Free Quasi-Symmetric Functions, Product Actions and Quantum Field Theory of Partitions

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Introduction

In a relatively recent paper, Bender, Brody and Meister introduce a special Field Theory described by

$$G(z) = \left(e^{\left(\sum_{n\geq 1} L_n \frac{z^n}{n!}\frac{\partial}{\partial x}\right)}\right)\left(e^{\left(\sum_{m\geq 1} V_m \frac{x^m}{m!}\right)}\right)\Big|_{x=0}$$

in order to prove that any sequence of numbers $\{a_n\}$ can be generated by a suitable set of rules applied to some type of Feynman diagrams . These diagrams actually are bicoloured multigraphs with no isolated vertex.

Actions of a direct product of permutation groups

Direct product actions

Two pairs (G_1, X_1) and (G_2, X_2), each G_i is a permutation group acting on X_i.

Intransitive action of $G_1 \times G_2$ on $X_1 \sqcup X_2$:

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases}.$$

$(G_1, X_1) \nrightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.

Cartesian action of $G_1 \times G_2$ on $X_1 \times X_2$:

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2).$$

$(G_1, X_1) \boxtimes (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$.

Explicit realization

Denote

- by $\circ_N$ the natural action of $\mathfrak{S}_n$ on $\{0, \ldots, n-1\}$,
- by $\circ_I$ the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \cdots, n+m-1\}$
- by $\circ_C$ the cartesian action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \ldots, nm-1\}$.

More precisely,

$$(\sigma_1, \sigma_2) \circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \leq i \leq n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \leq i \leq n+m-1 \end{cases}.$$

and

$$(\sigma_1, \sigma_2) \circ_C (j + nk) = (\sigma_1 \circ_N j) + n(\sigma_2 \circ_N k)$$

for $0 \leq i \leq n+m-1$, $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$.

Let the map $\nrightarrow$: $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ defined by

$$\sigma_1 \nrightarrow \sigma_2 = \sigma_1 \sigma_2[n]$$

$\sigma_1 = 1320 \in \mathfrak{S}_4$, $\sigma_2 = 534120 \in \mathfrak{S}_6$.

$\sigma_1 \nrightarrow \sigma_2 = 1320978564$, $\sigma_2 \nrightarrow \sigma_1 = 5341207986$

Proposition
$(\sigma_1 \nrightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i$.

Let the map $\boxtimes$: $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ defined by

$$\sigma_1 \boxtimes \sigma_2 = \prod_{i,j} c_i \boxtimes c'_j$$

where $\sigma_1 = c_1 \cdots c_k$ and $\sigma_2 = c'_1 \cdots c'_{k'}$ are the decompositions of σ_1 and σ_2 in a product of cycles and

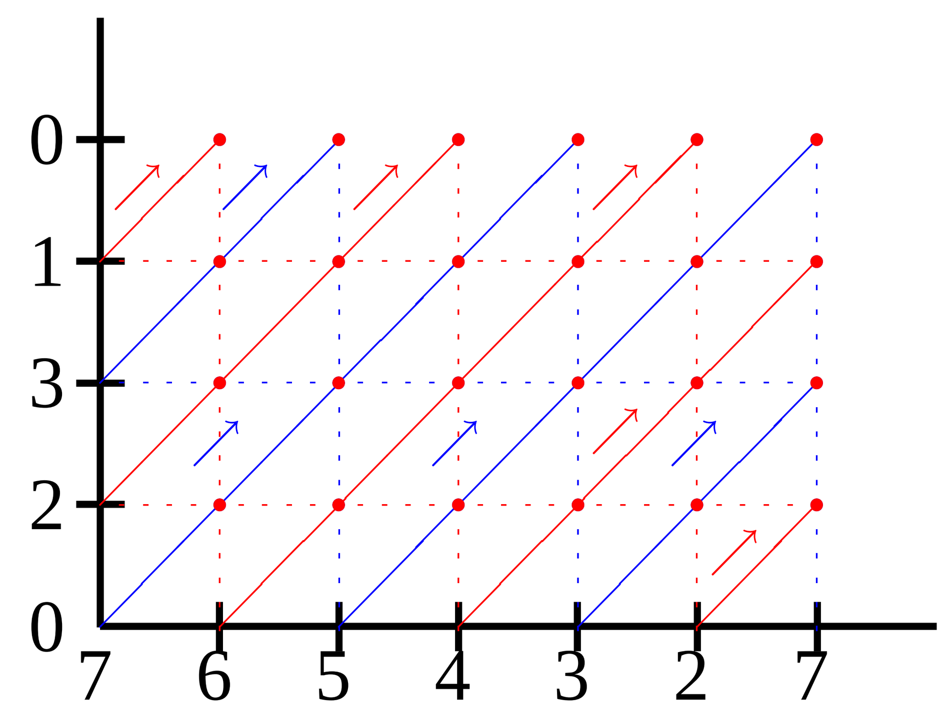
$$c \boxtimes c' = \prod_{s=0}^{l\wedge l'-1} (\phi(s,0), \phi(s+1,1) \cdots, \phi(s+l\vee l'-1, l\vee l'-1)),$$

($\wedge :=$ gcd, $\vee :=$ lcm, $c = (i_0, \cdots, i_{l-1})$, $c' = (j_0, \cdots, j_{l'-1})$ are two cycles and $\phi(k, k') = i_{k \bmod l} + nj_{k' \bmod l'}$.)

