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D.M.Wang simple systems and dynamic constructible closure

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1 Introduction

Dynamic evaluation is a general method for computing with parameters [6, 9]. In 1994, T. Gómez-Díaz implemented the dynamic constructible closure in the scientific computation system Axiom [17]: by simulating dynamic evaluation, it offers the possibility to compute with parameters in a very large way [13]. The outputs of a calculs with T. Gómez-Díaz programs are represented by a finite collection of constructible triangular systems defined in [12, definition p.106]. Though there are numerous applications of these programs (notably polynomial system solving with parameters [11], automatic geometric theorem proving [14, 15], computation of Jordan forms with parameters [16]), nobody gives theorical interest to this kind of triangular systems. The main reason of this phenomenon is that they are definied in [12] within the dynamic evaluation context. On the opposite, most notions of triangular systems (J.F. Ritt-W.T. Wu characteristic sets [24, 28], M. Kalkbrener regular chains [18], D. Lazard triangular sets [20], M. Moreno Maza regular sets [22], D.M. Wang simple systems [26, 27]) are definied in terms of commutative algebra. This problem is at the origin of the work done in [7] where we give a relevant algebraic model of T. Gómez-Díaz systems within commutative algebra terminology. This allows us to relate them to many concepts of triangular systems [7]. Thus, we give interest to the connections with D. Lazard triangular sets in [8]. In a way, this paper is the continuation of this previous work. This time, we study relationships between T. Gómez-Díaz systems and D.M. Wang simple systems. The paper is structured as follows. We have collected in section 2 some needed notations. In section 3, we give all the terminology related to our algebraic model of T. Gómez-Díaz systems. Thus, we define the notion of weak constructible triangular systems and introduce the properties of normalization and squarefreeness. Section 4 is more detailed. First of all, we study a weaker form of normalization called L-normalization. Then we give many properties of constructible triangular systems verifying this new notion. We obtain an algebraic and geometric framework which permits, in section 5, to explore the connections between T. Gómez-Díaz systems and D.M. Wang simple systems. In particular, this last section will demonstrate well the importance of our L-normalization property. Indeed, we show that simple systems and squarefree L-normalized constructible triangular systems are equivalent.

2 Preliminaries

Given a commutative ring A with identity and an ideal $\mathcal{I}$ of A, the residue class ring of A by $\mathcal{I}$ and the set of units of A are respectively denoted by $\frac{A}{\mathcal{I}}$ and $A^{\star}$. characteristic zero. We set $P_0 = K_0$. Let n be a positive integer. We denote respectively by $\mathbb{Z}_{\mathrm{n}}$ and $\mathbb{Z}_{\mathrm{n}}^{+}$ the sets $\{0, \ldots, n\}$ and $\{1, \ldots, n\}$. Then for all $i \in \mathbb{Z}_{\mathrm{n}}^{+}$, we define:

$$P_i = K_0[X_1, \ldots, X_i].$$

Moreover, for all $i \in \mathbb{Z}_{\mathrm{n}}^{+}$ and for all $f \in P_i - P_{i-1}$, we use the following terminology (nearly the one adopted by D. Lazard in [20]):

- the *main variable* of f is X_i,
- the *index* of f $(ind(f))$ is i,
- the *degree* of f $(deg(f))$ is its degree in X_i,
- the *leading coefficient* of f $(lc(f))$ is the coefficient of $X_i^{deg(f)}$ in $P_{i-1}[X_i]$,
- the *discriminant* of f $(Disc_{X_i}(f))$ is the resultant of the polynomials f and $\frac{\partial f}{\partial X_i}$ with respect to X_i.

Working with triangular systems gives rise to consider the following kind of ideal of polynomials (see the beginning of [2, section 2.3] for more references).

Definition 2.1 *[5, exercise 8 p.196] Let $I \subseteq P_n$ be an ideal and fix $f \in P_n$. Then the saturation of I with respect to f is the ideal of P_n:*

$$I : f^{\infty} = \{g \in P_n : f^m g \in I \text{ for some } m > 0\}.$$

Let A, B be two commutative rings with identity and $\sigma : A \to B$ a ring-homomorphism. Then by $\sigma[T]$ we mean the ring homorphism defined by:

$$\begin{array}{ccc} \mathcal{A}[T] & \xrightarrow{\sigma[T]} & \mathcal{B}[T] \\ \displaystyle\sum_{i=0}^{d} a_i T^i & \longmapsto & \displaystyle\sum_{i=0}^{d} \sigma(a_i) T^i. \end{array}$$

Moreover, if σ is one-to-one, we note:

$$\mathcal{A} \xhookrightarrow{\sigma} \mathcal{B},$$

and if σ is surjective, we note:

$$\mathcal{A} \overset{\sigma}{\twoheadrightarrow} \mathcal{B}.$$

From now on, the words ring and homomorphism mean respectively commutative ring with identity and ring homomorphism.
Let $\{a_k\}_{k \in I}$ be a finite set of elements of A and $\mathcal{I}$, S, respectively the ideal and the multiplicative set of A generated by the a_k $(k \in I)$. We write $\mathcal{I} = \langle a_k \rangle_{k \in I}$ and $S = \prec a_k \succ_{k \in I}$.

Moreover, given a multiplicative set T of A, we denote respectively by $\mathcal{S}at(T)$ and $T^{-1}A$ the saturation of T [1, exercise 7 p.44] and the ring of fractions of A with respect to T. Given $a \in A$, we write A_a the ring $T^{-1}A$ with $T = \prec a \succ$.
Finally, we set $\widetilde{K}_0$ to be an algebraic closure of K_0. Given an ideal $\mathcal{J}$ of P_n, we write $V(\mathcal{J})$ the affine variety of $\widetilde{K}_0^n$ definied by $\mathcal{J}$. By extension, given a polynomial $g \in P_n$, we denote by $V(g)$ the affine variety definied by the ideal $\langle g \rangle$ of P_n. For all subset W of $\widetilde{K}_0^n$, we write $\overline{W}$ the Zariski closure of W [5, definition 2 p.192].

