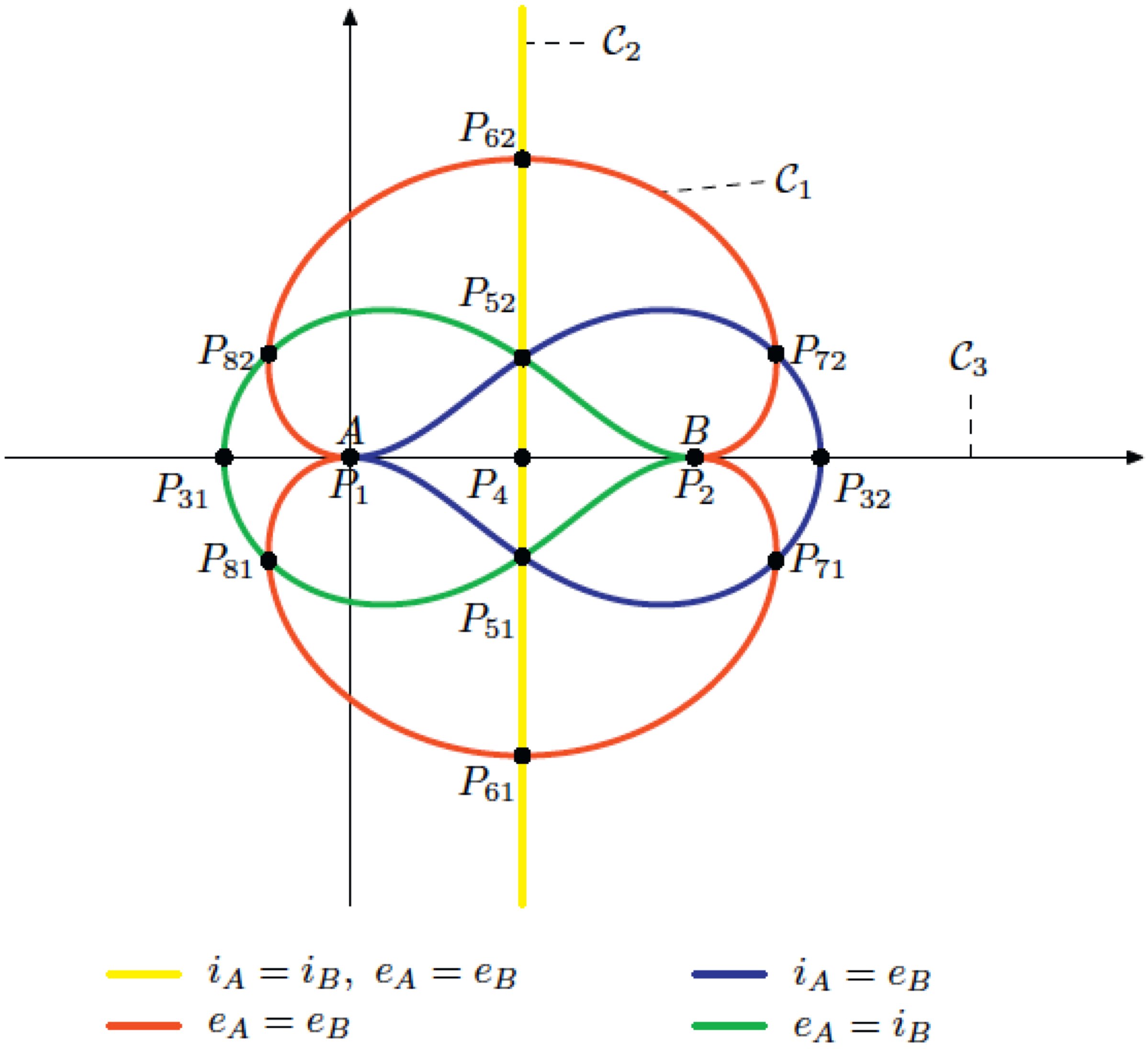
(

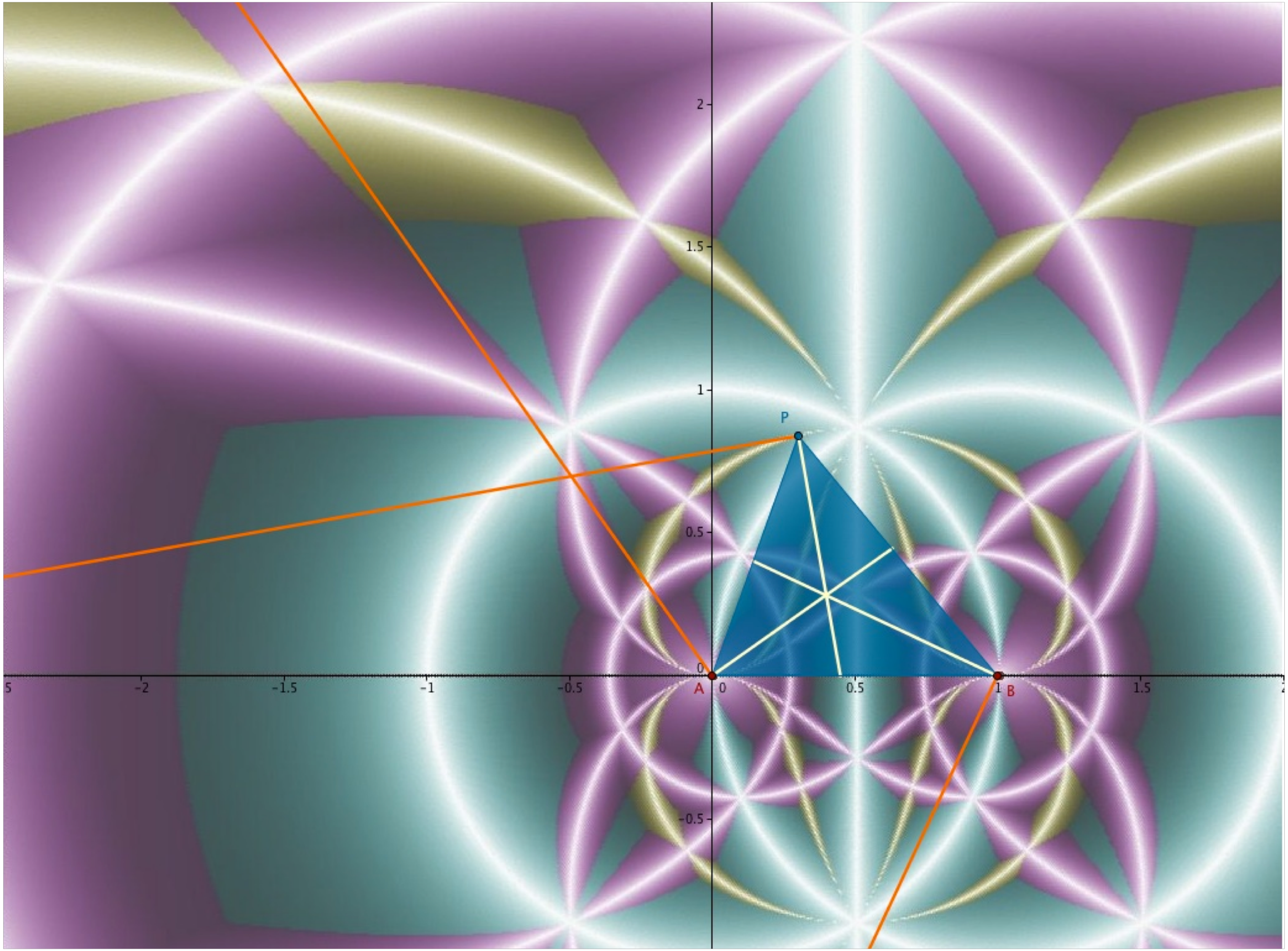

Implicit Curve
a: 863183691406
Line
f: −3.68x + 3.3
g: 0.44x + 3.26y =
h: −4.12x + 0.06y
i: −0.76x − 0.65y =
j: −0.34x + 0.94y =
Point
A = (−0.4, 1.68)
B = (2.86, 1.24)
C = (2.92, 5.36)
D = (2.88, 2.88)
E = (1.07, 3.31)
Segment
k = 2.74
l = 3.5
Implicit Curve a: LocusEquation(l ≟ k, C)
i
hf
C
E
D
A
B
g
j
l
k


i
C
h
f
E
D
l
k
g
A
B
j


h
f
i
C
D
E
l
g
A
k
B
j

A. Montes, T. Recio / Mathematics and Computers in Simulation 104 (2014) 67–81





2
1.5
1
P
0.5
0
A
0
B
-0.5
-2
-1.5
-1
-0.5
0.5
1
1.5

- Manuel Ladra and M.Pilar Páez-Guillán, Tomás Recio. "Dealing with negative conditions in automated proving: tools and challenges (The unexpected consequences of Rabinowitsch's trick)" (submitted)

