—— SOLUTIONS ——

King's College London

UNIVERSITY OF LONDON

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MSc / MSci EXAMINATION

7CCSMTSP/CSMTSP – TEXT SEARCHING AND PROCESSING

MAY 2009

TIME ALLOWED: TWO HOURS.

ANSWER THREE OF THE FIVE QUESTIONS.

NO CREDIT WILL BE GIVEN FOR ATTEMPTING ANY FURTHER QUESTIONS.

ALL QUESTIONS CARRY EQUAL MARKS.

THE USE OF ELECTRONIC CALCULATORS IS **NOT** PERMITTED.

BOOKS, NOTES OR OTHER WRITTEN MATERIAL MAY **NOT** BE BROUGHT INTO THIS EXAMINATION.

NOT TO BE REMOVED FROM THE EXAMINATION HALL

TURN OVER WHEN INSTRUCTED

1. Borders and overlaps

Given a word $x = x[0\,..\,m-1]$, its *Border* table is defined by: $Border[0] = -1$, and $Border[j]$ is the maximal length of (proper) borders of $x[0\,..\,j-1]$, for $0 < j \leq m$.

a. Give the *Border* table associated with the word aaabaaababa.

[10 marks]

Answer

i	0	1	2	3	4	5	6	7	8	9	10	11
$x[i]$	a	a	a	b	a	a	a	b	a	b	a	
$Border[i]$	-1	0	1	2	0	1	2	3	4	5	0	1

[unseen]

b. Design, and describe using pseudo-code, an algorithm that computes the *Border* table of a word x of length m in time $O(m)$.

[10 marks]

Answer

```
COMPUTE_BORDERS(string $x$; integer $m$)
1  $Border[0] \leftarrow -1$
2  for $i \leftarrow 0$ to $m - 1$
3  do $j \leftarrow Border[i]$
4      while $j \geq 0$ and $x[i] \neq x[j]$
5      do $j \leftarrow Border[j]$
6      $Border[i+1] \leftarrow j + 1$
7  return $Border$
```

[bookwork]

[10 marks]

c. Give a tight upper bound on the number of symbol comparisons executed during a run of the algorithm of Question 1.b, and prove the bound.

[10 marks]

Answer

The number of symbol comparisons is bounded by $2m$. [5 marks]
The running time is proportional to the number of symbol comparisons done at Line 4. Each positive comparison leads to an increment of variable i which values are in increasing order. Each negative comparison leads to an increment of expression $i - j$ which values are in increasing order. So, there are at most $2m$ comparisons, which proves the result. [5 marks]
[bookwork]

d. Let x and y be two strings on the alphabet $\{\mathsf{a}, \mathsf{b}\}$. Show how to find the positions of the occurrences of x in y using the *Border* table associated with the string $x\mathsf{c}y$ (c is a letter different from a and b).

[10 marks]

Answer

Note that $\mathit{Border}[i] \leq |x|$ because the letter c does not appear in y. If a positions i on $x\mathsf{c}y$ is such that $\mathit{Border}[i] = |x|$ then x is a suffix of $x\mathsf{c}y[0 \,..\, i-1]$ and then of $y[0 \,..\, i-2|x|-1]$. So, x occurs in y at position $i - |x|$. [5 marks]
Conversely, if x occurs in y at position j, then x is a border of $x\mathsf{c}y[0 \,..\, i-1]$ for $i = j + |x|$ because the border cannot be longer than x. Thus $\mathit{Border}[i] = |x|$. [5 marks]
Conclusion: the condition $\mathit{Border}[i] = |x|$ can be used to signal occurrences of x in y. [unseen]

e. The overlap between x and y, $ov(x, y)$, is the longest word that is both a prefix of x and a suffix of y. How would you find $ov(x, y)$ using the table *Border* associated with the string $x\mathsf{c}y$? How would you do it using the table *Border* associated with the string x?

