Algebraic invariants of 5-qubits

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Introduction

Quantifying entanglement in multipartite systems is a fundamental issue in Quantum Information Theory. However, for systems with more than two parts, very little is know in this respect. For system of up to 4 qubits, a complete classification of entanglement patterns and of corresponding invariants under local filtering operations (also known as SLOCC, Stochastic Local Operations assisted by Classical Communication) is know [16, 9]. Klyachko [5, 6] proposed to associate entanglement (of pure states) in a k-partite system with the mathematical notion of semi-stability, borrowed from geometric invariant theory, which means that at least one SLOCC invariant is non zero. However, even for system of k qubits, the complexity of these invariants grows very rapidly with the number of parts. For $k = 2$, they are given by simple linear algebra [15, 4]. The case $k = 3$ is already nontrivial but appears in the physics literature in [18] and boils down to a mathematical result which was known by 1880 [7]. The case $k = 4$ is quite recent [9], and to the best of our knowledge, few is known for 5-qubit systems [14].

Our main result is a closed expression of the Hilbert series of the algebra of SLOCC invariants of pure 5-qubit states. This result, which determines the number of linearly independent homogeneous invariants in any degree, was obtained through intensive symbolic computations relying on a very recent algorithm for multivariate residue calculations.

Hilbert series

Denote by $V = \mathbb{C}^2$ the local Hilbert space of a two state particle. The state space of a five particule system is $\mathcal{H} = V^{\otimes 5}$, which will be regarded as the natural representation of the group of invertible local filtering operations, also known as reversible stochastic local quantum operations assisted by classical communication

$$G = G_{\mathrm{SLOCC}} = \mathrm{SL}(2, \mathbb{C})^{\times 5},$$

that is, the group of 5-tuples of complex unimodular 2×2 matrices. We will denote by

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4,i_5=0}^{1} A_{i_1i_2i_3i_4i_5}|i_1\rangle|i_2\rangle|i_3\rangle|i_4\rangle|i_5\rangle$$

a state of the system. An element $\mathbf{g} = ({}^k g_i^j)$ of G maps $|\Psi\rangle$ to the state

$$|\Psi'\rangle = \mathbf{g}|\Psi\rangle$$

whose components are given by

$$A'_{i_1i_2i_3i_4i_5} = \sum_{\mathbf{j}} {}^1g_{i_1}^{j_1}\,{}^2g_{i_2}^{j_2}\,{}^3g_{i_3}^{j_3}\,{}^4g_{i_4}^{j_4}\,{}^5g_{i_5}^{j_5} A_{j_1j_2j_3j_4j_5} \qquad (1)$$

We are interested in the dimension of the space $\mathcal{I}_d$ of all G-invariant homogeneous polynomials of degree $d = 2m$ ($\mathcal{I}_d = 0$ for odd d) in the 32 variables $A_{i_1i_2i_3i_4i_5}$.

It is known that it is equal to the multiplicity of the trivial character of the symmetric group $\mathfrak{S}_{2m}$ in the fifth power of its irreducible character labeled by the partition $[m, m]$

$$\dim \mathcal{I}_d = \langle \chi^{2m} | (\chi^{mm})^5 \rangle. \qquad (2)$$

The generating function of these numbers

$$h(t) = \sum_{d \geq 0} \dim \mathcal{I}_d\, t^d \qquad (3)$$

is called the Hilbert series of the algebra $\mathcal{I} = \bigoplus_d \mathcal{I}_d$. Standard manipulations with symmetric functions allow to express it as a multi-dimensional residue:

$$h(t) = \oint \frac{du_1}{2\pi i u_1} \cdots \oint \frac{du_5}{2\pi i u_5} \frac{A(\mathbf{u})}{B(\mathbf{u}; t)} \qquad (4)$$

where the contours are small circles around the origin,

$$A(\mathbf{u}) = \prod_{i=1}^{5} \left(1 + 1/u_i^2\right) \qquad (5)$$

and

$$B(\mathbf{u}; t) = \prod_{a_i = \pm 1} (1 - t\, u_1^{a_1} u_2^{a_2} u_3^{a_3} u_4^{a_4} u_5^{a_5}) \qquad (6)$$

Such multidimensional residues are notoriously difficult to evaluate. After trying various approaches, we eventually succeded by means of a recent algorithm due to Guoce Xin [19], in a `Maple` implementation. The result can be cast in the form

$$Hilb(Inv) = \frac{P(t)}{Q(t)}$$

$P(t)$ is an even polynomial of degree 104 with non negative integer coefficients a_n

$$P(t) = \sum_{k=0}^{52} a_{2k} t^{2k}$$

given in table

n	a_n	n	a_n	n	a_n	n	a_n
0	1	30	24659	54	225699	78	9664
8	16	32	36611	56	214238	80	5604
10	9	34	52409	58	195358	82	3024
12	82	36	71847	60	172742	84	1659
14	145	38	95014	62	146849	86	770
16	383	40	119947	64	119947	88	383
18	770	42	14849	66	95014	90	145
20	1659	44	172742	68	71847	92	82
22	3024	46	195358	70	52409	94	9
24	5604	48	214238	72	36611	96	16
26	9664	50	225699	74	24659	104	1
28	15594	52	229752	76	15594		

Interpretation

On the expression of the Hilbert series: a complete description of the algebra by generators and relations is out of reach.

The structure of Cohen-Macaulay [2, 11, 1] of the algebra may be more relevant:

- There must exist a set of $17 = \dim \mathcal{H} - \dim SLOCC$ algebraically independent invariants.
- The denominator of the series is precisely a product of 17 factors. This makes plausible that these invariants can be choosen as five polynomials of degree 4, one polynomial of degree 6, five polynomials of degree 8), one polynomial of degree 10 and five polynomials of degree 12.
- These 17 polynomials are called the primary invariants.
- The numerator should then describe the secondary invariants: a set of 3014400 homogeneous polynomials such that any invariant polynomial can be uniquely expressed as a linear combination of secondary invariants, the coefficients being themselves polynomials in the primary invariants.

Using transvectants, which are classical tools of invariant theory (see [13]), we have computed a complete set of primary invariants of degree 4 and 6 [10].

Conclusion

This is the simplest kind of description to be expected but far too complex for physical applications. Furthermore, the knowledge of the invariants is not sufficient to classify entanglement patterns (see [16, 8, 9] for smaller case). The only known general approach for classifying orbits (entanglement patterns) requires the computation of the algebra of covariants (already almost intractable in the case of four qubits).

However, a closer look at the 4-qubit system, reveals that the classification of Verstraete et al. [16, 17] can be **reproduced by means of only a small set of covariants.** Finally, the investigation of entanglement measures requires an understanding of invariants under local unitary transformations (LUT) [3] . In a forthcoming paper, we will explain how to obtain LUT-invariants from SLOCC-covariants.

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