AG=Automated Geometer

Botana F., Kovács Z., Recio T. (2018): Towards an Automated Geometer. In: Fleuriot J., Wang D., Calmet J. (eds) Artificial Intelligence and Symbolic Computation. AISC 2018. Lecture Notes in Computer Science, vol 11110. Springer, Cham. pp 215-220.

http://www.math.rutgers.edu/~zeilberg/PG/gt.html

PLANE GEOMETRY:

AN ELEMENTARY TEXTBOOK

BY

SHALOSH B. EKHAD, XIV

(CIRCA 2050)

DOWNLOADED FROM THE FUTURE

BY

DORON ZEILBERGER

Foreword Introduction Definitions Theorems

RENE PictRENE

Foreword to Shalosh B. Ekhad XIV's Geometry Textbook

By Doron Zeilberger , the downloader.

Cover Introduction Definitions Theorems

One night, very late, I was browsing the internet, using my current computer, Shalosh B. Ekhad, III. I was searching for "Ekhad". All of a sudden, to my amazement, I chanced on a website whose *last update* was Sept. 30, 2050, and found this little Elementary Geometry textbook.

This text may seem a bit strange to 2001 humans. It appears that there are no proofs, only statements, in Maple, using English-based names for the definitions and theorems. But THE STATEMENT IS THE PROOF, ready to be run on Maple, that will output "true" if the proof-statement is correct, and "false" otherwise. The statement-proofs are collected in the accompanying Maple package RENE.

You don't have to know Maple to savor this book. The names of the commands are English-based, and the primitive definitions like Pt, Le, Ce, etc. are explained in the appropriate links in this completely hyper-texed text.

While this webbook is already a computer program, it must have been automatically generated by another "meta" computer program, as described in the author's Introduction. The output computer program, without the illustrations is RENE (in honor of Rene Descartes). To verify any given theorem, go into Maple, type: read RENE; followed by: TheoremName(); .

The beautiful illustrations were generated by another Maple package PictRENE. It too, must have been generated automatically. With it, you can draw many other diagrams for the theorems, using different input parameters. Once you downloaded PictRENE, get into Maple, type: "read PictRENE:" (without the quotes), and then type "ezra();" (w/o the quotes) to get on-line help.

September 21, 2050

Dear Children,

Do you know that until fifty years ago most of mathematics was done by humans? Even more strangely, they used human language to state and prove mathematical theorems. Even when they started to use computers to prove theorems, they always translated the proof into the imprecise human language, because, ironically, computer proofs were considered of questionable rigor!

…

All the theorems that were known to our grandparents, and most of what they called conjectures, can now be proved in a few nano-seconds on any PC.