[10 marks]

Answer

Let $k = \mathit{Border}[|x| + |y| + 1]$, then $ov(x, y) = x[0 \,..\, k-1]$. [5 marks]
With the table *Border* associated with the string x, apply MP algorithm until $j = |y|$; then $ov(x, y) = x[0 \,..\, i-1]$. [5 marks] [unseen]

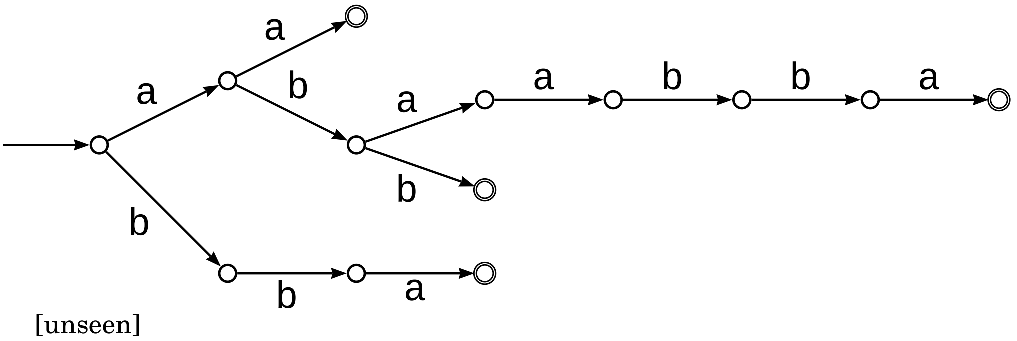
2. Dictionary-Matching Automaton

Let Σ be the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and X be a finite set of strings of Σ^*. The dictionary-matching automaton of X over Σ is denoted by $\mathcal{D}(X)$.

a. Draw the trie of the set $\{\mathtt{aa}, \mathtt{abaabba}, \mathtt{abb}, \mathtt{bba}\}$. Mark its terminal states.

[5 marks]

Answer



[unseen]

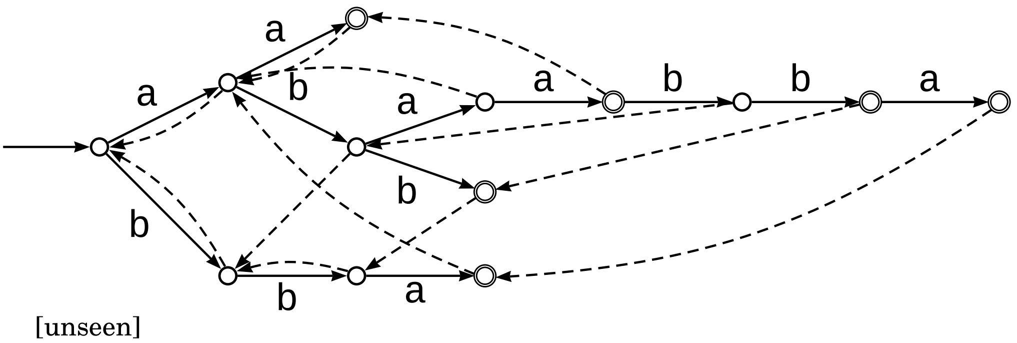
b. Define the notion of a failure link on a state (node) of the trie of X. Draw the implementation with failure links of the dictionary-matching automaton $\mathcal{D}(\{\mathtt{aa}, \mathtt{abaabba}, \mathtt{abb}, \mathtt{bba}\})$. Mark its terminal states.

[10 marks]

Answer

Let p be a state of the trie of X distinct from the root. Let $u \in \Sigma^+$, be the label of the path from the root to state p. Then the failure state $f(p)$ of state p is the state of the trie whose path from the root is labelled by the longest possible proper suffix of u.

[bookwork]



[unseen]

c. Describe in pseudo-code the next-state function for the implementation with failure links of $\mathcal{D}(X)$.

