Macdonald polynomials at $t = q^k$

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Abstract

We investigate the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ for the specialization $t = q^k$. We show an identity relying the polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe an operator whose eigenvalues characterize the polynomials $P_\lambda(\mathbb{X}; q, q^k)$.

1 Introduction

Macdonald polynomials are (q, t)-deformations of Schur functions which play an important rôle in the representation theory of the double affine Hecke algebra [8, 10] since they are the eigenfunctions of the Cherednik elements. The polynomials considered here are the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ and are the eigenfunctions of the Sekiguchi-Debiard operator. For (q, t)-generic, these polynomials are completely characterized by its eigenvalues, since the dimensions of the eigenspaces is 1. It is no longer the case when t is specialized to a rational power of q. Hence, rather than using eigenvalues, one characterizes the Macdonald (homogeneous symmetric) polynomials by orthogonality (*w.r.t.* to a (q, t)-deformation of the usual scalar product on symmetric functions) and by some conditions on their dominant monomials (see *e.g.* [9]). In this paper, we consider the specialization $t = q^k$ where k is a strictly positive integer. One of our motivations is to generalize an identity given in [1], which shows that even powers of the discriminant are rectangular Jack polynomials. Here, we show that this property follows from deeper relations between the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ (in the λ-ring notation). This result

is interesting in the context of the quantum fractional Hall effect[5], since it implies properties of the expansion of the powers of the discriminant in the Schur basis [3, 4, 11]. It implies also that the Macdonald polynomials (for $t = q^k$) are characterized by the eigenvalues of an operator $\mathfrak{M}$ described in terms of isobaric divided differences and whose the dimension of each eigenspace is 1.

The paper is organized as follow. After recalling notations and background (Section 2) for Macdonald polynomial, we give, in Section 3, some properties of the operator which substitutes a complete function to each power of a letter. These properties allow to show our main result in Section 4 which is an identity relying the polynomial $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe (Section 5) an operator $\mathfrak{M}$ whose eigenvalues characterize the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$. Finally, in Section 6, we give an expression of $\mathfrak{M}$ in terms of Cherednik elements.

2 Notations and background

Consider an alphabet $\mathbb{X}$ potentially infinite. We will use the notations of [7] for the generating function $\sigma_z(\mathbb{X})$ of the complete homogeneous functions $S^p(\mathbb{X})$,

$$\sigma_z(\mathbb{X}) = \sum_i S^i(\mathbb{X}) z^i = \prod_i \frac{1}{1 - xz}.$$

The algebra Sym of symmetric function has a structure of λ-ring [7]. We recall that the sum of two alphabets $\mathbb{X} + \mathbb{Y}$ is defined by

$$\sigma_z(\mathbb{X} + \mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S^i(\mathbb{X} + \mathbb{Y}) z^i.$$

In particular, if $\mathbb{X} = \mathbb{Y}$ one has $\sigma_z(2\mathbb{X}) = \sigma_z(\mathbb{X})^2$. This definition is extended for any complex number α by $\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha$. For example, the generating series of the elementary function is

$$\begin{aligned} \lambda_z(\mathbb{X}) &:= \textstyle\sum \Lambda_i(\mathbb{X}) z^i = \prod_x (1 + xz) \\ &= \sigma_{-z}(-\mathbb{X}) = \textstyle\sum_i (-1)^i S^i(-\mathbb{X}) z^i. \end{aligned}$$

The complete functions of the product of two alphabets $\mathbb{X}\mathbb{Y}$ are given by the Cauchy kernel

$$K(\mathbb{X}, \mathbb{Y}) := \sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S^i(\mathbb{X}\mathbb{Y}) = \prod_{x\in\mathbb{X}} \prod_{y\in\mathbb{Y}} \frac{1}{1 - xyt} = \sum_\lambda S_\lambda(\mathbb{X}) S_\lambda(\mathbb{Y}),$$

where S_λ denote, as in [7], a Schur function. More generally, one has

$$K(\mathbb{X},\mathbb{Y}) = \sum_\lambda A_\lambda(\mathbb{X})B_\lambda(\mathbb{Y})$$

for any pair of basis $(A_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ in duality for the usual scalar product $\langle\, ,\, \rangle$.

2.1 Macdonald polynomials

One considers the (q,t)-deformation (see *e.g.* [9]) of the usual scalar product on symmetric functions defined for a pair of power sum functions Ψ^λ and Ψ^μ (in the notation of [7]) by

$$\langle \Psi^\lambda, \Psi^\mu\rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{1}$$