3 Squarefree normalized constructible triangular systems

In [7], we build a relevant algebraic model of the triangular systems involved in the dynamic constructible closure programs: it is the concept of *squarefree normalized constructible triangular systems*. This section introduces all the terminology related to these systems and gives one of their properties (very important for the rest of this paper) without the squarefree hypothesis.

Definition 3.1 *Let n be a positive integer and E be a subset of $\mathbb{Z}_n^+$. A* weak constructible triangular system *in P_n is a set $\{g_j\, \xi_j\, 0\}_{j \in E \cup F}$ verifying for all $j \in E \cup F$:*

1. *the polynomial g_j belongs to P_j with index j,*
2. *ξ_j is the symbol "$=$" or the symbol "$\neq$" according as $j \in E$ or $j \in F$.*

Remark. Let $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ be a weak constructible triangular system in P_n. One can easily check that E and F are two disjoint subsets of $\mathbb{Z}_n^+$. We write then $E \sqcup F$ the disjoint union of E and F.

Definition 3.2 *Let $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ be a weak constructible triangular system in P_n. We set $L_0 = K_0$ and we denote by Ψ_0 the identity homomorphism of L_0. For all $i \in \mathbb{Z}_n^+$, we recursively define a ring L_i and a homomorphism $\Psi_i : P_i \to L_i$ in the following way:*

- *if $i \notin E \sqcup F$, we set:*
$$L_i = L_{i-1}[X_i] \text{ and } \Psi_i = \Psi_{i-1}[X_i],$$
- *if $i \in F$, we set:*
$$L_i = (L_{i-1}[X_i])_{\Psi_{i-1}[X_i](g_i)} \text{ and } \Psi_i = inj_{L_i} \circ \Psi_{i-1}[X_i],$$
where inj_{L_i} is the canonical homomorphism:
$$\begin{array}{ccc} L_{i-1}[X_i] & \xrightarrow{inj_{L_i}} & (L_{i-1}[X_i])_{\Psi_{i-1}[X_i](g_i)} \\ f & \mapsto & \dfrac{f}{1}, \end{array}$$
- *if $i \in E$, we set:*
$$L_i = \frac{L_{i-1}[X_i]}{\langle \Psi_{i-1}[X_i](g_i) \rangle} \text{ and } \Psi_i = \pi_{L_i} \circ \Psi_{i-1}[X_i],$$
where π_{L_i} is the projection of $L_{i-1}[X_i]$ over $\dfrac{L_{i-1}[X_i]}{\langle \Psi_{i-1}[X_i](g_i) \rangle}$.

Example. Let us study the unit circle example. Let $K_0 = L_0 = \mathbb{Q}$ and g_2 be the polynomial $X_2^2 + X_1^2 - 1$ of P_2. We consider the system $\{g_2 = 0\}$. It is obviously a weak constructible triangular system in P_2. Then we can construct the rings L_i $(i = 1, 2)$ associated with this system. By definition, since $E = \{2\}$ and $F = \emptyset$, we have:

$$L_1 = L_0[X_1] = \mathbb{Q}[\mathbb{X}_{\nVdash}] \text{ and } \mathbb{L}_{\nvDash} = \frac{\mathbb{L}_{\nVdash}[\mathbb{X}_{\nvDash}]}{\langle \gneqq_{\nVdash}[\mathbb{X}_{\nvDash}](\mho_{\nvDash})\rangle} = \frac{\mathbb{Q}[\mathbb{X}_{\nVdash}, \mathbb{X}_{\nvDash}]}{\langle \mathbb{X}_{\nvDash}^{\nvDash} + \mathbb{X}_{\nVdash}^{\nvDash} - \nVdash \rangle}$$

where Ψ_1 is the identity homomorphism of $\mathbb{Q}[\mathbb{X}_{\nVdash}]$.

Notation. Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. For all $i \in \mathbb{Z}_{\mathrm{n}}^{+}$, we write $E_i = E \cap \mathbb{Z}_{\mathrm{i}}^{+}$ and $F_i = F \cap \mathbb{Z}_{\mathrm{i}}^{+}$. Furthermore, we set $G_0 = \{1\}$ and for all $i \in \mathbb{Z}_{\mathrm{n}}^{+}$, we define a multiplicative set G_i of P_i by:

$$G_i = \prec g_k \succ_{k\in F_i} .$$

Definition 3.3 *A* normalized constructible triangular system *in P_n is a weak constructible triangular system $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ in P_n such that for all $j \in E \sqcup F$:*

$$lc(g_j) \in \mathcal{S}\mathrm{at}(G_{j-1}).$$

Example. Consider the weak constructible triangular system in P_3 with $K_0 = \mathbb{Q}$:

$$T = \begin{cases} g_3 = (X_1\,X_2 - 1)\,X_3^2 + X_2 = 0 \\ g_2 = X_1\,X_2^2 - X_2 \neq 0 \\ g_1 = X_1^2 - X_1 \neq 0 \end{cases}$$

By definition, the sets G_1 and G_2 are respectively equal to $\prec X_1^2 - X_1 \succ$ and $\prec X_1^2 - X_1, X_1\,X_2^2 - X_2 \succ$. Then we have obviously that $lc(g_2) = X_1 \in \mathcal{S}at(G_1)$ and $lc(g_3) = X_1\,X_2 - 1 \in \mathcal{S}at(G_2)$. Thus T is a normalized constructible triangular system in P_3.

The most important property of normalized constructible triangular systems (next section will demonstrate this) is given by the following proposition.

