SEQUENTIALIZATION AND UNAMBIGUITY OF $(\mathrm{MAX},+)$ RATIONAL SERIES OVER ONE LETTER

Sylvain Lombardy

Ecole Nationale Supérieure des Télécommunications,
46, rue Barrault, 75013 Paris, France
lombardy@enst.fr

Abstract: We present an algorithm to decide whether a (max,+)-rational series over one letter is sequential. We discuss the relation between sequentiality and unambiguity of rational series.

Keywords: Automata, Rational series, (max,+) Algebra

1. INTRODUCTION

Determinization of automata with multiplicity is a well-known problem solved for many semirings. The classical theory of automata corresponds to the Boolean semiring and we know that every automaton can be determinized. Transducers correspond to automata with multiplicity in the semiring of subsets of a free monoid; if the relation is functional, we can decide sequentiality (Choffrut, 1977).

The problem in the case of $(\max,+)$ or $(\min,+)$-semirings is open, except if the input alphabet is a singleton :

Theroem. *Let $\mathbb{K}$ be a $(\max,+)$-semiring. Determinizability of a unary $\mathbb{K}$-automaton is decidable.*

We present in a first part two properties of sequential series with values in a $(\max,+)$-semiring. As in the case of functions with values in a free monoid (Raney, 1958), we show that a function is sequential if, and only if, the number of its translated series is finite.

The second property is uniform divergence, which is often called bounded variations. This is a classical characterization of sequential functions with values in a monoid of words. This can not be extended to $(\max,+)$-automata. In fact, contrary to transducers, there are $(\max,+)$-automata which realize functions which can not be realized unambiguously. This is what makes the problem harder than for transducers.

In the second part, we prove that the sequentiality of a function from a^* in a $(\max,+)$-semiring can be decided on the automaton which realizes it.

The third part is devoted to algorithms which use results of the second part for the decidability of sequentiality and the determinization.

2. NOTATIONS AND DEFINITIONS

Let $\mathbb{G}$ be an additive subgroup of $\mathbb{R}$ and $\mathbb{K} = \mathbb{G}$, $\mathbb{G} \cap \mathbb{R}_+$ or $\mathbb{G} \cap \mathbb{R}_-$. In the last two cases, we say that $\mathbb{K}$ is respectively positive or negative.
Let $\mathbb{K}_\mathsf{m}$ be the commutative and idempotent semiring $(\mathbb{K} \cup \{-\infty\}, \max, +)$.

We use the additive notations for the sum (which is actually the multiplication law of the semiring).

Any function or series $\varphi : X \to \mathbb{K}_\mathsf{m}$, the support of φ, denoted by $\mathbf{Supp}(\varphi)$ is the set of elements $x \in X$ such that $x\varphi \neq -\infty$.

Definition 1. A $\mathbb{K}$-automaton $\mathcal{A}$ is a sextuple $(Q, A, \mathbb{K}_\mathsf{m}, E, I, T)$, where $E : Q \times A \times Q \to \mathbb{K}_\mathsf{m}$, $I : Q \to \mathbb{K}_\mathsf{m}$ and $T : Q \to \mathbb{K}_\mathsf{m}$ are functions.

For any $e = (p, a, q) \in Q \times A \times Q$, e is a transition if $e \in \mathbf{Supp}(E)$. The elements of $\mathbf{Supp}(I)$

and $\mathbf{Supp}(T)$ are respectively called initial and terminal states.

The value of a path is the product of values of its transitions which is, with sum notations,

$$E_P = \sum_{i\in[1;k]} E_{e_i}.$$

A computation is a path C which begins in an initial state i and ends in a final state t. Its value is $\widehat{E}_C = I_i + E_C + T_t$.

For any subset X of Q, any $u \in A^*$, $X \cdot u$ is the set of states of Q that can be reached from a state of X by a path labeled by u; $X * u$ is the maximum value of all these paths and $\widehat{X * u}$ the maximum value of corresponding computations. The maximum of an empty set is -∞.