The familly of Macdonald polynomials $(P_\lambda(\mathbb{X};q,t))_\lambda$ is the unique basis of symmetric functions orthogonal for $\langle\ ,\ \rangle_{q,t}$ and such that

$$P_\lambda(\mathbb{X};q,t) = m_\lambda(\mathbb{X}) + \sum_{\mu\leq\lambda} u_{\lambda\mu} m_\mu(\mathbb{X}), \tag{2}$$

where m_λ denote, as usual, a monomial function [7, 9]. The reproducing kernel associated to this scalar product is

$$K_{q,t}(\mathbb{X},\mathbb{Y}) := \sum_\lambda \langle \Psi^\lambda, \Psi^\lambda\rangle_{q,t}^{-1} \Psi_\lambda(\mathbb{X})\Psi_\lambda(\mathbb{Y}) = \sigma_1\left(\frac{1-t}{1-q}\mathbb{X}\mathbb{Y}\right)$$

see *e.g.* [9] (VI. 2). In particular, one has

$$K_{q,t}(\mathbb{X},\mathbb{Y}) = \sum_\lambda P_\lambda(\mathbb{X};q,t)Q_\lambda(\mathbb{Y};q,t), \tag{3}$$

where $Q_\lambda(\mathbb{X};q,t)$ is the dual basis of $P_\lambda(\mathbb{Y};q,t)$ for $\langle\, ,\, \rangle_{q,t}$,

$$Q_\lambda(\mathbb{X};q,t) = \langle P_\lambda, P_\lambda\rangle_{q,t}^{-1} P_\lambda(\mathbb{X};q,t). \tag{4}$$

The coefficient $b_\lambda(q,t) = \langle P_\lambda, P_\lambda\rangle_{q,t}^{-1}$ is known to be

$$b_\lambda(q,t) = \prod_{(i,j)\in\lambda} \frac{1-q^{\lambda_j-i+1}t^{\lambda'_i-j}}{1-q^{\lambda_j-i}t^{\lambda'_i-j+1}} \tag{5}$$

see [9] VI.6. Writing

$$K_{q,t}\left(\left(\frac{1-q}{1-t}\right)\mathbb{X},\mathbb{Y}\right)=K(\mathbb{X},\mathbb{Y}), \tag{6}$$

one finds that $\left(P_\lambda\left(\left(\frac{1-q}{1-t}\right)\mathbb{X};q,t\right)\right)_\lambda$ is the dual basis of $(Q_\lambda(\mathbb{X};q,t))_\lambda$ for the usual scalar product $\langle\ ,\ \rangle$.

Note that there exists an other Kernel type formula which reads

$$\lambda_1(\mathbb{X}\mathbb{Y})=\sum_\lambda P_{\lambda'}(\mathbb{X};t,q)P_\lambda(\mathbb{Y};q,t)=\sum_\lambda Q_{\lambda'}(\mathbb{X};t,q)Q_\lambda(\mathbb{Y};q,t). \tag{7}$$

where λ' denotes the conjugate partition of λ. This formula can be found in [9] VI.5 p 329.

From Equalities (6) and (3) , one has

$$\sigma_1(\mathbb{X}\mathbb{Y})=K_{q,t}\left(\frac{1-q}{1-t}\mathbb{X},\mathbb{Y}\right)=\sum_\lambda Q_\lambda\left(\frac{1-q}{1-t}\mathbb{X};q,t\right)P_\lambda(\mathbb{Y};q,t). \tag{8}$$

Applying (7) to

$$\sigma_1(\mathbb{X}\mathbb{Y})=\lambda_{-1}(-\mathbb{X}\mathbb{Y}),$$

one obtains

$$\sigma_1(\mathbb{X}\mathbb{Y})=\sum_\lambda(-1)^{|\lambda|}Q_{\lambda'}(-\mathbb{X};t,q)Q_\lambda(\mathbb{Y};q,t). \tag{9}$$

Identifying the coefficient of $P_\lambda(\mathbb{Y};t,q)$ in (8) and (9), one finds the property below.

Lemma 2.1

$$Q_\lambda(-\mathbb{X};t,q)=(-1)^{|\lambda|}P_{\lambda'}\left(\frac{1-q}{1-t}\mathbb{X};q,t\right). \tag{10}$$

Unlike the usual ($q=t=1$) scalar product, there is no expression as a constant term for the product $\langle\ ,\ \rangle_{q,t}$ when $\mathbb{X}=\{x_1,\ldots,x_n\}$ is finite. But the Macdonald polynomials are orthogonal for an other scalar product defined by

$$\langle f,g\rangle'_{q,t;n}=\frac{1}{n!}\mathrm{C.T.}\{f(\mathbb{X})g(\mathbb{X}^\vee)\Delta_{q,t}(\mathbb{X})\} \tag{11}$$

where C.T. means constant term, $\Delta_{q,t}(\mathbb{X}) = \prod_{i\neq j} \frac{(x_i x_j^{-1};q)_\infty}{(t x_i x_j^{-1};q)_\infty}$, $(a;b)_\infty = \prod_{i\geq 0}(1-ab^i)$ and $\mathbb{X}^\vee = \{x_1^{-1},\dots,x_n^{-1}\}$. The expression of $\langle P_\lambda, Q_\lambda\rangle'_{q,t;n}$ is given by ([9] VI.9)

$$\langle P_\lambda, Q_\lambda\rangle'_{q,t;n} = \frac{1}{n!}\text{C.T.}\{\Delta_{q,t}(\mathbb{X})\} \prod_{(i,j)\in\lambda} \frac{1-q^{i-1}t^{n-j+1}}{1-q^i t^{n-j}}. \tag{12}$$

2.2 Skew symmetric functions

Let us define as in [9] VI 7, the skew Q functions by

$$\langle Q_{\lambda/\mu}, P_\nu\rangle_{q,t} := \langle Q_\lambda, P_\mu P_\nu\rangle_{q,t}. \tag{13}$$