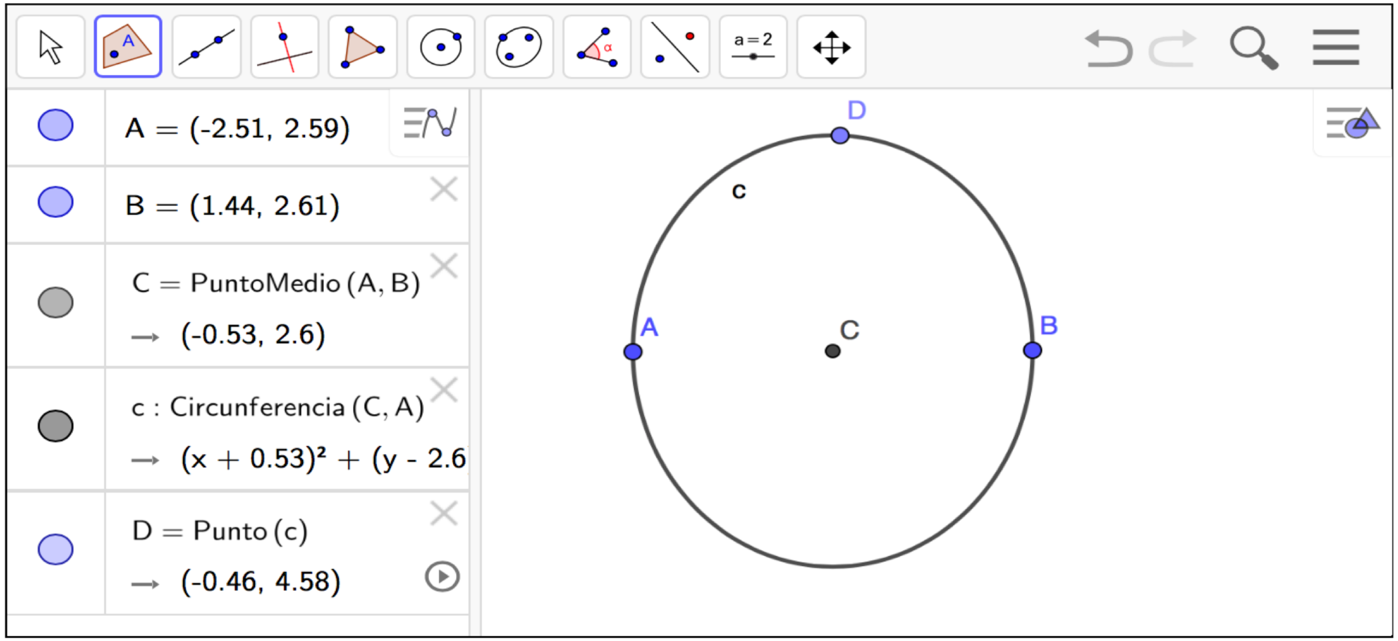
…

All the theorems in this textbook were automatically discovered (and of course proved) by computer. The discovery program started with 3 generic points in the plane, and iteratively constructed new points, lines, and circles using a few primitives. Whenever a new point (or line, or circle, or whatever) coincided with an old one, a "theorem" was born. ..

- http://prover-test.geogebra.org/~kovzol/ag/automated-geometer.html
- http://prover-test.geogebra.org/~kovzol/ag/automated-geometer.html?offline=1
-

Welcome to the Automated Geometer!

Let us consider this initial input construction (you may freely edit the construction or upload another one as well; only the visible points will be observed):



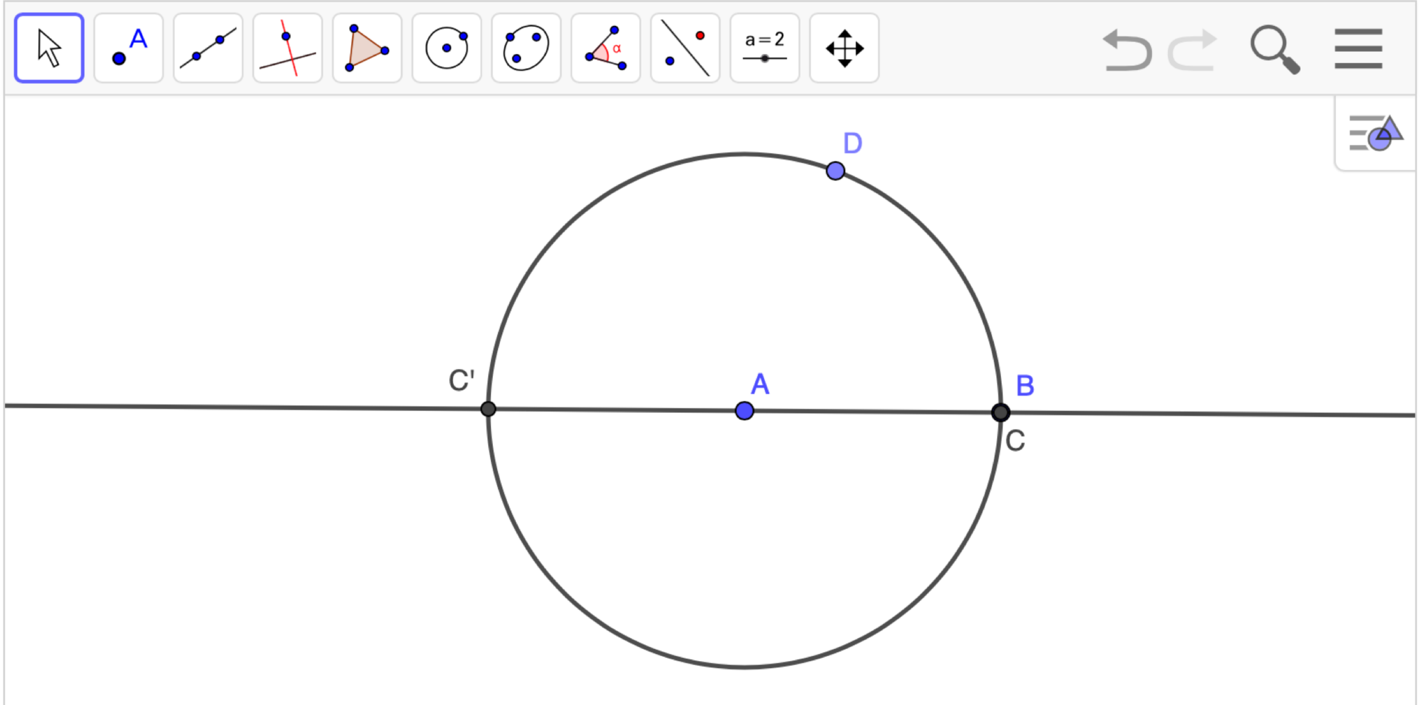
Select relations to check:

- Collinearity of three points
- Equality of distances between two points
- Perpendicularity of segments defined by two points
- Parallelism of segments defined by two points

Start discovery

Using GeoGebra 5.0.524.0.