Proposition 3.1 *[7, proposition 1.2.2 p.43] Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a normalized constructible triangular system in P_n. Then for all $i \in \mathbb{Z}_{\mathrm{n}}^{+}$ and for all $g \in \mathcal{S}\mathrm{at}(G_i)$:*

$$\Psi_i(g) \in L_i^\star.$$

Proof. By induction on i. The main ingredient of the proof is that, since the leading coefficients of the g_j belong to $\mathcal{S}at(G_{j-1})$ $(j \in E_i \sqcup F_i)$, the homomorphism Ψ_i preserves identity.$\diamond$

Definition 3.4 *A* squarefree *normalized constructible triangular system in P_n is a normalized constructible triangular system $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ in P_n such that for all $j \in E \sqcup F$:*

$$Disc_{X_j}(g_j) \in L_{j-1}^\star.$$

Example. Consider again the system T of example 3. Is it squarefree ? The discriminant of g_3 is equal to $4\,X_2\,(X_1\,X_2 - 1)^2$ and so we have:

$$\frac{1}{4}\,X_2\,Disc_{X_3}(g_3) = g_2^2.$$

Therefore it belongs to $\mathcal{S}at(G_2)$. The discriminant of g_2 is equal to $-X_1$ and so satisfies the equality:

$$-(X_1 - 1)\, Disc_{X_2}(g_2) = g_1.$$

Therefore it belongs to $\mathcal{S}at(G_1)$. Since the discriminant of g_1 is different from 0, we conclude then that T is a squarefree normalized constructible triangular system in P_3.

Remark. We show in [7, section 4.1] that these squarefree normalized constructible triangular systems form a relevant algebraic model of the T. Gómez-Díaz systems involved in the dynamic constructible closure programs. The example 5 will give an illustration of this result.

4 Squarefree L-normalized constructible triangular systems

This section introduces a weaker notion of normalization called L-normalization. The origin of this property provides from the polynomial triangular systems theory. First of all, D. Lazard defines in the polynomial frame the notion of normalization [20, definition 3.2 p.150]. But later on, it appears with the works of P. Aubry and M. Moreno Maza that this notion is less interesting than the so-called property of regularity [22]. Indeed, regularity is weaker than normalization and furthermore, allows to make comparaison with many others triangular sets notions [2, 3]. In particular, P. Aubry shows that the concepts of M. Kalkbrener regular chain and regular triangular polynomial system are equivalent (see for example [4, proposition 4.4.12]). Our aim is to mimic their approach. Thus, next section will show that our notion of L-normalization permits us to relate algebraic models of T. Gómez-Díaz systems to D.M. Wang simple systems. More precisely, this section explores in details the properties of L-normalized constructible triangular systems [7, section 1.2]. We give only one result with the additional squarefreeness hypothesis (lemma 4.1). As in previous section, all the results are given with sketches of proofs (details can be found in [7]).

Definition 4.1 *A* L-normalized constructible triangular system *in* P_n *is a weak constructible triangular system* $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ *in* P_n *such that for all* $j \in E \sqcup F$*:*

$$\Psi_{j-1}(lc(g_j)) \in L^{\star}_{j-1}.$$

The following result is an immediate consequence of proposition 3.1. Nevertheless, it will be the key fact in the proof of the main result of this paper (theorem 5.2).

Proposition 4.1 *[7, proposition 1.2.2 p.43] Every normalized constructible triangular system in* P_n *is* L*-normalized.*

Remark. The converse is false as it will be shown in remark 4.

Notation. Given a L-normalized constructible triangular system in P_n, for all $i \in \mathbb{Z}_n$ we define q_i to be the projection of P_i over $\frac{P_i}{ker\,\Psi_i}$.

Remark. Given a L-normalized constructible triangular system in P_n, one can check that for all $i \in \mathbb{Z}_n$, the image $q_i(G_i)$ of G_i is a multiplicative set of $\frac{P_i}{ker\,\Psi_i}$ which does not contain 0.

Notation. Given a L-normalized constructible triangular system in P_n, for every $i \in \mathbb{Z}_n$ we define $can_{\overline{G}_i}$ to be the canonical homomorphism:

$$\begin{array}{ccccc} P_i & \xrightarrow[q_i]{} & \dfrac{P_i}{ker\ \Psi_i} & \xrightarrow[can_{\overline{G}_i}]{} & q_i(G_i)^{-1}\left(\dfrac{P_i}{ker\ \Psi_i}\right) \\ f & \longmapsto & q_i(f) & \longmapsto & \dfrac{q_i(f)}{1} \end{array}$$

The following results (theorems 4.1 and 4.2) give two main properties of L-normalized constructible triangular systems in P_n (a weaker form of these two theorems is given in [8]). They are very closed to the polynomial triangular systems theory [7, chapter 1]. In particular, a "polynomial version" of theorem 4.2 is presented in [2, proposition 5.1 p.121].

Theorem 4.1 *[7, theorem 1.2.1 p.45] Let $\{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ be a L-normalized constructible triangular system in P_n. Then for all $i \in \mathbb{Z}_n^+$, we have the following commutative diagram:*

$$\begin{tikzcd} P_i \arrow[r, "\Psi_i"'] \arrow[d, two heads, "q_i"'] \arrow[dr, phantom, "\circ"] & L_i \\ \dfrac{P_i}{ker\ \Psi_i} \arrow[r, hook, "can_{\overline{G}_i}"'] & q_i(G_i)^{-1}\left(\dfrac{P_i}{ker\ \Psi_i}\right) \arrow[u, "\psi_i", "\simeq"'] \end{tikzcd}$$

where for all $f, g \in P_i$, with $q_i(g) \in q_i(G_i)$, the isomorphism ψ_i is defined by:

$$\psi_i\left(\frac{q_i(f)}{q_i(g)}\right) = \frac{\Psi_i(f)}{\Psi_i(g)}.$$

Proof. By induction on i. Using proposition 3.1 and [1, proposition 3.1 p.37], it is easy to show the existence of ψ_i making the diagram commutative. The injectivity of ψ_i is obvious. Proving that the homomorphism ψ_i is surjective is more difficult: the case $i \notin E \sqcup F$ is easy; if $i \in E \sqcup F$, the key fact is the isomorphism:

$$L_{i-1}[X_i] \simeq q_{i-1}(G_{i-1})^{-1}\left(\frac{P_{i-1}}{ker\ \Psi_{i-1}}\right)[X_i].$$