Straightforwardly, one has

$$Q_{\lambda/\mu}(\mathbb{X};q,t) = \sum_\nu \langle Q_\lambda, P_\nu P_\mu\rangle_{q,t} Q_\nu(\mathbb{X};q,t). \tag{14}$$

Classically, one has [1]

Proposition 2.2 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets, one has*

$$Q_\lambda(\mathbb{X}+\mathbb{Y};q,t) = \sum_\mu Q_\mu(\mathbb{X};q,t) Q_{\lambda/\mu}(\mathbb{Y};q,t),$$

or equivalently

$$P_\lambda(\mathbb{X}+\mathbb{Y};q,t) = \sum_\mu P_\mu(\mathbb{X};q,t) P_{\lambda/\mu}(\mathbb{Y};q,t).$$

The following identities generalize Equalities (3) and (7) and can be found in [9] example 6 p.352

$$\sum_\rho P_{\rho/\lambda}(\mathbb{X};q,t) Q_{\rho/\mu}(\mathbb{Y};q,t) = K_{qt}(\mathbb{X},\mathbb{Y}) \sum_\rho P_{\mu/\rho}(\mathbb{X};q,t) Q_{\lambda/\rho}(\mathbb{Y};q,t), \tag{15}$$

$$\sum_\rho Q_{\rho'/\lambda'}(\mathbb{X};t,q) Q_{\rho/\mu}(\mathbb{Y};q,t) = \lambda_1(\mathbb{X}\mathbb{Y}) \sum_\rho Q_{\mu'/\rho'}(\mathbb{X},t,q) Q_{\lambda/\rho}(\mathbb{Y};q,t). \tag{16}$$

[1] See *e.g.* [9] VI.7 for a short proof of this identity

3 The substitution $x^p \to S^p(\mathbb{Y})$ and the Macdonald polynomials

Let $\mathbb{X} = \{x_1, \ldots, x_n\}$ be a finite alphabet and $\mathbb{Y}$ be an other (potentially infinite) alphabet. For simplicity we will denote by $\int_{\mathbb{Y}}$ the substitution

$$\int_{\mathbb{Y}} x^p = S^p(\mathbb{Y}), \tag{17}$$

for each $x \in \mathbb{X}$ and each $p \in \mathbb{Z}$.

3.1 Substitution formula

Set $\mathbb{Y}^{tq} = \frac{1-t}{1-q}\mathbb{Y}$ and consider the substitution

$$\int_{\mathbb{Y}^{tq}} x^p = S^p\left(\mathbb{Y}^{tq}\right) = Q_p(\mathbb{Y}; q, t). \tag{18}$$

Setting

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}; q, t) := \frac{1}{n!}\int_{\mathbb{Y}} P_\lambda(\mathbb{X}; q, t) Q_\mu(\mathbb{X}^\vee; q, t)\Delta(\mathbb{X}, q, t) \tag{19}$$

where $\mathbb{X}^\vee = \{x_1^{-1}, \ldots, x_n^{-1}\}$, one has the following property.

Theorem 3.1 *Let $\mathbb{X} = \{x_1, \ldots, x_n\}$ be an alphabet and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $\mu \subset \lambda$. The polynomial $\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq}; q, t)$ is the Macdonald polynomial*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq}; q, t) = \frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1 - q^{i-1}t^{n-j+1}}{1 - q^i t^{n-j}}\text{C.T.}\{\Delta(\mathbb{X}, q, t)\}Q_{\lambda/\mu}(\mathbb{Y}, q, t) \tag{20}$$

Proof From the definition of the Q_λ, one has

$$\int_{\mathbb{Y}^{tq}} x^p = Q_p(\mathbb{Y}; q, t) = \text{C.T.}\{x^{-p}K_{q,t}(x, \mathbb{Y})\}. \tag{21}$$

Hence, the polynomial $\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{qt}, q, t)$ is the constant term

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq}; q, t) = \frac{1}{n!}\text{C.T.}\{P_\lambda(\mathbb{X}^\vee; q, t)Q_\mu(\mathbb{X}; q, t)K_{q,t}(\mathbb{X}, \mathbb{Y})\Delta(\mathbb{X}, q, t)\}.$$

Equality (15) admits

$$K_{qt}(\mathbb{X},\mathbb{Y})Q_\mu(\mathbb{X};q,t)=\sum_\rho P_{\rho/\mu}(\mathbb{Y};q,t)Q_\rho(\mathbb{X};q,t),$$

as a special case. Hence,

$$\begin{aligned}\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq},q,t) &= \langle P_\lambda(\mathbb{X}^\vee;q,t),\sum_\rho P_{\rho/\mu}(\mathbb{Y};q,t)Q_\rho(\mathbb{X};q,t)\rangle'_{q,t;n}\\ &= \langle P_\lambda(\mathbb{X}^\vee;q,t),Q_\lambda(\mathbb{X};q,t)\rangle'_{q,t;n}Q_{\lambda/\mu}(\mathbb{Y},q,t).\end{aligned}\tag{22}$$

Equality (12) allows to conclude. □

3.2 Substitution dual formula

Setting $\overline{\mathbb{Y}}=\{-y_1,\dots,-y_m,\dots\}$ if $\mathbb{Y}=\{y_1,\dots,y_m,\dots\}$ [2], one observes the following propery.