Let us consider this initial input construction (only the visible points will be observed) :



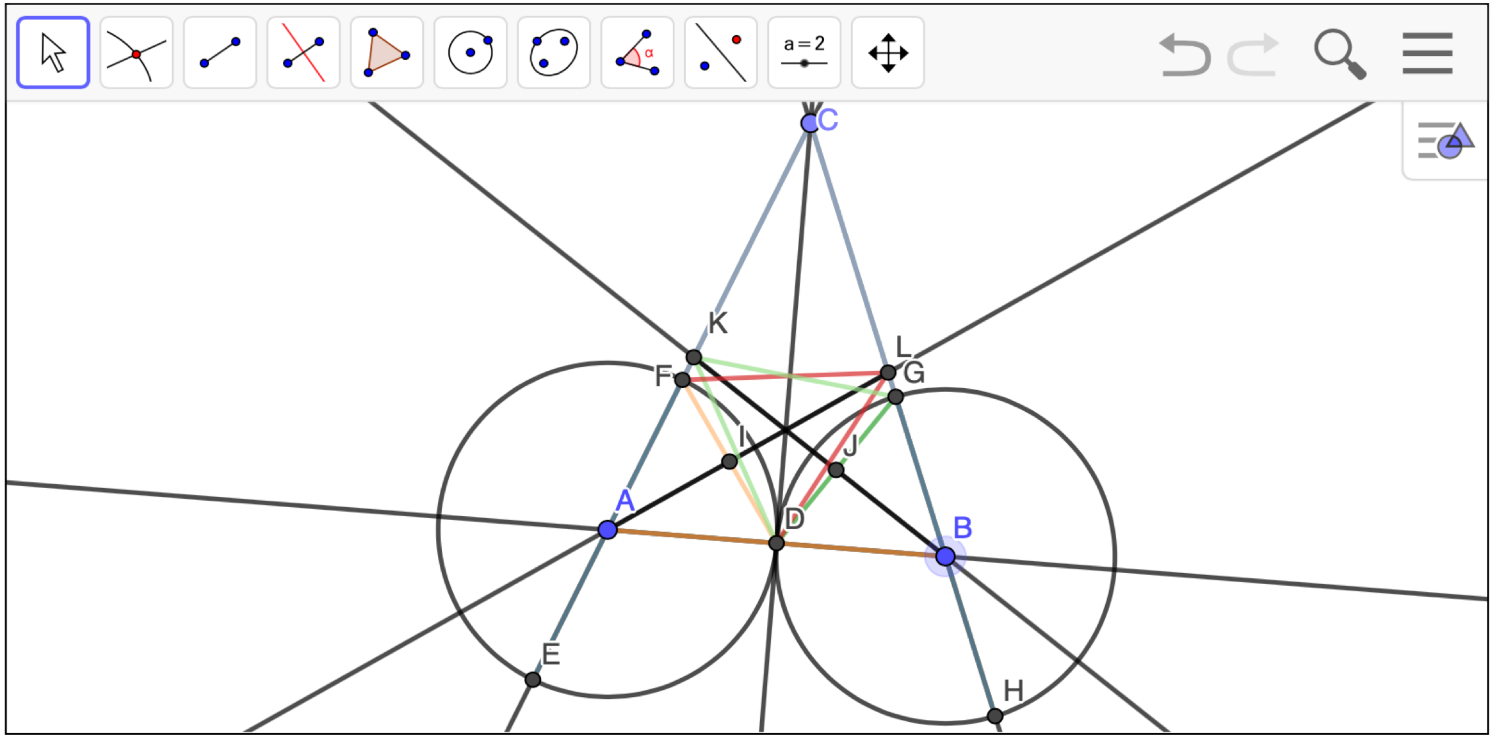
Select relations to check:

Collinearity of three points
Equality of distances between two points
Perpendicularity of segments defined by two points
Parallelism of segments defined by two points