Given $f \in L_{i-1}[X_i]$, one can check then that there exists $h \in P_i$ and $g \in G_{i-1}$ such that $f = \frac{\Psi_{i-1}[X_i](h)}{\Psi_{i-1}(g)}$. The result follows from the commutativity of the diagram.$\diamond$

Notation. Given a weak constructible triangular system $T = \{g_j\, \xi_j\, 0\}_{j \in E \sqcup F}$ in P_n, we set for all $i \in \mathbb{Z}_n^+$:

$$h_i = \prod_{j \in E_i} lc(g_i).$$

Furthermore, for all $f \in P_n$, we denote by $prem(f, \{g_j\}_{j \in E_i})$ the generalized pseudo-remainder of f by the triangular set $\{g_j\}_{j \in E_i}$ (see for example [21, theorem 5.2.2 p.170]).

Theorem 4.2 *[7, theorem 1.2.2 p.49] Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a L-normalized constructible triangular system in P_n. Then for all $i \in \mathbb{Z}_n^+$:*

$$ker\Psi_i = \langle g_j\rangle_{j\in E_i} : h_i^\infty = \{g \in P_i;\, prem(g, \{g_j\}_{j\in E_i}) = 0\}.$$

Proof. There are two main ingredients in the proof. Fix a positive integer $i \in \mathbb{Z}_n^+$. The first is that the image $\Psi_i(h_i)$ of h_i is a unit of L_i (since the constructible triangular system is normalized). The second is that, given $g \in ker\Psi_i$, we have $prem(g, \{g_j\}_{j\in E_i}) = 0$ [7, lemma 1.2.10 p.48].$\diamond$

Remark. Using theorems 4.1 and 4.2, we can prove that the converse of proposition 4.1 is false. Let us consider the following system in $\mathbb{Q}[\mathbb{X}_{\not\Vdash}, \mathbb{X}_{\not\models}, \mathbb{X}_{\not\Vdash}, \mathbb{X}_{\not\trianglerighteq}]$:

$$T = \left\{ \begin{array}{l} g_4(X_1, X_2, X_3, X_4) = (X_2\,X_3 + X_1^2 - 1)\,X_4 - 1 \neq 0 \\ g_3(X_1, X_2, X_3) = X_1\,X_3 + X_2 = 0 \\ g_2(X_1, X_2) = X_2^2 + X_1 = 0 \\ g_1(X_1) = X_1^2 - X_1 \neq 0 \end{array} \right.$$

The constructible triangular system $\{g_1 \neq 0,\, g_2 = 0,\, g_3 = 0\}$ in $\mathbb{Q}[\mathbb{X}_{\not\Vdash}, \mathbb{X}_{\not\models}, \mathbb{X}_{\not\Vdash}]$ is obviously normalized and so by proposition 4.1, L-normalized. Moreover, it is clear that $lc(g_4) = X_2\,X_3 + X_1^2 - 1$ does not belong to $\mathcal{S}at(X_1^2 - X_1)$. Therefore, the weak constructible triangular system T is not normalized. However, one can check using theorem 4.2 that:

$$ker\;\Psi_3 = \langle g_2, g_3\rangle : X_1^\infty = \langle X_2^2 + X_1, X_1\,X_3 + X_2, X_2\,X_3 - 1\rangle.$$

Thus, this implies that $lc(g_4)$ is equal to X_1^2 modulo $ker\;\Psi_3$ and so invertible in L_3 by theorem 4.1. Then the weak constructible triangular system T is L-normalized.

For the next definition, we need further notations.

Notation. Given a weak constructible triangular system $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ in P_n, for all $i \in \mathbb{Z}_n^+$, we define $\mathcal{J}_i$ to be the ideal $\langle g_j\rangle_{j\in E_i}$ of P_i and we set:

$$H_i = \prod_{j\in F_i} g_j.$$

Definition 4.2 *Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$, we define a subset Z_i of $\widetilde{K}_0^i$ by:*

$$Z_i = V(\mathcal{J}_i) - V(H_i).$$

It is the set of the zeros *of the constructible system* $\{g_j\,\xi_j\,0\}_{j\in E_i\sqcup F_i}$.

This definition is very natural. Indeed, given a weak constructible triangular system $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ in P_n, it is logic to give interest to the set $V(g_j)_{j\in E} - V(g_j)_{j\in F}$. In fact, under the L-normalization hypothesis, it can be characterized by a less trivial property. For all $i \in \mathbb{Z}_n^+$, the set Z_i is the standard open set [23, definition 4.13 p.21] of $V(ker\;\Psi_i)$ definied by the class of H_i in the ring $\frac{P_i}{\sqrt{ker\;\Psi_i}}$ (see [7, proposition 2.3.2 p.84] for more details). This result leads to the following theorem.

Theorem 4.3 *[7, theorem 2.3.1p.86] Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a normalized constructible triangular system in P_n. Then for all $i \in \mathbb{Z}_n^+$ and for all $f \in P_i$, the following conditions are equivalent:*

1. $\Psi_i(f) \in L_i^\star$,
2. $\forall\, a \in Z_i, f(a) \neq 0$.