[10 marks]

Answer

```
NextState($p, a$)
    if there is an edge $(p, a, q)$ in the trie
        return $q$
    else if $f(p)$ is defined
        return NextState($f(p), a$)
    else return the root of the trie
```

[bookwork]

d. What data structure would you use to implement a state of the dictionary-matching automaton?

[10 marks]

Answer

For each node in the automaton, one can use a structure comprising two pointers, one for the failure link and one for the list of next nodes defined by the transition function, a boolean field to mark terminal states, and possibly a field for storing some data associated with the state (for example, the letter labelling the incoming edge).

[unseen]

e. Describe in your own words or in pseudo-code how the failure links of the implementation of a dictionary-matching automaton can be computed. What are the terminal states of the automaton?

[15 marks]

Answer

The computation is done top-down in a width-first traversal of the trie. The root has no failure link. The failure target of the children of the root is the root.

Assume that we want to compute the failure link of node q (the failure links of states at a lower level have already been computed). Also assume that the parent of q is the node p and that the arc connecting the nodes p and q is labelled by the letter a. Then, the failure link $f(q)$ of the node q is given by $f(q) = \text{NextState}(f(p), a)$ if $f(p)$ is defined (where NextState is the function described in 2.c), otherwise $f(q)$ is the root.

The set of terminal states comprises the terminal states of the trie and states q for which $f(q)$ is a terminal state.

[bookwork]

3. BM-Horspool

Let x be a string of length m, $x = x[0 \,.\,.\, m-1]$.

a. The *DA* table of a string implements the bad-character rule for the BM algorithm. How do you define the *DA* table for the string x? What is the length of the shift inferred from *DA* when the rule applies after comparing the text symbol $y[j]$ and the pattern symbol $x[i]$?

[10 marks]

Answer

$DA[\sigma] = \min\{|z| > 0 \mid \sigma z \text{ suffix of } x\} \cup \{|x|\}$,
$shift = DA[y[j]] - m + i + 1$.

[bookwork]

b. On the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$, give the *DA* table associated with the word $x = \mathsf{acbabaaba}$

[5 marks]

Answer

a	a	b	c	d
$DA[a]$	2	1	7	9

[unseen]

c. Describe in pseudo-code the computation of the *DA* table for the word x and the alphabet A.

[15 marks]

Answer

```
COMPUTE_DA(string $x$; integer $m$)
    for all $a$ in $A$ do
        $DA[a] = m$
    for $i \longleftarrow 0$ to $m-2$ do
        $DA[x[i]] = m - i - 1$
    return $DA$
```

[bookwork]

d. Describe in pseudo-code a string-matching algorithm, searching for x in y, using the DA table of x.

[10 marks]

Answer

```
BMH(string $x, y$; integer $m, n$);
    $pos \longleftarrow 0$
    while $pos \leq n - m$ do
        $i \longleftarrow m - 1$
        while $i \geq 0$ and $x[i] = y[pos + i]$ do
            $i \longleftarrow i - 1$
        if $i = -1$ then
            output('$x$ occurs in $y$ at position ', $pos$)
        $pos \longleftarrow pos + \max\{1, DA[y[pos + i]] - m + i + 1\}$
```

[unseen]

e. What is the minimum and the maximum number of symbol comparisons for the algorithm of Question 3.d applied to a text of length n and pattern of length m?

[5 marks]

Consider the pattern x = acbabaaba of Question 3.a. Give a text y on the alphabet $\{a, b, c\}$ for which the number of symbol comparisons equals the minimum number of symbol comparisons you have just given. Give a text y on the alphabet $\{a, b, c\}$ for which the number of symbol comparisons equals the maximum number of symbol comparisons you have just given.

[5 marks]

Answer

Minimum is n/m, maximum is $m(n - m + 1)$.
Text cccccccccc.. gives the minimum (around $n/7$) and text $(\text{bcbabaaba})^k$ gives the maximum.