Theorem 3.2 *Let $\mathbb{X}=\{x_1,\dots,x_n\}$ be an alphabet and $\lambda=(\lambda_1,\dots,\lambda_n)$ be a partition and $\mu\subset\lambda$. One has*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(-\overline{\mathbb{Y}};q,t)=\mathfrak{H}^{n,k}_{\lambda'/\mu'}(\mathbb{Y}^{qt};t,q)\tag{23}$$

where $\mathbb{Y}^{qt}=\frac{1-q}{1-t}\mathbb{Y}$.

Proof It suffices to show that

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(-\overline{\mathbb{Y}};q,t)=\frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{\lambda'/\mu'}(\mathbb{Y},t,q).$$

The proof of this identity is almost the same than the proof of (20) except than one uses the formula

$$\prod(1+x_iy_j)Q_\mu(\mathbb{X};t,q)=\sum_\rho Q_\rho(\mathbb{X};q,t)Q_{\rho'/\mu'}(\mathbb{Y};t,q),$$

which is a special case of identity (16). □

Note that in the case of partitions, one has

[2] The operation $\mathbb{Y}\leftarrow\overline{\mathbb{Y}}$ makes sense for virtual alphabet since it sends any homogeneous symmetric polynomial $P(\mathbb{Y})$ of degree p to $(-1)^pP(\mathbb{Y})$.

Corollary 3.3

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}},q,t)=\frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^{i}t^{n-j}}\text{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{\lambda'}(\mathbb{Y},t,q) \quad (24)$$

Example 3.4 Consider the following equality

$$\mathfrak{H}_{41/3}^{2,3}(-\overline{\mathbb{Y}};q,t)=(*)\text{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{2111/111}(\mathbb{Y};t,q).$$

where $\mathbb{X}=\{x_1,x_2\}$. The coefficient $(*)$ is computed as follows. One writes the partition 41 in a rectangle of height 2 and length 4.

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Each $\times$ of coordinate (i,j) is interpreted as the faction $[i,j]:=\frac{1-q^{i-1}t^{3-j}}{1-q^{i}t^{2-j}}$. Hence

$$(*)=[1,2][1,1][2,1][3,1][4,1]=\frac{(1-t)(1-t^2)(1-qt^2)(1-q^2t^2)(1-q^3t^2)}{(1-q)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)}$$

4 A formula relying the polynomials $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right)$ and $P_\lambda\left(\mathbb{X};q,q^k\right)$

When $t=q^k$ with $k\in\mathbb{N}$, Corollary 3.3 gives

Corollary 4.1

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}},q,q^k)=\beta_{\lambda}^{n,k}(q)Q_{\lambda'}(\mathbb{Y};q^k,q). \quad (25)$$

where

$$\beta_{\lambda}^{n,k}(q)=\prod_{i=0}^{n-1}\begin{bmatrix}\lambda_{n-i}-1+k(i+1)\\ k-1\end{bmatrix}_q$$

and $\begin{bmatrix}n\\p\end{bmatrix}_q=\frac{(1-q^n)\ldots(1-q^{n-r+1})}{(1-q)\ldots(1-q^r)}$ *denotes the q-binomial.*

Proof From Corollary 3.3, it remains to compute C.T.$\{\Delta(\mathbb{X},q,t)$. The evaluation of this term is deduced from the q-Dyson conjecture [3]

$$\text{C.T.}\{\Delta(x;q,q^k)\} = n!\prod_{i=1}^{n}\begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}_q,$$

and can be found in [9] examples 1 p 372.

Hence,

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}},q,q^k) = \beta_{\lambda}^{n,k}(q)Q_{\lambda'}(\mathbb{Y},q^k,q),$$

where

$$\beta_{\lambda}^{n,k}(q)\prod_{(i,j)\in\lambda}\frac{1-q^{i+k(n-j+1)-1}}{1-q^{i+k(n-j)}}\prod_{i=1}^{n}\begin{bmatrix} ik-1 \\ k-1 \end{bmatrix}_q. \tag{26}$$

But,

$$\prod_{(i,j)\in\lambda}\frac{1-q^{i+k(n-j+1)-1}}{1-q^{i+k(n-j)}} = \prod_{i=0}^{n-1}\prod_{j=1}^{\lambda_{n-i}}\frac{1-q^{j+k(i+1)-1}}{1-q^{j+ki}}.$$

Hence, rearranging the factors appearing in the right hand side of Equality (26), one obtains

$$\begin{aligned}\beta_{\lambda}^{n,k}(q) &= \prod_{i=0}^{n-1}\left(\begin{bmatrix} (i+1)k-1 \\ ik \end{bmatrix}_q\prod_{j=1}^{\lambda_{n-i}}\frac{1-q^{j+k(i+1)-1}}{1-q^{j+ki}}\right) \\ &= \prod_{i=0}^{n-1}\begin{bmatrix} \lambda_{n-i}-1+k(i+1) \\ k-1 \end{bmatrix}_q.\end{aligned} \tag{27}$$

This ends the proof.□

Example 4.2 Set $k=2, n=3$ and consider the polynomial

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}};q,q^2) = \frac{1}{n!}\int_{-\overline{\mathbb{Y}}}P_{[32]}(x_1+x_2+x_3;q,q^2\underbrace{\quad}\,)\prod_{i\neq j}(1-x_ix_j^{-1})(1-qx_ix_j^{-1}).$$

One has,

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}};q,q^2) = \frac{(1-q^5)(1-q^8)}{(1-q)^2}Q_{[221]}(\mathbb{Y};q^2,q).$$

[3] see [12] for a proof.