Proof. There are three steps in the proof. First, given a L-normalized constructible triangular system in P_n, one can show that for all $i \in \mathbb{Z}_n^+$ and for all $f \in P_i$, we have $\Psi_i(f) \in L_i^\star$ if and only if f does not vanish on $V(\mathcal{J}_i) - V(H_i\,h_i)$ [7, corollary 2.3.1 p.84]. The second point is that for all $i \in \mathbb{Z}_n^+$, the set $V(\mathcal{J}_i) - V(H_i\,h_i)$ is the standard open set of $V(ker\,\Psi_i)$ definied by the class of H_i in the ring $\frac{P_i}{\sqrt{ker\,\Psi_i}}$ [7, proposition 2.3.1 p.82]. Finally, it suffices to apply [7, proposition 2.3.2 p.84] mentionned above to conclude.$\diamond$

It follows immediately that we can give a characterization in geometric terms of L-normalized constructible triangular systems in P_n.

Corollary 4.1 *Let $T = \{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. Then the following conditions are equivalent:*

1. *T is a L-normalized constructible triangular system in P_n,*
2. *for all $j \in E \sqcup F$ and for all $a \in Z_{j-1}$, $lc(g_j)(a) \neq 0$.*

The next result forms the second part of an algebra-geometry dictionary started in theorem 4.3.

Theorem 4.4 *Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a L-normalized constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$ and for all $f \in P_i$, the following conditions are equivalent:*

1. $f \in \sqrt{ker\,\Psi_i}$,
2. $\forall\, a \in Z_i, f(a) = 0$.

Proof. By induction on i. Let $f \in \sqrt{ker\,\Psi_i}$. One can check that this ideal is equal to $\sqrt{\mathcal{J}_i} : H_i$ [7, lemma 2.3.3 p.86]. The result is then obvious. On the opposite, let f be a polynomial of P_i with $Z_i \subseteq V(f)$. It implies that f vanishes on $\overline{Z_i}$ and so on $V(\mathcal{J}_i : H_i^\infty)$ by [4, proposition A.1.16 p.142]. Then, since the ideal $\sqrt{\mathcal{J}_i : H_i^\infty}$ is equal to $\sqrt{\mathcal{J}_i} : H_i$ by [4, proposition A.1.14 p.141], it sufficies to apply the Nullstellensatz and again [7, lemma 2.3.3 p.86] to conclude.$\diamond$

The following proposition is a nice property of L-normalized constructible triangular systems in P_n. Keeping only the polynomials defining their equations, it states that we obtain a so-called regular chain [18, definition p.111].

Proposition 4.2 *[7, proposition 2.4.1 p.90] Let $\{g_j\,\xi_j\,0\}_{j\in E\sqcup F}$ be a L-normalized constructible triangular system in P_n. Then $\{g_j\}_{j\in E}$ is a regular chain in P_n.*

Proof. The key fact is that the polynomial triangular system $\{g_j = 0\}_{j\in E}$ is regular [7, proposition 1.3.3 p.54]. The result is then a corollary of [22, theorem III.6 p.107] which proves the equivalence between regular chain and regular polynomial triangular systems.$\diamond$

In particular, this results allows us to give an algebraic property of the ideals ker Ψ_i ($i \in \mathbb{Z}_n^+$) associated with a L-normalized constructible triangular system in P_n. For this purpose, we first recall a well-known result concerning regular chains (details of the proof can be found in [7, proposition 2.1.1 p.74]).

Proposition 4.3 *Let R be a regular chain in P_n. For all $i \in \mathbb{Z}_n^+$, the ideal $Rep_i(R)$ [18, definition p.112] of P_i represented by the regular chain $R \cap P_i$ is unmixed of dimension $i - |R \cap P_i|$.*

Corollary 4.2 *Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a L-normalized constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$, the ideal $ker\, \Psi_i$ is unmixed of dimension $i - |E_i|$.*

Proof. Fix $i \in \mathbb{Z}_n^+$. By proposition 4.2, we know that $\{g_j\}_{j \in E_i}$ is a regular chain in P_i. Furthermore, the ideal $ker\, \Psi_i$ is equal to $\mathcal{J}_i : h_i^\infty$. Given an ideal $\mathcal{I}$ of P_n, let us denote by $Ass(\mathcal{I})$ the prime ideals associated with $\mathcal{I}$. In his thesis, P. Aubry shows first that $\sqrt{\mathcal{J}_i : h_i^\infty}$ is equal to $Rep_i(\{g_j\}_{j \in E_i})$ [4, theorem 4.4.11 p.57] and that $Ass(\sqrt{\mathcal{J}_i : h_i^\infty})$ is equal to $Ass(\mathcal{J}_i : h_i^\infty)$ [4, corollary 4.1.5 p.44]. We conclude then with the previous proposition.$\diamond$

Given a L-normalized constructible triangular system in P_n, propositions 4.4 and 4.5 give two geometric properties of the sets Z_i ($i \in \mathbb{Z}_n^+$). The first guarantees that the set of zeros is non-empty. The second relates the set Z_i to $V(ker\, \Psi_i)$ ($i \in \mathbb{Z}_n^+$).

Proposition 4.4 *[7, proposition 2.4.3 p.93] Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a L-normalized constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$ we have $Z_i \neq \emptyset$.*

Proof. One can note that the result is obvious if $E = \emptyset$. On the opposite, there are two ingredients in the proof. Fix $i \in \mathbb{Z}_n^+$. First the set $\{g_j\}_{j \in E_i}$ is a regular chain of P_i by proposition 4.2. The second is the inclusion (with the notations adopted in [18, definition p.111]): $RZ_i(\{g_j\}_{j \in E}) \subseteq Z_i$ [7, lemma 2.4.3 p.90].$\diamond$

Proposition 4.5 *Let $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a L-normalized constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$:*

$$\overline{Z}_i = V(ker\, \Psi_i).$$

Proof. Fix $i \in \mathbb{Z}_n^+$. By theorem 4.2, we know that the ideal $ker\, \Psi_i$ is equal to $\mathcal{J}_i : h_i^\infty$. In fact, one can check that $ker\, \Psi_i$ is also equal to $\mathcal{J}_i : (H_i \, h_i)^\infty$ [7, lemma 2.3.2 p.82]. We conclude then using the equality $Z_i = V(\mathcal{J}_i) - V(H_i \, h_i)$ [7, propositions 2.3.1 p.82 and 2.3.2 p.84].$\diamond$

Definition 4.3 *A* squarefree *L-normalized constructible triangular system in P_n is a L-normalized constructible triangular system $\{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ in P_n such that for all $j \in E \sqcup F$:*

$$Disc_{X_j}(g_j) \in L_{j-1}^\star.$$

As in corollary 4.1, we can characterize squarefree L-normalized constructible triangular systems in geometric terms.