[unseen]

4. Suffix Array and Suffix Tree

a. Define the data structure called the Suffix Array of a string y of length n.

[10 marks]

Answer

It is composed of two tables p and LCP such that
$y[\mathrm{p}[0]\,..\,n-1] < y[\mathrm{p}[1]\,..\,n-1] < \ldots < y[\mathrm{p}[n-1]\,..\,n-1]$
and $\mathrm{LCP}[i] = |\mathbf{lcp}(y[\mathrm{p}[i-1]\,..\,n-1], y[\mathrm{p}[i]\,..\,n-1])|$.
[bookwork]

b. Give the Suffix Array of the string $y =$ abaabbabb.

[10 marks]

Answer

i	0	1	2	3	4	5	6	7	8
$y[i]$	a	b	a	a	b	b	a	b	b
$\mathrm{p}[i]$	2	0	6	3	8	1	5	7	4
$\mathrm{LCP}[i]$	0	1	2	3	0	1	2	1	2

[unseen]

c. How do you compute the maximal length of prefixes common to the suffixes of y at positions i and j ($i < j$) using its Suffix Array? What is the running time of your solution?

[10 marks]

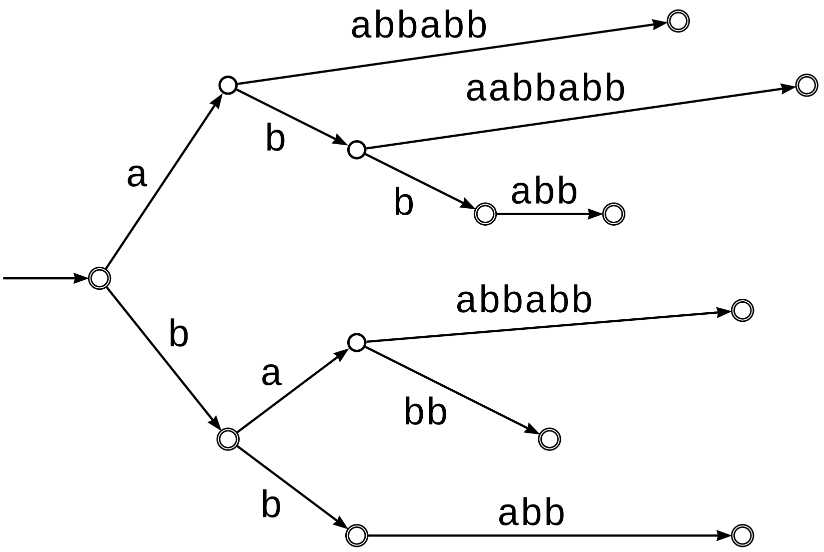
Answer

The value is $\min\{\mathrm{LCP}[k] \mid i < k \leq j\}$. It can be computed by traversing the sub-array $\mathrm{LCP}[i+1\,..\,j]$ in time $O(j-i)$. If the pair (i, j) is a pair of the binary search tree the running time is $O(\log(j-i))$.
[unseen]

d. Design the Suffix Tree of the string $y =$ abaabbabb without the suffix links but with the final states.

[10 marks]

Answer



[unseen]

e. Describe in your own words how to compute the LCP (longest common prefix) table associated with the Suffix Array of y using its suffix tree.

[10 marks]

Answer

The depth of the deepest internal node between the consecutive suffixes is their LCP. Its computation is done during a traversal of the tree in which letters are processed in lexicographic order.

[unseen]

5. Word transformation

Let $x = x[0\,.\,.\,m-1]$ be a word of length m. For an integer i, $0 \leq i < m$, the i-rotation of x is the word $x[i\,.\,.\,m-1]x[0\,.\,.\,i-1]$. We assume in this question that the m rotations of x are pairwise distinct and that x is the smallest of them according to the lexicographic order.

The BW matrix of x, denoted by $BW(x)$, is the $m \times m$ matrix whose lines are the rotations of x in lexicographic order.