Now the following theorems can be obtained:

1. AB⊥BC
2. AD⊥BC
3. AC⊥BC
4. AC⊥BC'
5. AC'⊥BC
6. BD⊥BC
7. BD⊥DC
8. BC⊥BC'
9. BC⊥DC
10. BC⊥DC'
11. BC⊥CC'
12. BC'⊥DC
13. BC'⊥CC'
14. DC⊥DC'

Finished, found 14 theorems among 45 possible statements.
Found theorems that are true only on parts.
Elapsed time: 0h 0m 1s



Select relations to check:

- Collinearity of three points
- Equality of distances between two points
- Perpendicularity of segments defined by two points
- Parallelism of segments defined by two points

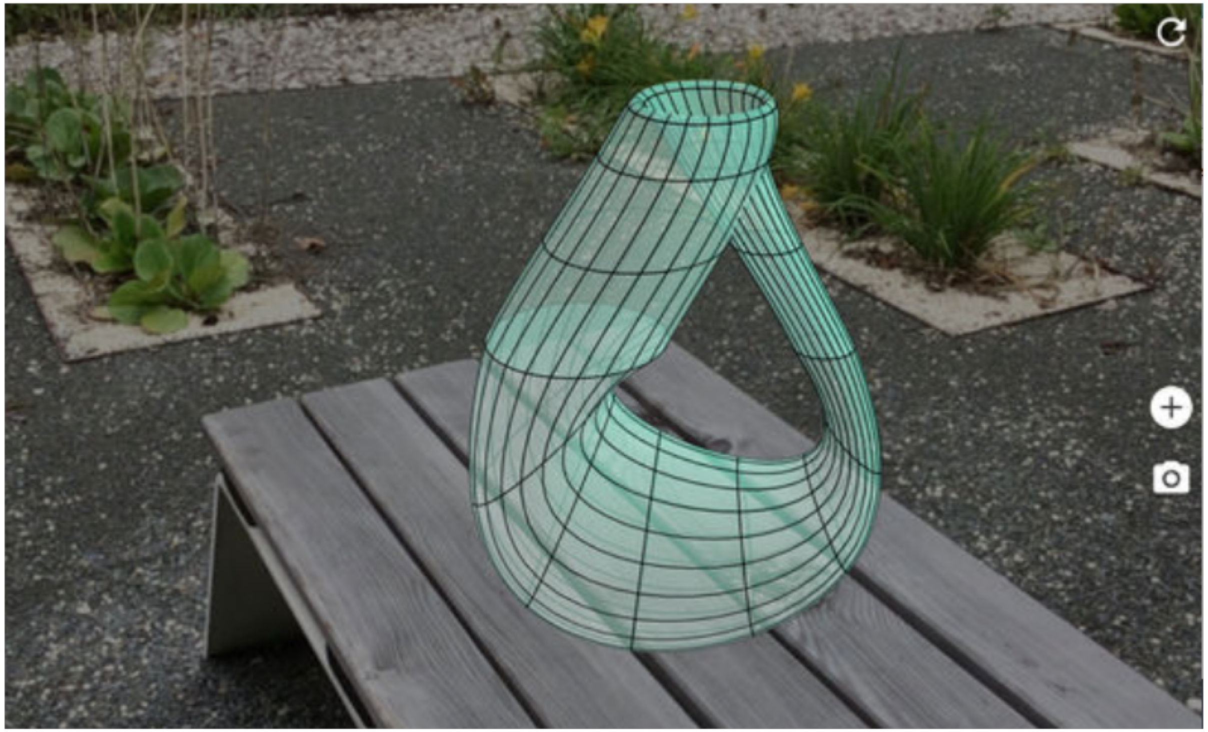
Now the following theorems can be obtained:

1. AB=EF	5. AD=AF	9. AE=AF	13. AF=BD	17. AJ=EI	21. CE=CG	25. DK=GK	29. EH=FG
2. AB=GH	6. AD=BD	10. AE=BD	14. AF=BG	18. BD=BG	22. DE=DG	26. DL=FL	30. EI=HJ
3. AC=BC	7. AD=BG	11. AE=BG	15. AF=BH	19. BD=BH	23. DI=FI	27. EF=GH	31. EJ=HI
4. AD=AE	8. AD=BH	12. AE=BH	16. AG=BE	20. BG=BH	24. DJ=GJ	28. EG=FH	

Finished, found 31 theorems among 2145 possible statements.
Found theorems that are true only on parts.
Elapsed time: 0h 3m 56s