Proposition 4.6 *Let $T = \{g_j \, \xi_j \, 0\}_{j \in E \sqcup F}$ be a weak constructible triangular system in P_n. Then the two following conditions are equivalent:*

1. *T is a squarefree L-normalized constructible triangular system in P_n,*

2. for all $j \in E \sqcup F$ and for all $a \in Z_{j-1}$:

$$lc(g_j)(a) \neq 0,\ Disc_{X_j}(g_j)(a) \neq 0.$$

Proof. The result follows immediately from theorem 4.3.$\diamond$

We end this section with an interesting algebraic property of the ideals $ker\,\Psi_i\,(i \in \mathbb{Z}_n^+)$ associated with a squarefree L-normalized constructible triangular system in P_n.

Lemma 4.1 *Let $\{g_j\ \xi_j\ 0\}_{j \in E \sqcup F}$ be a squarefree L-normalized constructible triangular system in P_n. For all $i \in \mathbb{Z}_n^+$, the ideal $ker\,\Psi_i$ is radical.*

Proof. Fix $i \in \mathbb{Z}_n^+$. First, one can check that the polynomial triangular system $\{g_j = 0\}_{j \in E_i}$ is regular and squarefree (in the sense of D. Lazard triangular sets [20, definition 3.2 p.150]) [7, proposition 3.2.2 p.107]. Moreover, by theorem 4.2, we know that the ideal $ker\,\Psi_i$ is equal to $\mathcal{J}_i : h_i^\infty$. We conclude then using [7, corollary 3.1.2 p.101] which states in particular that, under squarefree hypothesis, this ideal is radical.$\diamond$

5 D.M. Wang simple systems

In [26, 27], D.M. Wang presents an algorithm that decomposes a constructible system $[P, Q]$ (where P is a set of equations and Q a set of inequations) into finitely many constructible systems called *simple systems.* More precisely, his algorithm returns e simple systems $[T_1, \widetilde{T}_1], \ldots, [T_e, \widetilde{T}_e]$ such that:

$$Zero(P/Q) = \bigcup_{i=1}^{e} Zero(T_i/\widetilde{T}_i)$$

where, for all $1 \leq i \leq e$, $Zero(P/Q)$ and $Zero(T_i/\widetilde{T}_i)$ denote respectively the set of zeros of the systems $[P, Q]$ and $[T_i, \widetilde{T}_i]$. This concept of simple system is due to J.M. Thomas [25].
There is a natural question to ask: is it possible to define simple systems in terms of constructible triangular systems (studied in the last two sections) ? Proposition 5.1 will show that such a characterization is possible. In fact, the notions of D.M. Wang simple system and squarefree L-normalized constructible triangular system are equivalent (theorem 5.1). This allows us in theorem 5.2 to relate D.M. Wang simple systems to algebraic models of T. Gómez-Díaz systems.
On the other hand, two main results of [26] can be then easily proved (proposition 5.2) since they have been showed in the previous section within our terminology. Moreover, theorem 5.3 generalizes [26, theorem 6 p.312] as it states in particular that the Zariski closure of the sets $Zero(T_i/\widetilde{T}_i)$ $(1 \leq i \leq e)$ are equidimensionnal. Finally, we relate D.M. Wang simple systems to M. Kalkbrener regular chains (theorem 5.4).

Remark. This section is essentially [7, section 3.2.3]. All the results are given with self-contained proofs. For this reason, for every result, we do not mention the explicit reference to [7].

Notation. From now on, simple system means D.M. Wang simple systems.

In [26], D.M. Wang works with the classical monomial order: $X_1 < \ldots < X_n$. He first introduces the following squarefree definition.

Definition 5.1 *[26, definition p.300] Let f be a polynomial of P_n, $\hat{K}_0$ an extension field of K_0 and $a = (a_1, \ldots, a_{n-1})$ a point of $\hat{K}^{n-1}$. The polynomial $f(a, X_n)$ is said to be* squarefree *with respect to X_n if:*

$$gcd\left(f(a, X_n), \frac{\partial f}{\partial X_n}(a, X_n)\right) \in \hat{K}.$$

Remark. This definition can be restated as follows. The polynomial $f(a, X_n)$ is said to be squarefree with respect to X_n if:

$$Disc_{X_n}(f)(a) \neq 0.$$

Moreover D.M. Wang uses the following terminology. A finite non-empty ordered set $[f_1, \ldots, f_r]$ of non-constant polynomials of P_n is called a triangular set [26, definition 4 p.301] if:

$$ind(f_1) < \ldots < ind(f_r).$$

Given two sets of polynomials $P = [f_1, \ldots, f_r]$, $Q = [g_1, \ldots, g_s]$ in P_n, he denotes by $[P, Q]$ the following constructible system in P_n:

$$\begin{cases} f_r = 0 \\ \vdots \\ f_1 = 0 \\ g_s \neq 0 \\ \vdots \\ g_1 \neq 0 \end{cases}$$

He writes then $Zero(P/Q)$ the set of zeros of this system. For all set P of polynomials in P_n, for all constructible system $\mathcal{B} = [P, Q]$ in P_n and for all $k \in \mathbb{Z}_{\mathrm{n}}^{+}$, he sets:

$$\begin{array}{ll} P^{(k)} = \{f \in P \mid ind(f) \leq k\}, & P^{\langle k \rangle} = \{f \in P \mid ind(f) = k\}, \\ \mathcal{B}^{(k)} = [P^{(k)}, Q^{(k)}], & \mathcal{B}^{\langle k \rangle} = [P^{\langle k \rangle}, Q^{\langle k \rangle}]. \end{array}$$