The BW transform of x, denoted by $L(x)$, is the last column of the BW matrix. (It is a word of length m.)

a. Give the BW matrix of the word $x = \texttt{aabbab}$. Give $L(\texttt{aabbab})$.

[5 marks]

Answer

$$BW(\texttt{aabbab}) = \begin{pmatrix} \texttt{a} & \texttt{a} & \texttt{b} & \texttt{b} & \texttt{a} & \texttt{b} \\ \texttt{a} & \texttt{b} & \texttt{a} & \texttt{a} & \texttt{b} & \texttt{b} \\ \texttt{a} & \texttt{b} & \texttt{b} & \texttt{a} & \texttt{b} & \texttt{a} \\ \texttt{b} & \texttt{a} & \texttt{a} & \texttt{b} & \texttt{b} & \texttt{a} \\ \texttt{b} & \texttt{a} & \texttt{b} & \texttt{a} & \texttt{a} & \texttt{b} \\ \texttt{b} & \texttt{b} & \texttt{a} & \texttt{b} & \texttt{a} & \texttt{a} \end{pmatrix}$$

$L(\texttt{aabbab}) = \texttt{bbaaba}$ [unseen]

b. How would you compute the BW matrix of x considering the Suffix Array of the string xx?
What would be the running time of the algorithm both if the alphabet is bounded and if it is unbounded?

[15 marks]

Answer

Rotations of x are segments of length m of the word $x' = x[0]x[1]\cdots x[m-1]x[0]x[1]\cdots x[m-1]$. Sorting the suffixes of this word gives the answer. [5 marks]
On an unbounded alphabet, suffixes can be sorted either by using the suffix tree or the suffix automaton of x', which is done in $O(m \times \log a)$ time, where a is the size of the alphabet of x. [5 marks]
On a bounded alphabet, suffixes can be sorted with the suffix array of x' which requires $O(m)$ time. [5 marks] [unseen]

c. Let a be a letter and u, v be two different strings of the same length. Prove that $au < av$ if and only $ua < va$.

[10 marks]

Answer

Let w be the longest common prefix of u and v. Since $u \neq v$, w is a proper prefix of u and of v. Then, $u = wbu'$ and $v = wcv'$ for some letters b, c and some words u', v'. The condition $au < av$ is equivalent to $b < c$, because aw is the longest prefix of au and av, which is equivalent to $u < v$ and to $ua < va$, because none of u and v is a prefix of the other. [unseen]

d. Let $F(x)$ be the first column of $BW(x)$. Let a be a letter occurring at two positions i and j on x: $a = x[i] = x[j]$. Show that the two occurrences a appear in the same relative order in $F(x)$ and in $L(x)$. [Hint: use Question 5.c.]

[10 marks]

Answer

The occurrences of a in $F(x)$ are associated with two rotations $au = x[i]u$ and $av = x[j]v$ of x. If $x[i]u < x[j]v$, by Question 5.c, we have $ux[i] < vx[j]$. If $x[i]u > x[j]v$, by Question 5.c, we have $ux[i] > vx[j]$. Therefore occurrences of a appear in the same relative order in $F(x)$ and in $L(x)$. [unseen]

e. Note that the letters in $F(x)$ are in non-decreasing order and that, if $L(x)[k] = x[i]$, $0 \leq i < m - 2$, then $x[i + 1] = F(x)[k]$. Describe how to compute x from $L(x)$.

[10 marks]

Answer

First compute $F(x)$ by sorting the letter of $L(x)$.
The first letter of x is $L(x)[0]$.
Assume that the letter $x[i]$ has been computed. The next letter $x[i + 1]$ can be computed as follows: let $L(x)[k]$ be the letter having the same rank among occurrences of $x[i]$'s in $L(x)$ as its rank among occurrences of $x[i]$'s in $F(x)$; then $x[i + 1] = F(x)[k]$. [unseen]