Let

$$\Omega_S := \frac{1}{n!}\int_{\mathbb{X}} \prod_{i\neq j}(1-x_i x_j^{-1}) \tag{28}$$

and for each $v \in \mathbb{Z}^n$,

$$\tilde{S}_\lambda(\mathbb{X}) = \det\left(x_i^{v_j+n-j}\right)\prod_{i<j}(x_i-x_j)^{-1}.$$

Lemma 4.3 *If v is a any vector of $\mathbb{Z}^n$, one has*

$$\Omega_S \tilde{S}_v(\mathbb{X}) = S_v(\mathbb{X}) := \det(S^{v_i-i+j}(\mathbb{X})) \tag{29}$$

Proof The identity is obtain by the direct computation:

$$\begin{aligned}
\frac{1}{n!}\int_{\mathbb{X}} \tilde{S}_v(\mathbb{X})\prod_{i<j}\left(1-x_i x_j^{-1}\right) &= \frac{1}{n!}\int_{\mathbb{X}} \det\left(x_i^{v_j-j+1}\right)\det(x_i^{j-1}) \\
&= \frac{1}{n!}\int_{\mathbb{X}} \sum_{\sigma_1,\sigma_2\in\mathfrak{S}_n} \operatorname{sign}(\sigma_1\sigma_2)\prod_i x^{v_{\sigma_1(i)}-\sigma_1(i)+\sigma_2(i)-1} \\
&= \frac{1}{n!}\sum_{\sigma_1\sigma_2} \operatorname{sign}(\sigma_1\sigma_2)\prod_i S_{v_{\sigma_1(i)}-\sigma_1(i)+\sigma_2(j)} \\
&= \det(S^{v_i-i+j}(\mathbb{X})).
\end{aligned}$$

□

In particular, Ω_S lets invariant any symmetric polynomial. The operator

$$\mathfrak{A}_m := \Omega_S \Lambda^n(\mathbb{X})^{-m} \tag{30}$$

acts on symmetric polynomials by expanding theim in the Schur basis and substracting m on each part.

Example 4.4 If $\mathbb{X} = \{x_1, x_2, x_3\}$ consider the polynomial, and $\lambda = [320]$. One has

$$P_{32}(\mathbb{X};q,t) = S_{32}(\mathbb{X}) + \frac{(-q+t)\,S_{311}(\mathbb{X})}{qt-1} + \frac{(q+1)\,(qt^2-1)\,(-q+t)\,S_{221}(\mathbb{X})}{(qt-1)^2\,(qt+1)}.$$

Hence,

$$\begin{aligned}
\mathfrak{A}_1 P_{32}(\mathbb{X};q,t) &= \tfrac{(-q+t)S_2(\mathbb{X})}{qt-1} + \tfrac{(q+1)\left(qt^2-1\right)(-q+t)S_{11}(\mathbb{X})}{(qt-1)^2(qt+1)} \\
&= \tfrac{(-q+t)(t+1)\left(q^2t-1\right)P_{11}(\mathbb{X};q,t)}{(qt-1)^2(qt+1)} + \tfrac{(-q+t)P_2(\mathbb{X};q,t)}{qt-1}.
\end{aligned}$$

Theorem 4.5 *If λ denotes a partition of length at most n, one has*

$$\mathfrak{A}_{(k-1)(n-1)} P_\lambda(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right) \quad (31)$$

Proof From the definitions of the operators $\mathfrak{A}_m$ (30) and Ω_S (28), one obtains

$$\mathfrak{A}_{(k-1)(n-1)} P_\lambda(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \frac{1}{n!} \int_{\mathbb{X}} P_\lambda(\mathbb{X}; q, q^k) \Delta(\mathbb{X}, q, q^k).$$

Corollary 4.1 implies

$$\begin{aligned} \frac{1}{n!} \int_{\mathbb{X}} P_\lambda(\mathbb{X}; q, q^k) \Delta(\mathbb{X}, q, q^k) &= \mathfrak{H}_\lambda^{n,k}(\mathbb{X}; q, q^k) \\ &= \beta_\lambda^{n,k}(q) Q_{\lambda'}(-\overline{\mathbb{X}}; q^k, q) \end{aligned}$$

But, from Lemma 2.1, one has

$$Q_{\lambda'}(-\overline{\mathbb{X}}; q^k, q) = (-1)^{|\lambda|} Q_{\lambda'}(-\mathbb{X}; q^k, q) = P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right).$$