Moreover, he sets:

$$\breve{\mathcal{B}} = P \cup Q, \ Zero(\mathcal{B}) = Zero(P/Q).$$

Definition 5.2 *[26, definition 5 p.302] A pair $\mathcal{G} = [T, \widetilde{T}]$ of triangular sets in P_n is called a* simple system *if:*

1. *$T \cap \widetilde{T} = \emptyset$ and $\breve{\mathcal{G}}$ can be reordered as a triangular set,*
2. *For any $k \in \mathbb{Z}_{\mathrm{n}}^{+}$, $f \in \breve{\mathcal{G}}^{\langle k \rangle}$ and $a \in Zero(\mathcal{G}^{(k-1)})$:*

$$lc(f)(a) \neq 0 \textit{ and } f(a, X_k) \textit{ is squarefree with respect to } X_k.$$

Remark. D.M. Wang defines also the notions of primitive and reduced simple systems. We restrict our study to simple systems in P_n.

What we have to do is translating as best as possible this definition within our algebraic framework developped in the previous sections. It is the purpose of the following lemma.

Lemma 5.1 *Let $T = \{g_j\}_{j\in E}$ and $\widetilde{T} = \{g_j\}_{j\in F}$ be two sets of polynomials in P_n such that for all $j \in E \cup F, ind(g_j) = j$. Then the following conditions are equivalent:*

1. *T and $\widetilde{T}$ are two triangular sets in P_n. Moreover, the set $\breve{\mathcal{G}} = T \cup \widetilde{T}$ can be reordered as a triangular set,*
2. *the system $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ is a weak constructible triangular system in P_n.*

Proof. If T and $\widetilde{T}$ are triangular sets then $\{g_j = 0\}_{j\in E}$ and $\{g_j \neq 0\}_{j\in F}$ are two weak constructible triangular systems in P_n. Moreover, if $\breve{\mathcal{G}}$ can be reordered as a triangular set, it clearly means that the sets E, F are disjoint and the result is then obvious. On the opposite, let $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. By definition, for all $j \in E \sqcup F$, the index of g_j is j. It implies, since E and F are disjoint, that $\breve{\mathcal{G}} = \{g_j\}_{j\in E\sqcup F}, T = \{g_j\}_{j\in E}$ and $\widetilde{T} = \{g_j\}_{j\in F}$ are triangular sets.$\diamond$

This lemma allows us to rephrase the definition of simple systems in terms of weak constructible triangular systems. For this purpose, we first make a remark.

Remark. Given a simple system $[T, \widetilde{T}]$ in P_n, the set $Zero(T/\widetilde{T})$ is simply the set of zeros (definition 4.2) of the weak constructible triangular system $[T, \widetilde{T}]$ (with D.M. Wang terminology):

$$Zero(T/\widetilde{T}) = Z_n.$$

Proposition 5.1 *Let $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. Then the following conditions are equivalent:*

1. *$[[g_j]_{j\in E}, [g_j]_{j\in F}]$ is a simple system,*
2. *for all $j \in E$ and for all $a \in Z_{j-1}$:*

$$lc(g_j)(a) \neq 0,\ Disc_{X_j}(g_j)(a) \neq 0.$$

Proof. It is an immediate consequence of lemma 5.1 and remark 5.$\diamond$

The following theorem shows that simple systems are well-known in our framework. Indeed, the second statement in previous proposition is exactly the one which appears in proposition 4.6. Next theorem is then an obvious corollary of propositions 4.6 and 5.1.

Theorem 5.1 *Let $\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ be a weak constructible triangular system in P_n. Then the following conditions are equivalent:*

1. *$[[g_j]_{j\in E}, [g_j]_{j\in F}]$ is a simple system in P_n,*
2. *$\{g_j\, \xi_j\, 0\}_{j\in E\sqcup F}$ is a squarefree L-normalized constructible triangular system in P_n.*

Thus, we have obtained a characterization of simple systems within our terminology. We can now relate them to algebraic models of T. Gómez-Díaz systems. It is the purpose of the following theorem which states the main result of this paper.

Theorem 5.2 *Every algebraic model of T. Gómez-Díaz systems, i.e. every squarefree normalized constructible triangular system, is a simple system.*

Proof. As simple systems and squarefree L-normalized constructible triangular systems are equivalent, this result follows immediately from proposition 4.1.$\diamond$

Remark. Note that the converse is false: it suffices to combine remark 4 with theorem 5.1.

Example. The reader must be careful with theorem 5.2. *It does not state that a triangular system obtained by a calculus with the dynamic contuctible closure programs forms a simple system, but its model does.* Consider for example the following T. Gómez-Díaz system with $K_0 = \mathbb{Q}$ and parameters x, y, z (see [7, chapter 4] for detailed definitions):

```
 [ 1 --- y 2 (x) 1 z + --------- = 0 and y /=
0 , y /= --- and x /= 1 , x /= 0 1 (x) (y - ---) (x) ]
```

There is no need to ask whether it is or not a simple system : they are not polynomials of $\mathbb{Q}[\curvearrowleft, \curvearrowright, \digamma]$! However, one can easily check that an algebraic model of this system is :

$$T = \begin{cases} (X_1 X_2 - 1) X_3^2 + X_2 = 0 \\ X_1 X_2^2 - X_2 \neq 0 \\ X_1^2 - X_1 \neq 0 \end{cases}$$

It is the system studied in example 3. We have shown that T is a squarefree normalized constructible triangular system in $\mathbb{Q}[\mathbb{X}_{\nVdash}, \mathbb{X}_{\nvDash}, \mathbb{X}_{\nVDash}]$ in examples 3 and 3. Thus T is also L-normalized by proposition 4.1. Therefore we conclude using theorem 5.1 that:

$$[[(X_1 X_2 - 1) X_3^2 + X_2],\ [X_1^2 - X_1,\ X_1 X_2^2 - X_2]]$$

is a simple system of $\mathbb{Q}[\mathbb{X}_{\nVdash}, \mathbb{X}_{\nvDash}, \mathbb{X}_{\nVDash}]$.