The cartesian action is compatible with the natural action.

Proposition
$(\sigma_1 \boxtimes \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i$.

$c_1 = (0, 2, 3, 1)$, $c_2 = (7, 6, 5, 4, 3, 2)$.
$c_1 \boxtimes c_2$:



$(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)$
$(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$.

Algebraic structure

Proposition *Associativity*
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$
2. $\sigma_1 \boxtimes (\sigma_2 \boxtimes \sigma_3) = (\sigma_1 \boxtimes \sigma_2) \boxtimes \sigma_3$

Proposition *Semi-distributivity*
$\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$

$$\sigma_1 \boxtimes (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \boxtimes \sigma_2) \nrightarrow (\sigma_1 \boxtimes \sigma_3)$$

Cycle index algebra

Cartesian product in Sym

Let $c_j(\sigma)$ be the number of cycles of σ of length j

$$\mathfrak{z}(\sigma) = \prod_{j=0}^{\infty} \psi_j^{c_j(\sigma)} \qquad (1)$$

where ψ_i is a power sum symmetric function.
One defines

$$\prod_{1\le i\le\infty} \psi_i^{\alpha_i} \star \prod_{1\le j\le\infty} \psi_j^{\beta_j} = \prod_{1\le i,j\le\infty} \psi_{i\vee j}^{\alpha_i\beta_j(i\wedge j)}$$

Proposition i) $\mathfrak{z} : \oplus_{n\ge0}\mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ is a morphism sending the two laws $+\!\!\!\!\rightarrow$ $\times\!\!\!\!\searrow$ respectively to $\times$; $\star$.
More precisely, for $\sigma,\ \tau \in \sqcup_{n\ge0}\mathfrak{S}_n$ one has

$$\mathfrak{z}(\sigma +\!\!\!\!\rightarrow \tau) = \mathfrak{z}(\sigma)\mathfrak{z}(\tau)\ ;\ \mathfrak{z}(\sigma \times\!\!\!\!\searrow \tau) = \mathfrak{z}(\sigma)\star\mathfrak{z}(\tau) \qquad (2)$$

ii) $\star$ is commutative and distributive over $\times$.

Proposition $P := \{\prod_{i=1}^{\infty}\psi_i^{\alpha_i}\}_{(\alpha_i)_{i\ge1}\in\mathbb{N}^{(\mathbb{N}^*)}}$ is closed by $\times$ and $\star$.
More precisely: $(P,\times,\star)$ is isomorphic to a subsemiring of the semiring $\mathbb{N}[(\mathbb{N}^{(\mathfrak{p})}, sup)]$ ($\mathfrak{p}$:= the set of prime numbers).

Cycle index polynomial

Polyà cycle index polynomial :

$$Z(G) = \mathfrak{z}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right).$$

1. Symmetric group $\mathfrak{S}_n$: $Z(\mathfrak{S}_n) = h_n$.
2. Alternating group A_n : $Z(A_n) = h_n + e_n$.

As $\mathfrak{z}$ is a morphism of 2-associative algebra, one recovers the classical relations

$$Z(G_1 +\!\!\!\!\rightarrow G_2) = Z(G_1)Z(G_2),\ Z(G_1 \times\!\!\!\!\searrow G_2) = Z(G_1)\star Z(G_2)$$

1. Intransitive product of $\mathfrak{S}_n$ and $\mathfrak{S}_m$:

$$Z(\mathfrak{S}_n +\!\!\!\!\rightarrow \mathfrak{S}_m) = h_n h_m.$$

2. Cartesian product of $\mathfrak{S}_n$ and $\mathfrak{S}_m$:

$$Z(\mathfrak{S}_n \times\!\!\!\!\searrow \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} \frac{1}{z_\lambda z_\rho}\prod_{i,j}\psi_{\lambda_i\vee\rho_j}^{\lambda_i\wedge\rho_j},$$

where $z_\lambda = \prod i^{n_i} n_i!$ if n_i is the number of parts of λ equal to i.