Definition 2. A series α in $\mathbb{K}_{\mathsf{m}}\langle\langle A^*\rangle\rangle$ is rational if there exists a $\mathbb{K}$-automaton $\mathcal{A}$ such that, for every $u \in A^*$, $\langle \alpha, u\rangle = \widehat{Q * u}$. We say that $\mathcal{A}$ realizes the series α. A series is sequential if there exists a deterministic $\mathbb{K}$-automaton that realizes it.

3. TWO PROPERTIES OF SEQUENTIAL SERIES

3.1 *Translated Series of a Series*

The series $\overset{\circ}{\alpha}$ is defined such that $\langle \overset{\circ}{\alpha}, u\rangle$ is the "biggest common part" of elements of $\langle \alpha, A^*\rangle$. This definition is inspired by the case of functions into a free monoid (Raney, 1958).

Definition 3. We define the series $\overset{\circ}{\alpha}$ in $\mathbb{K}_{\mathsf{m}}\langle\langle A^*\rangle\rangle$ according to the sign of $\mathbb{K}$. For every u in A^*, let $D = u^{-1}\mathbf{Supp}(\alpha)$.

- If $\mathbb{K} > 0$, $\langle \overset{\circ}{\alpha}, u\rangle = \min\{\langle uv, \alpha\rangle \mid v \in D\}$.
- If $\mathbb{K} < 0$, $\langle \overset{\circ}{\alpha}, u\rangle = \max\{\langle uv, \alpha\rangle \mid v \in D\}$.
- Else, $\langle \overset{\circ}{\alpha}, u\rangle = \langle \alpha, uv\rangle$, where v is the smallest element of D according to the radix order [1].

For every $u \in A^*$, The left translated series of α by u is defined by :

$$\forall v,\ \langle \alpha/u, v\rangle = \langle \alpha, uv\rangle - \langle \overset{\circ}{\alpha}, u\rangle \qquad (1)$$

Remark. $\mathbf{Supp}(\alpha/u) = u^{-1}\mathbf{Supp}(\alpha)$.

These translated series give, as for transducers, a criterium of sequentiability:

Theorem 4. A series α in $\mathbb{K}_{\mathsf{m}}\langle\langle A^*\rangle\rangle$ is sequential if, and only if, the set of its translated series, $\{\alpha/u \mid u \in A^*\}$ is finite.

[1] $x < y$ iff $(|x| < |y|)$ or $(|x| = |y|$ and $x <_{\text{lex}} y)$

T he proof of the theorem makes use of the following lemma:

Lemma 5. Let $u, u' \in A^*$. If $u^{-1}\mathbf{Supp}(\alpha) = u'^{-1}\mathbf{Supp}(\alpha)$ and, for all $v \in u^{-1}\mathbf{Supp}(\alpha)$, $\delta_{u,u'}(v) = \langle \alpha, uv\rangle - \langle \alpha, u'v\rangle$ does not depend on v, then $\delta_{u,u'} = \langle \overset{\circ}{\alpha}, u\rangle - \langle \overset{\circ}{\alpha}, u'\rangle$ and $\alpha/u = \alpha/u'$.

• If the set of translated series is finite, let $Q = \{\alpha/u \mid u \in A^*\}$ and $\mathcal{A}_\alpha = (Q, A, \mathbb{K}, E, I, T)$ be the following automaton :

$$\begin{aligned} I_{\alpha/\mathbf{1}_{A^*}} &= \langle \overset{\circ}{\alpha}, \mathbf{1}_{A^*}\rangle,\ \mathbf{Supp}(I) = \{\alpha/\mathbf{1}_{A^*}\}, \\ T_{\alpha/u} &= \langle \alpha/u, \mathbf{1}_{A^*}\rangle, \\ E_{(\alpha/u, a, \alpha/(ua))} &= \langle \overset{\circ}{\alpha}, ua\rangle - \langle \overset{\circ}{\alpha}, u\rangle \end{aligned}$$

$\langle \overset{\circ}{\alpha}, ua\rangle - \langle \overset{\circ}{\alpha}, u\rangle \in \mathbb{K}$ by definition of $\overset{\circ}{\alpha}$. We can check that this automaton is deterministic and that it realizes α.