Furthermore theorem 5.2 shows that all our results related to squarefree L-normalized constructible triangular systems can be rephrasing in terms of simple systems. In particular, we can establish easily next proposition which gather together two results of [26, 27]. The first point refers to last part of [26, theorem 1 p.310], the second is exactly [26, theorem 4 p.312].

Proposition 5.2 *Let $[T, \widetilde{T}]$ be a simple system in P_n. Then:*

1. *the set $Zero(T/\widetilde{T})$ is non empty,*
2. *for all $f \in P_n$, the following conditions are equivalent:*
   - *(a) $\forall\, a \in Zero(T/\widetilde{T}), f(a) = 0$,*
   - *(b) $prem(f, T) = 0$.*

Proof. We know that $[T, \widetilde{T}]$ forms in particular a L-normalized constructible triangular system in P_n. Thus, the first point is simply proposition 4.4. Moreover $[T, \widetilde{T}]$ is also squarefree. Then, since $ker\,\Psi_n$ is radical by lemma 4.1, it suffices to apply theorems 4.2 and 4.4 to conclude.$\diamond$

Let $[P, Q]$ be a constructible system of P_n (with D.M. Wang terminology) . In [26], D.M. Wang presents a method that computes finitely many simple systems $[T_i, \widetilde{T_i}]$ $(1 \leq i \leq e)$ of P_n such that:

$$Zero(P/Q) = \bigcup_{i=1}^{e} Zero(T_i/\widetilde{T}_i).$$

In [26, theorem 6 p.312], for all $1 \leq i \leq e$, D.M. Wang calculates the Gröbner basis of the following ideal $\mathcal{I}_i$ of P_n:

$$\mathcal{I}_i = \langle T_i, Y \prod_{f \in T_i} lc(f) - 1 \rangle \cap P_n.$$

Then he shows that the condition $Q = \emptyset$ implies:

$$V(P) = \bigcup_{i=1}^{e} V(\mathcal{I}_i).$$

Moreover, he says that for all $1 \leq i \leq e$, the affine variety $V(\mathcal{I}_i)$ is $(n - |T_i|)$-equidimensional. Next theorem establishes an analogous result in the general case (i.e. without the hypothesis $Q = \emptyset$) and shows in particular that the Zariski closure of $Zero(T_i/\widetilde{T}_i)$ is $(n-|T_i|)$-equidimensional $(1 \leq i \leq e)$.

Theorem 5.3 *Let $[P, Q]$ be a constructible system in P_n (with D.M Wang terminology) and:*

$$Zero(P/Q) = \bigcup_{i=1}^{e} Zero(T_i/\widetilde{T}_i)$$

a zero decomposition obtained by his algorithm SimSys [26, section 3 p.302]. If for all $1 \leq i \leq e$, we denote by $\mathcal{I}_i$ the following ideal of P_n:

$$\mathcal{I}_i = \langle T_i, Y \prod_{f \in T_i} lc(f) - 1 \rangle \cap P_n$$

then:

$$\overline{Zero(P/Q)} = \bigcup_{i=1}^{e} V(\mathcal{I}_i).$$

Furthermore for all $1 \leq i \leq e$, the affine variety $V(\mathcal{I}_i)$ is equal to $\overline{Zero(T_i/\widetilde{T}_i)}$ and is $(n-|T_i|)$-equidimensional.

Proof. For all $1 \leq i \leq e$, we denote respectively by $\mathcal{J}_{i,n}, h_{i,n}$ and $\Psi_{i,n}$ the ideal of P_n generated by the polynomials of T_i, the product $\prod_{f \in T_i} lc(f)$ and the homomorphism associated with the weak constructible triangular system $[T_i, \widetilde{T}_i]$ in P_n. Applying [10, theorem 1.5.4 p.19], the Zariski closure of $Zero(P/Q)$ is equal by hypothesis to:

$$\overline{Zero(P/Q)} = \bigcup_{i=1}^{e} \overline{Zero(T_i/\widetilde{T}_i)}.$$

Fix a positive integer i $(1 \leq i \leq e)$. The simple system $[T_i, \widetilde{T}_i]$ is a L-normalized constructible triangular system in P_n. Then, by proposition 4.5, we have:

$$\overline{Zero(T_i/\widetilde{T}_i)} = V(ker\, \Psi_{i,n}).$$

Moreover, applying [5, exercise 9 p.196], the ideal $\mathcal{I}_i$ is equal to:

$$\mathcal{I}_i = \mathcal{J}_{i,n} : h_{i,n}^{\infty}.$$

Therefore, by theorem 4.2, we obtain:

$$\mathcal{I}_i = ker\, \Psi_{i,n}$$

and so the equality:

$$\overline{Zero(T_i/\widetilde{T}_i)} = V(\mathcal{I}_i).$$

The result follows then from corollary 4.2.$\diamond$

Furthermore, D.M. Wang notes that simple systems without inequation (i.e. simple systems $[T/\widetilde{T}]$ with $\widetilde{T} = \emptyset$) form polynomial triangular systems and refers the reader to the works of D. Lazard [20] and M. Kalkbrener [19]. The following result justifies and extends this observation to general simple systems.

Theorem 5.4 *Let $[T, \widetilde{T}]$ be a simple system in P_n. Then the set T is a regular chain in P_n.*

Proof. By theorem 5.1, the simple system $[T, \widetilde{T}]$ is in particular a L-normalized constructible triangular system in P_n. The result is then a direct consequence of proposition 4.2.$\diamond$

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