The result follows. □

Example 4.6 Set $k = 2$, $n = 3$ and $\lambda = [2]$. One has

$$\begin{aligned} &P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - q x_j) = -q^3 S_{[6,2]} + q^2 \frac{q^3 - 1}{q - 1} S_{[6,1,1]} \\ &+ \frac{q^2(q^5 - 1)}{q^3 - 1} S_{[5,3]} - \frac{q(q^2 + 1)(q^5 - 1)}{q^3 - 1} S_{[5,2,1]} - \frac{q(q^7 - 1)}{q^3 - 1} S_{[4,3,1]} + \frac{q^7 - 1}{q - 1} S_{[4,2,2]}. \end{aligned}$$

And,

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - q x_j) = \frac{q^7 - 1}{q - 1} S_{[2]}.$$

Since,

$$P_{[2]}\left(\frac{x_1 + x_2 + x_3}{1 + q}; q, q^2\right) = \frac{1 - q}{1 - q^3} S_{[2]}$$

one obtains

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2) \prod_{i \neq j} (x_i - q x_j) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 7 \\ 1 \end{bmatrix}_q P_{[2]}\left(\frac{x_1 + x_2 + x_3}{1 + q}; q, q^2\right).$$

As a consequence, one has

Corollary 4.7 *If* $\lambda = \mu + [((k-1)(n-1))^n]$,

$$P_\mu(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_\lambda \left(\frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right)$$

Proof Since the size of $\mathbb{X}$ is n,

$$P_\lambda(\mathbb{X}; q, q^k) = P_\mu(\mathbb{X}; q, q^k)(x_1 \dots x_n)^{(k-1)(n-1)}.$$

Then, the result is a direct consequence of Theorem 4.5.□

Example 4.8 Set $k = 2$, $n = 3$ and $\lambda = [5, 2]$. One has,

$$\begin{aligned} &P_{[5,2]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2 x_2)(x_2 - qx_1)(x_2 - q^2 x_1) = \\ &q^3 S_{[9,2]} + \frac{(1-q^7)(1+q^4)}{1-q^5} S_{[7,4]} - \frac{(1-q^2)(1+q)(1+q^2)(1+q^4)}{1-q^5} S_{[8,3]}. \end{aligned}$$

This implies

$$\begin{aligned} &\mathfrak{A}_2 P_{[5,2]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2 x_2)(x_2 - qx_1)(x_2 - q^2 x_1) = \\ &(x_1 x_2)^{-2} P_{[5,2]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2 x_2)(x_2 - qx_1)(x_2 - q^2 x_1) = \\ &P_{[3]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2 x_2)(x_2 - qx_1)(x_2 - q^2 x_1). \end{aligned}$$

One verifies that

$$\begin{aligned} &P_{[3]}(x_1 + x_2; q, q^3)(x_1 - qx_2)(x_1 - q^2 x_2)(x_2 - qx_1)(x_2 - q^2 x_1) = \\ &\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \begin{bmatrix} 10 \\ 2 \end{bmatrix}_q P_{[5,2]}(\frac{x_1 + x_2}{1 + q + q^2}; q, q^3). \end{aligned}$$

Remark 4.9 If μ is the empty partition, Corollary 4.7 gives

$$\prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_{[((k-1)(n-1))^n]} \left(\frac{1-q}{1-q^k} \mathbb{X}; q, q^k \right).$$

This equality generalizes an identity given in [1]:

$$\prod_{i<j} (x_i - x_j)^{2(k-1)} = \frac{(-1)^{\frac{((k-1)n(n-1)}{2}}}{n!} \binom{kn}{k, \dots, k} P^{(k)}_{n^{(n-1)(k-1)}}(-\mathbb{X}),$$

where $P_\lambda^{(k)}(\mathbb{X}) = \lim_{k\to 1} P_\lambda^{(\alpha)}(\mathbb{X}; q, q^k)$ denotes a Jack polynomial (see *e.g.* [9]). The expansion of the powers of the discriminant and their q-deformations in the different basis of symmetric functions is a difficult problem having many applications, for example, in the study of Hua-type integrals (see *e.g.* [**?**, **?**]) or in the context of the factional quantum Hall effect (*e.g.* [3, 4, 5, 11]). Note that in [2], we gave an expression of an other q-deformation of the powers of the discriminant as staircase Macdonald polynomials. This deformation is also relevant in the study of the expansion of $\prod_{i<j}(x_i - x_j)^{2k}$ in the Schur basis, since we generalized [2] a result of [4].

5 Macdonald polynomials at $t = q^k$ as eigenfunctions

Let $\mathbb{Y} = \{y_1, \ldots, y_{kn}\}$ be an alphabet of cardinality kn with $y_1 = x_1, \ \ldots, \ y_n = x_n$. One considers the symmetrizer π_ω defined by

$$\pi_\omega f(y_1, \ldots, y_{kn}) = \prod_{i<j}(x_i - x_j)^{-1} \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma) f(y_{\sigma(1)}, \ldots, y_{\sigma(kn)}) y_{\sigma(1)}^{kn-1} \ldots y_{\sigma(kn-1)}.$$

Note that π_ω is the isobaric divided difference associated to the maximal permutation ω in $\mathfrak{S}_{kn}$.