• If α is sequential, there exists a deterministic automaton $\mathcal{A} = (Q, A, \mathbb{K}, E, I, T)$ realizing α.
Let i be its initial state. For all $u, v \in A^*$,

$$\langle \alpha, uv\rangle = I_i + i * u + (i \cdot u) * v + T_{(i\cdot u)\cdot v}. \qquad (2)$$

Then, for all u, u' in A^*, if $i \cdot u = i \cdot u'$, from Lemma 5, $\alpha/u = \alpha/u'$. There is a surjection from states of $\mathcal{A}$ into the set of translated series of α which is, therefore, finite. □

Remark. As for transducers, this proof shows that if there exists a deterministic automaton, there exists a minimal and canonical one.

3.2 *Topological Property and Sequentiality*

For $(u, v) \in A^*$, we define $u \wedge v$ as the longest prefix common to u and v. d_p is the prefix distance defined by $d_p(u, v) = |u| + |v| - 2|u \wedge v|$.

Definition 6. A series α in $\mathbb{K}_{\mathsf{m}}\langle\langle A^*\rangle\rangle$ is uniformly diverging [2] if

$$\begin{aligned} &\forall k \in \mathbb{N}, \exists N,\ \forall u, v \in \mathbf{Supp}(\alpha), \\ &d_p(u, v) \leqslant k \Rightarrow |\ \langle \alpha, u\rangle - \langle \alpha, v\rangle| \leqslant N. \end{aligned} \qquad (3)$$

Proposition 7. If α is a sequential series, it is uniformly diverging.

A ctually $\{\alpha/u \mid u \in A^*\}$ is finite. For all k, we define

$$N_k = \max_{u\in A^*} \max_{v\in\mathbf{Supp}(\alpha/u), |v|\leqslant k} \{|\langle \alpha/u, v\rangle|\}.$$

Then, for all $u, v \in A^*$ such that $d_p(u, v) \leqslant k$, it is easy to see that

$$|\langle \alpha, u\rangle - \langle \alpha, v\rangle| \leqslant 2N_k. \qquad (4)$$

[2] We follow (Béal, Carton, Prieur and Sakarovitch, 2000), while the usual terminology is "series with bounded variation"(Choffrut, 1977; Mohri, 1997).

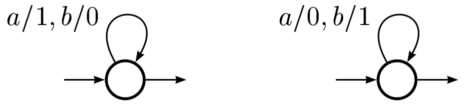

Fig. 1. An ambiguous rational series

Then α is uniformly diverging. $\square$

Contrary to the case of transducers (Choffrut, 1977), the property 3 is not characteristic.

Proposition 8. A rational and uniformly diverging series can be a non-sequential series.

Example 9. Let α be the series realized by the automaton of figure 9. For all $u \in A^*$,

$$\langle \alpha, u \rangle = \begin{cases} |u|_a & \text{if } |u|_a \geqslant |u|_b \\ |u|_b & \text{otherwise.} \end{cases}$$

On the one hand, this series is 1-Lipschitzian, thus uniformly diverging, On the other hand, for all n, $\langle \mathring{\alpha}, a^n \rangle = \langle \alpha, a^n \rangle = n$,and, for all m,

$$\langle \alpha / a^n, b^m \rangle = \begin{cases} 0 & \text{if } m \leqslant n \\ m - n & \text{otherwise.} \end{cases}$$

The set of translated series of α is infinite and α is not sequential.

This fact emphasizes the difference between the theory of sequentiality of rational functions in a monoid and the sequentiality of rational series in a semiring.

4. DECIDABILITY OF SEQUENTIALITY FOR A UNARY ALPHABET

Let $\mathcal{A} = (Q, \{a\}, \mathbb{K}, E, I, T)$ be a trim automaton realizing a rational series α. Let n be the number of states of $\mathcal{A}$ and $M = \max\{|E_e|, e \in \mathbf{Supp}(E)\}$.