Enumeration of a type of Feynman diagrams related to quantum field theory of partitions

The *type* $t(f)$ of $f : X \to L = \{l_0 < l_1 \cdots < l_k \cdots\}$ is the vector $(i_0,\dots,i_p,\dots)$ where i_k is the number of elements of X whose image by f is l_k.
The *shape* $s(f)$ of f is the partition obtained by sorting in the decreasing order $t(f)$ and erasing the zeroes.
The number $d_\lambda^s(G,L)$ of G-classes on L^X with the shape λ is the coefficient of m_λ in the expansion of $Z(G)$ in the basis of monomial symmetric functions:

$$Z(G) = \sum_\lambda d_\lambda^s(G,L)m_\lambda.$$

Let us define the generating series of the type of our Feynman diagrams

$$F(n,m) := \sum_{I=(i_0,\dots,i_p,\dots)} f_I^t(n,m)\prod_{k=0}^{\infty} y_k^{i_k},$$

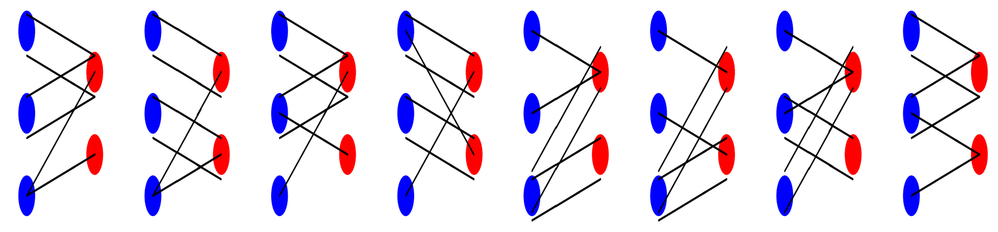
where $f_I^t(n,m)$ denotes the number of Feynman diagrams of type I.

Theorem One has the following decomposition of the cycle index polynomial.

$$Z(\mathfrak{S}_n \times\!\!\!\!\searrow \mathfrak{S}_m) = \sum_{(1,1)\le_{lex}(k,p)\le_{lex}(n,m)} F(k,p)y_0^{nm+kp-np-mk} + y_0^{nm}.$$

$$\begin{aligned}F(2,3) &= Z(\mathfrak{S}_3 \times\!\!\!\!\searrow \mathfrak{S}_2) - F(1,3)y_0^3 - F(2,2)y_0^2\\ &\quad -F(2,1)y_0^4 - F(1,2)y_0^4 - F(1,1)y_0^5 - y_0^6\\ &= y_2^6 + y_2^5y_1 + 3y_2^4y_1 + 3y_2^4y_1y_0 + 2y_2^4y_0^2\\ &\quad +3y_2^3y_1^3 + 6y_2^3y_1^2y_0 + 5y_2^3y_1y_0^2 + y_2^3y_0^3 + 3y_2^2y_1^4\\ &\quad +3y_2^2y_1^3y_0 + 8y_2^2y_1^2y_0^2 + 3y_2^2y_1y_0^3 + y_2y_1^5 + 3y_2y_1^4y_0\\ &\quad +5y_2y_1^3y_0^2 + 3y_2y_1^2y_0^3 + y_1^6 + y_1^5y_0 + y_1^3y_0^3 + 2y_1^2y_0^4.\end{aligned}$$

For example, there is 8 $(2,2,2)$- Feynman diagrams:



Non commutative realizations

Free quasi-symmetric cycle index algebra

Recall that the algebra **FQSym** is defined by one of its bases, indexed to $\mathfrak{S}$ and defined as follows

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A\rangle\rangle$$

One has

$$F^\sigma F^\tau = F^{\sigma +\!\!\!\!\rightarrow \tau}.$$

This induces naturally a morphism of algebra

$$\begin{aligned}\underline{\mathfrak{z}} : \left(\bigoplus\mathbb{Q}[\mathfrak{S}_n], +\!\!\!\!\rightarrow, +\right) &\to (\mathbf{FQSym}, ., +)\\ \sigma &\to \mathbf{F}^\sigma.\end{aligned}$$

One defines the product $\star$ on **FQSym** by $F^\sigma \star F^\tau := F^{\sigma\times\!\!\!\!\searrow\tau}$. By this way, $\underline{\mathfrak{z}}$ becomes a morphism of 2-associative algebra. Furthermore, $\star$ is associative, distributive over the sum and semi-distributive over the shifted concatenation .

Free quasi-symmetric Polyà cycle index polynomial

$$\underline{Z}(G) := \underline{\mathfrak{z}}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right).$$

Proposition
Let $G_1, G_2 \in \mathfrak{S}_{sg}$ be two permutation groups, one has

1. $\underline{Z}(G_1 +\!\!\!\!\rightarrow G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
2. $\underline{Z}(G_1 \times\!\!\!\!\searrow G_2) = \underline{Z}(G_1)\star\underline{Z}(G_2)$.

$$\begin{array}{ccc}\mathfrak{S}_{sg} & \xrightarrow{\underline{Z}} & \mathbf{FQSym}\\ Z\downarrow & {}^{z}\swarrow & \uparrow\underline{\mathfrak{z}}\\ Sym & \xleftarrow[\mathfrak{z}]{} & \bigoplus\mathbb{Q}[\mathfrak{S}_n]\end{array}$$

1. Free quasi-symmetric cycle index of $\mathfrak{S}_n$:

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} F^\sigma.$$

   Analogous of $z(\mathbf{H_n}) = Z(\mathfrak{S}_n) = h_n$.

2. Free cycle index polynomial of the alternative groups:

$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n).$$

   analogous of elementary symmetric functions :

$$z(\mathbf{E_n}) = Z(A_n) - Z(\mathfrak{S}_n) = e_n.$$