This operator applied to a symmetric function of the alphabet $\mathbb{X}$ increases its alphabet from $\mathbb{X}$ to $\mathbb{Y}$ in its expansion in the Schur basis, since

$$\pi_\omega S_\lambda(\mathbb{X}) = S_\lambda(\mathbb{Y}). \tag{32}$$

Indeed, the image of the monomial $y_1^{i_1} \ldots y_{kn}^{i_{kn}}$ is the Schur function $S_I(\mathbb{Y})$. Since

$$\pi_\omega S_\lambda(\mathbb{X}) = \pi_\omega x_1^{\lambda_1} \ldots x_n^{\lambda_n} = \pi_\omega y_1^{\lambda_1} \ldots y_n^{\lambda_n} y_{n+1}^0 \ldots y_{kn}^0,$$

one recovers Equality (32).

One defines the operator π^{tq} which consists in applying π_ω and specializing the result to the alphabet

$$\mathbb{X}^{tq} := \{x_1, \ldots, x_n, qx_1, \ldots, qx_n, \ldots, q^{k-1}x_1, \ldots, q^{k-1}x_n\}.$$

From Equality (32), one has

$$\pi_\omega^{tq} S_\lambda(\mathbb{X}) = S_\lambda\left((1 + q + \cdots + q^{k-1})\mathbb{X}\right). \tag{33}$$

Furthermore, the expansion of $S_\lambda\left((1+q+\cdots+q^{k-1})\mathbb{X}\right)$ in the Schur basis being triangular, the operator π^{tq} defines an automorphism of the space $Sym_{\leq n}$ generated by the Schur functions indexed by partitions whose length are less or equal to n, *i.e.* for each function $f \in Sym_{\leq n}$, one has

$$\pi^{tq} f(\mathbb{X}) = f(\mathbb{X}^{tq}). \tag{34}$$

In particular, one has

Lemma 5.1 *Let λ be a partition such that $l(\lambda) \leq n$ then*

$$\pi_\omega^{tq} P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, t = q^k\right) = P_\lambda(\mathbb{X}, q, q^k). \tag{35}$$

Proof It suffices to remark that $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right) \in Sym_{\leq n}(\mathbb{X})$.[4]

It follows from (34),

$$\pi_\omega^{tq} P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right) = P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}^{tq}; q, q^k\right) = P_\lambda(\mathbb{X}, q, q^k).$$

□

Consider the operator $\mathfrak{M} : f \to \mathfrak{M} f$ defined by

$$\mathfrak{M} := (x_1 \ldots x_n)^{(k-1)(1-n)} \pi_\omega^{tq} \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j).$$

The following theorem shows that the Macdonald polynomials are the eigenfunctions of the operator $\mathfrak{M}$.

Theorem 5.2 *The Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ are eigenfunctions of $\mathfrak{M}$. The eigenvalue associated to $P_\mu(\mathbb{X}; q, q^k)$ is $\beta^{n,k}_{\mu+((k-1)(n-1))^n}(q)$. Furthermore, if $k > 1$, the dimension of each eigenspace is 1.*

Proof From Corollary 4.7, one has

$$P_\mu(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \beta^{n,k}_\lambda(q) P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$$

[4] This can be seen as a consequence of the determinantal expression of the expansion of $P_\lambda(\mathbb{X}, q, t)$ in the Schur basis evaluated on the alphabet $\mathbb{X}^{tq}$ (see [6]).

where $\lambda = \mu + ((k-1)(n-1))^n$. Applying π_ω^{tq} to the left and the right hand sides of this equality, one obtains from Lemma 5.1

$$\begin{aligned}\pi_\omega^{tq} P_\mu(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) &= \beta_\lambda^{n,k}(q) \pi_\omega^{tq} P_\lambda \left(\tfrac{1-q}{1-q^k} \mathbb{X}; q, q^k \right) \\ &= \beta_\lambda^{n,k}(q) P_\lambda \left(\mathbb{X}; q, q^k \right).\end{aligned}$$

Since the cardinality of $\mathbb{X}$ is n, one has

$$P_\lambda \left(\mathbb{X}; q, q^k \right) = (x_1 \dots x_n)^{(k-1)(n-1)} P_\lambda \left(\mathbb{X}; q, q^k \right),$$

and

$$\mathfrak{M} P_\mu(\mathbb{X}; q, q^k) = \beta_{\mu + ((k-1)(n-1))^n}^{n,k}(q) P_\mu(\mathbb{X}; q, q^k). \tag{36}$$

Suppose now that $k > 1$. It remains to prove that the dimensions of the eigenspaces equal 1. More precisely, It suffices to show that $\beta_\lambda(q) = \beta_\mu(q)$ implies $\lambda = \mu$. The denominators of $\beta_\lambda(q)$ and $\beta_\mu(q)$ being the same, one needs to examine the numerators, that is the products $\gamma_\lambda = \prod_{i=0}^{n-1} (q^{\lambda_{n-i}+ki}; q)_{k-1}$ and $\gamma_\mu = \prod_{i=0}^{n-1} (q^{\mu_{n-i}+ki}; q)_{k-1}$. One needs the following lemma.