Definition 10. For any simple cycle $\mathcal{C}$ of length l, we define the weight of $\mathcal{C}$ by $\varrho = E_{\mathcal{C}}/l$.

Notations. Let ϱ_m be the maximum weight of simple cycles of $\mathcal{A}$ and let R be the set of states that belong to these cycles. Let $\mathcal{L}_R(\mathcal{A})$ be the set of words that label a path going through at least one state of R. $\mathcal{L}_{\overline{R}}(\mathcal{A})$ is the intersection of the complement of $\mathcal{L}_R(\mathcal{A})$ with the support of α. Let $\delta = min\{\varrho_m - \varrho | \varrho \neq \varrho_m\}$.

Example 11. Let $\mathcal{A}_1$ be the automaton of figure *2*. $n = 5$ and $M = 3$. Simple cycles are $[q, q]$ and $[s, t, s]$, with respective weight 1 and 3/2. Thus $R = \{s, t\}$, $\varrho_m = 3/2$ and $\delta = 1/2$.
The support of α_1 realized by $\mathcal{A}_1$ is $a^2 a^*$, $\mathcal{L}_R(\mathcal{A}_1) = a^2(a^2)^*$, hence $\mathcal{L}_{\overline{R}}(\mathcal{A}_1) = a^3(a^2)^*$.

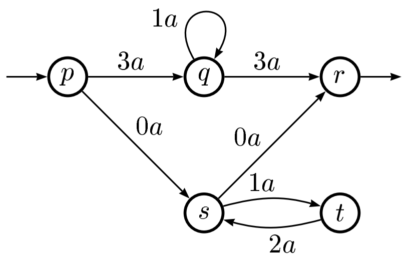

Fig. 2. The $\mathbb{N}$-automaton that realizes $\mathcal{A}_1$.

From these sets we can decide the sequentiality of the realized series; that is our main theorem :

Theorem 12. A rational series α realized by an automaton $\mathcal{A}$ is sequential if, and only if, $\mathcal{L}_{\overline{R}}(\mathcal{A})$ is finite.

The both sides of this theorem will be proved in this section (Propositions 14 and 17).

Proposition 13. Let α be a rational series realized by an automaton $\mathcal{A}$.
i) For all k, $\langle \alpha, a^k \rangle \leqslant \varrho_m k + M(n+2)$.
ii) There exists (r, s) such that, for all k,

$$\langle \alpha, a^{rk+s} \rangle \geqslant \varrho_m (rk + s) - M(4n + 2).$$

i) is easily proved by induction on k. To prove ii), we chose a cycle of maximum weight and show then the inequality. $\square$

These inequalities allow to prove the following proposition:

Proposition 14. Let α be a rational series realized by an automaton $\mathcal{A}$. If α is sequential, then $\mathcal{L}_{\overline{R}}(\mathcal{A})$ is finite.

I f α is sequential, it is uniformly diverging, therefore there exists η such that

$$|i - j| \leqslant r \Rightarrow |\langle \alpha, a^i \rangle - \langle \alpha, a^j \rangle| \leqslant \eta. \qquad (5)$$

If $\mathcal{L}_{\overline{R}}(\mathcal{A})$ is infinite, there exists l arbitrary large such that $a^l \in \mathcal{L}_{\overline{R}}(\mathcal{A})$. Let k be such that $l \in [r(k-1) + s, rk + s]$. A maximum path realizing a^l goes through cycles with weight smaller than $\varrho_m - \delta$, hence $\langle \alpha, a^l \rangle \leqslant M(n+2) + (\varrho_m - \delta)l$. It then holds:

$$\langle \alpha, a^{rk+s} \rangle - \langle \alpha, a^l \rangle) > l\delta - M(5n + 4). \qquad (6)$$

For $l > (M(5n+4) + \eta)/\delta$, this inequality in contradiction with sequentiality. $\square$

To prove the converse proposition, the key lemma is the following.

Lemma 15. For all f accepted by $\mathcal{A}$ for which there exists a computation going through R, there exists a computation labeled by f with less than n^2 transitions out of cycles of maximum weight.