Augmented reality (to explore math objects in a real world context)

https://www.facebook.com/geogebra/videos/10155662725038232/



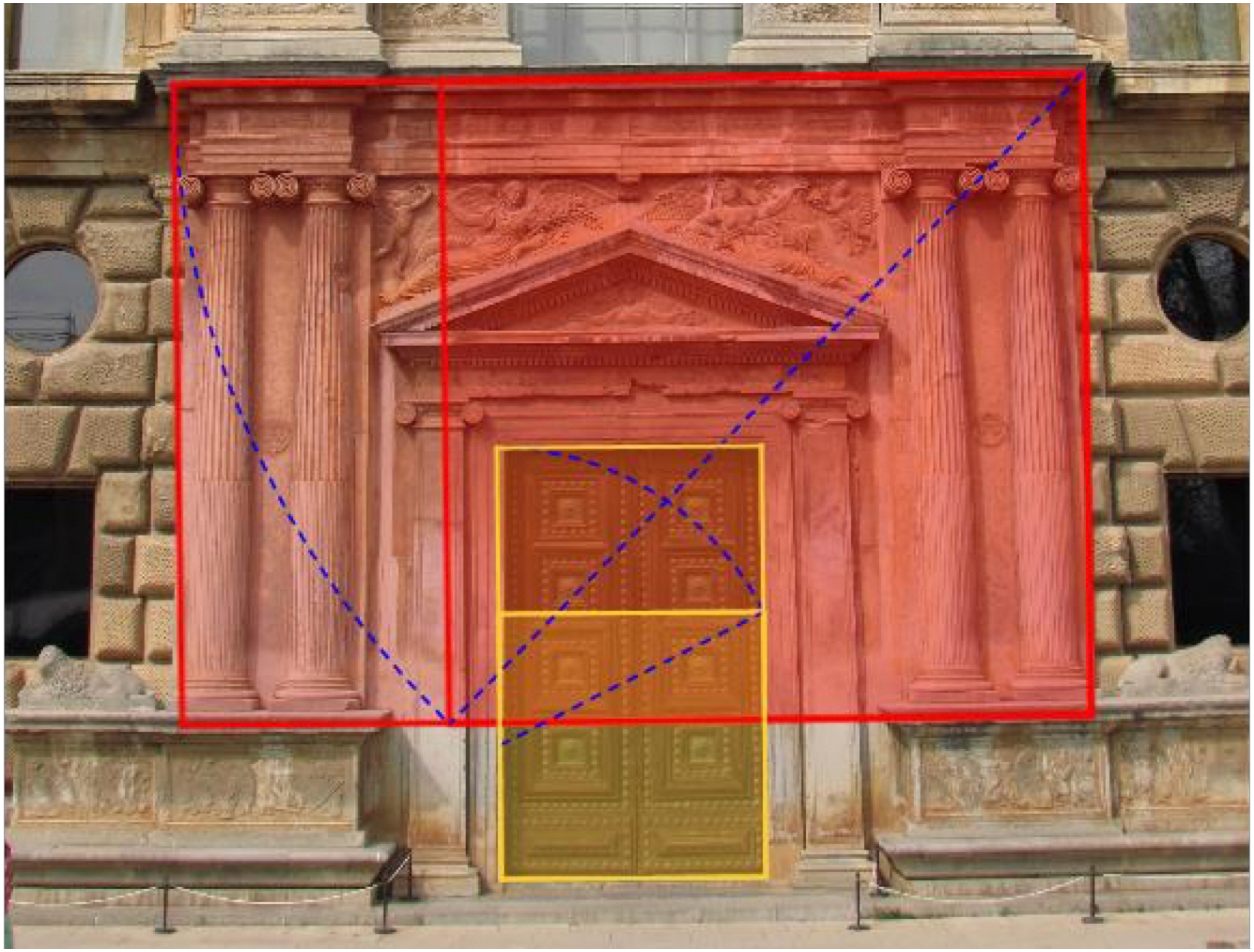
A new scenery

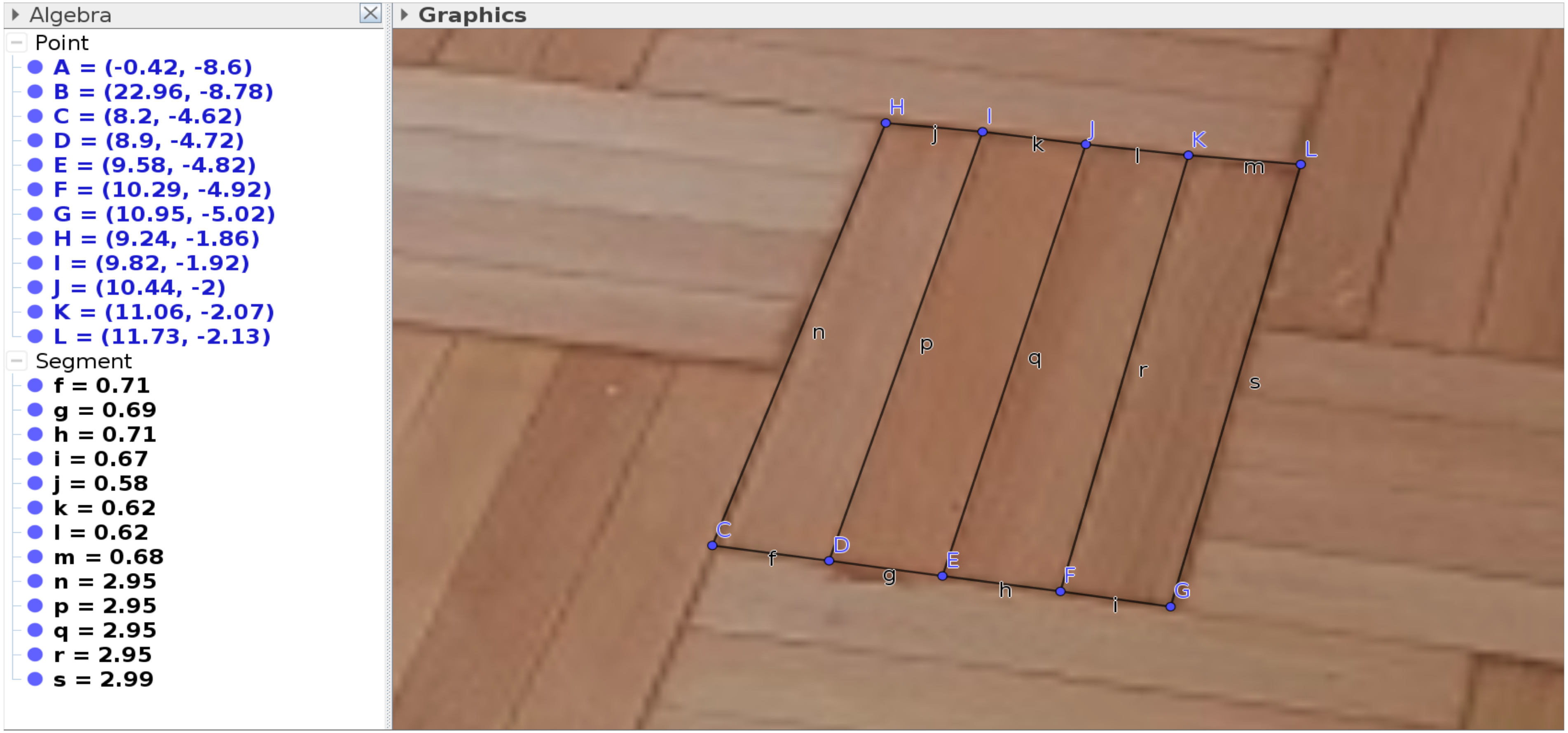
- **Real world** (Smart phones, robotic vision, sensors) – mathematical layer

  1) Pre-processed, associated by geo-possitioning, markers;

  2) Measures {image — Hough transform – recognizing geometric elements } — input GeoGebra – systematic and automatic derivation of geometric properties — generalizing, proving, discovering;

- **Automatically augmented geometric reality**





Imaginary detection of reality with an internal representation of a parquet floor being photographed by a smartphone and translated into GeoGebra

- Recio, T., Richard, P., Vélez, M.P. Designing tasks supported by GeoGebra Automated Reasoning Tools for the development of mathematical skills. International Journal of Technology in Mathematics Education. (to appear)

- Hohenwarter, M., Kovács, Z., Recio, T.: Using GeoGebra Automated Reasoning Tools to explore geometric statements and conjectures. In: Proof Technology in Mathematics Research and Teaching, Eds: Hanna, G. , de Villiers, M., Reid, D. Springer. (to appear).

- Botana, F.; Kovács, Z.; Martínez-Sevilla, A.; Recio, T.: Automatically Augmented Reality with GeoGebra. In: Augmented reality in educational settings. Leiden, (The Netherlands): Brill|Sense. (to appear).