Lemma 5.3 *Let $I = \{i_1, \dots, i_n\}$ and $J = \{j_1, \dots, j_m\}$ be two finite subsets of $\mathbb{N} \setminus \{0\}$. Then, $I \neq J$ implies $\prod_{i \in I}(1 - q^i) \neq \prod_{j \in J}(1 - q^j)$.*

Proof Without lost of generalities, one can suppose $I \cap J = \emptyset$. Suppose that $i_1 \leq \dots \leq i_n$ and $j_1 \leq \dots \leq j_m$. Then, expanding the two products, one finds

$$\prod_{i \in I}(1 - q^i) = 1 - q^{i_1} + \sum_{l > i_1} (*) q^l \neq 1 - q^{j_1} + \sum_{l > j_1} (*) q^l = \prod_{j \in J}(1 - q^j).$$

□

Each term $(q^{\lambda_{n-i}+ki}; q)_{k-1}$ is characterized by the degree of its factor of lower degree : $\lambda_{n-i} + ki$. Hence, from Lemma 5.3, $\beta_\lambda(q) = \beta_\mu(q)$ implies that it exists a permutation σ of $\mathfrak{S}_n$ verifying

$$\lambda_i + k(n-i) = \mu_{\sigma_i} + k(n - \sigma_i),$$

for each i. But, since λ is decreasing, one has

$$\lambda_i + k(n-i) - \lambda_{i-1} - k(n-i+1) \leq 0.$$

And then,

$$\mu_{\sigma_i} + k(n - \sigma_i) - \mu_{\sigma_{i-1}} - k(n - \sigma_{i-1}) \leq 0. \tag{37}$$

But, since μ is decreasing, $\sigma_{i-1} - \sigma_i$ has the same sign than $\mu_{\sigma_i} - \mu_{\sigma_{i-1}}$. As a consequence, Inequality (37) implies $\sigma_i > \sigma_{i-1}$ for each i. The only possibility is $\sigma = Id$, which ends the proof.□

Example 5.4 If $n = 5$, the eigenvalues associated to the partitions of 4 are

$$\begin{aligned}
\beta^{4,k}_{[4\,k,4\,k-4,4\,k-4,4\,k-4,4\,k-4]} &= \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-5\\k-1\end{bmatrix}_q \begin{bmatrix}9k-1\\k-1\end{bmatrix}_q,\\
\beta^{4,k}_{[4\,k-1,4\,k-3,4\,k-4,4\,k-4,4\,k-4]} &= \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-2\\k-1\end{bmatrix}_q,\\
\beta^{4,k}_{[4\,k-2,4\,k-2,4\,k-4,4\,k-4,4\,k-4]} &= \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-3\\k-1\end{bmatrix}_q \begin{bmatrix}9k-3\\k-1\end{bmatrix}_q,\\
\beta^{4,k}_{[4\,k-2,4\,k-3,4\,k-3,4\,k-4,4\,k-4]} &= \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-4\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-3\\k-1\end{bmatrix}_q,\\
\beta^{4,k}_{[4\,k-3,4\,k-3,4\,k-3,4\,k-3,4\,k-4]} &= \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-4\\k-1\end{bmatrix}_q \begin{bmatrix}7k-4\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-4\\k-1\end{bmatrix}_q.
\end{aligned}$$

6 Expression of $\mathfrak{M}$ in terms of Cherednik elements

In this paragraph, we restate Proposition 5.2 in terms of Cherednik operators. Cherednik's operators $\{\xi_i; i \in \{1, \ldots, n\}\} =: \Xi$ are commutative elements of the double affine Hecke algebra. The Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ are eigenfunctions of symmetric polynomials $f(\Xi)$ and the eigenvalues are obtained substituting each occurrence of ξ_i in $f(\Xi)$ by $q^{\lambda_i} t^{n-i}$ (see [8] for more details).
Suppose that $k > 1$ and consider the operator $\tilde{\mathfrak{M}} : f \to \tilde{\mathfrak{M}} f$ defined by

$$\tilde{\mathfrak{M}} := \prod_{i=1}^{k-1} (1 - q^i)^n \mathfrak{M}. \tag{38}$$

From Proposition 5.2, one has

$$\tilde{\mathfrak{M}} P_\lambda(\mathbb{X}; q, q^k) = \prod_{i=0}^{n-1} \prod_{j=1}^{k-1} (1 - q^{\lambda_{n-i} + k(i+1) - j}) P_\lambda(\mathbb{X}; q, q^k). \tag{39}$$

The following proposition shows that $\tilde{\mathfrak{M}}$ admits a closed expression in terms of Cherednick elements.

Proposition 6.1 *One supposes that $k > 1$. For any symmetric function f, one has*

$$\tilde{\mathfrak{M}} f(\mathbb{X}) = \prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+k} \xi_i) f(\mathbb{X}). \tag{40}$$

Proof From Theorem 5.2, it suffices to prove the formula (40) for $f = P_\lambda$. The polynomial $P_\lambda(\mathbb{X}; q, t)$ is an eigenfunction of the operator $\prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+k} \xi_i)$ and its eigenvalues is $\prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+\lambda_i} t^{n-i+1}) P_\lambda(\mathbb{X}; q, t)$. Hence, setting $t = q^k$, we obtain

$$\prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+k} \xi_i) P_\lambda(\mathbb{X}; q, q^k) = \prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+\lambda_i + k(n-i+1)}) P_\lambda(\mathbb{X}; q, q^k).$$

Comparing this expression to Equality (38), one finds the result. □

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