T he proof makes use of the following application of the pigeon hole principle:

Lemma 16. For all $n \in \mathbb{N}$, for all $k \geqslant n$, for any set $(x_i)_{i \in [1;k]}$ of elements of $\mathbb{Z}/n\mathbb{Z}$, there exists a non-empty set $J \subset [1;k]$ such that $\sum_{i \in J} x_i = 0$.

By contradiction, let f be the shortest word such that for any computation there is more than n^2 transitions out of cycles of maximum weight and such that there is a computation going through a state $p \in R$. By minimality, this computation has no cycle of maximum weight. Let l be the length of a cycle of maximum weight around p.
$|f| \geqslant n^2$, thus this computation goes through at least n simple cycles; $l \leqslant n$, by the previous lemma, we can choose k of these cycles such that their total length is multiple of l. We can replace them by cycles of maximum weight, which is in contradiction with minimality of f. $\square$

If $\mathcal{L}_R(\mathcal{A})$ is finite, for word in $\mathbf{Supp}(\alpha)$ that is long enough, there is a computation with a bounded number of transitions out of cycle of maximum weight. We shall show that, for a “long” word, this kind of computation has a maximum value.

Proposition 17. Let α be a rational series realized by an automaton $\mathcal{A}$. If $\mathcal{L}_{\overline{R}}(\mathcal{A})$ is finite, then α is sequential.

L et $N_0 = \max_{f \in \mathcal{L}_{\overline{R}}}(|f|)$. Let B be the lcm of lengths of simple cycles of maximum weight. Let $N = \max(N_0, n^2 + 2M(n^2+2)/\delta))$.
By lemma 15, for all $k > N$,

$$\langle \alpha, a^k \rangle > \varrho_m(k - n^2) - M(n^2+2). \qquad (7)$$

Let r be the smallest integer greater than N such that there exists $a^k \in \mathbf{Supp}(\alpha)$ and a computation with a maximum value with r transitions out of cycles of maximum weight. Nevertheless, this computation has cycles, thus the value of α is smaller than $\varrho_m(k-n^2) - M(n^2+2)$, which is in contradiction with inequality 7. Thus, for all words longer than N, for all computations with maximum value, there is less than N transitions out of cycles of maximum weight.
We will show now that, for any $k > N + Bn$, $\langle \alpha, a^{k+B} \rangle = \langle \alpha, a^k \rangle + B\varrho_m$.
If a^k belongs to $\mathbf{Supp}(\alpha)$, there exists a computation realizing $\langle \alpha, a^k \rangle$. It goes through a state $p \in R$. Turning around p by a cycle of maximum weight, we can obtain a computation labeled by a^{k+B} and with value equal to $\langle \alpha, a^k \rangle + B\varrho_m \leqslant \langle \alpha, a^{k+B} \rangle$.
Conversely, if $a^{k+B} \in \mathbf{Supp}(\alpha)$, a computation realizing $\langle \alpha, a^{k+B} \rangle$ has cycles of maximum weight of lengths $l_1, \ldots, l_k$, $(k \leqslant n)$, let λ_i the number of cycle of length l_i. As the length out of cycles of maximum weight is smaller than N, $\sum_i \lambda_i l_i \geqslant Bn$; therefore, there exists i such that $\lambda_i l_i \geqslant B$; l_i divide B, thus there exists $\lambda'_i \leqslant \lambda_i$ such that $\lambda'_i l_i = B$. Going λ'_i times less through cycles of length l_i, a^k is accepted and the value of computation is $\langle \alpha, a^{k+B} \rangle - B\varrho_m \leqslant \langle \alpha, a^k \rangle$.
Hence the equality. For all $r \geqslant 0$, $\langle \alpha, a^{k+B} a^r \rangle = \langle \alpha, a^k a^r \rangle + B\varrho_m$, then $\alpha / a^k = \alpha / a^{k+B}$. $\square$

Actually, for rational series over a unary alphabet, the uniform divergence is characteristic of sequential series.

Proposition 18. Let α in $\mathbb{K}_{\mathsf{m}}\langle\langle a^* \rangle\rangle$ be a rational series. α is sequential if, and only if, α is uniformly diverging.

L et α be a rational and uniformly diverging series realized by an automaton $\mathcal{A}$. Let n, M, $\mathcal{L}_{\overline{R}}$, ϱ_m and δ be defined as below. For all $u \in \mathcal{L}_{\overline{R}}$, $u\alpha \leqslant M(n+2) + (\varrho_m - \delta)|u|$. By proposition 14, there exists k such that for all $u \in \mathcal{L}_{\overline{R}}$, $\exists v \in A^*$ such that $d_p(u,v) \leqslant k$ and $v\alpha \geqslant \varrho_m|v| - M(4n+2)$. α is uniformly diverging, thus there exists η (depending only of k) such that :

$$\begin{aligned} \eta &\geqslant |v\alpha - u\alpha| \\ &\geqslant \varrho_m k - M(5n+4) + \delta|u| \end{aligned} \qquad (8)$$

Hence, the length of u is bounded. Therefore $\mathcal{L}_{\overline{R}}$ is finite and α is sequential. $\square$

5. UNAMBIGUITY OF RATIONAL SERIES OVER ONE LETTER

We first use a result which is a particular case of a theorem in (Gaubert, 1994).

Proposition 19. Let α be a rational series of $\mathbb{K}_{\mathsf{m}}\langle\langle a^* \rangle\rangle$. There exist sequential series $\alpha_1, \ldots, \alpha_k$ such that, for every $n \in \mathbb{N}$, $\langle \alpha, a^n \rangle = \max\{\langle \alpha_i, a^n \rangle \mid i \leqslant k\}$.

Proposition 20. Any rational series $\alpha : a^* \to \mathbb{K}$ is unambiguous.

L et $\alpha_1, \ldots, \alpha_k$ be sequential series such that, for every $n \in \mathbb{N}$, $\langle \alpha, a^n \rangle = \max\{\langle \alpha_i, a^n \rangle \mid i \leqslant k\}$. Let $\varrho_1, \ldots, \varrho_k$ be the respective weights of the cycles these series, and $p_1, \ldots, p_k$ be there respective periods. If $\varrho_i = \varrho_j$, the series $\max(\alpha_i, \alpha_j)$ is sequential by the characterization of the previous section. Hence, we can suppose $\varrho_1 < \ldots < \varrho_k$. Therefore, if i is smaller than j, there exists

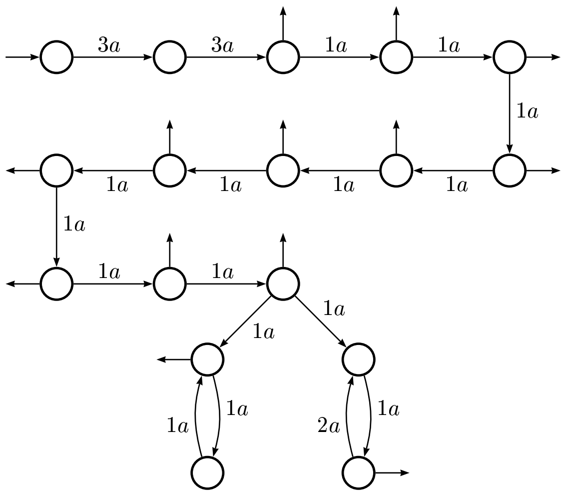

Fig. 3. An unambiguous automaton realizing α_1

N_{ij} such that for every $n > N_{ij}$, $\langle \alpha_i, a^{n+p_i} \rangle = \langle \alpha_i, a^n \rangle + p_i \varrho_i$, $\langle \alpha_j, a^{n+p_j} \rangle = \langle \alpha_j, a^n \rangle + p_j \varrho_j$ and, if $a^n \in \mathbf{Supp}(\alpha_i) \cap \mathbf{Supp}(\alpha_j)$, $\langle \alpha_i, a^n \rangle < \langle \alpha_j, a^n \rangle$. Let $N = \max_{i<j}\{N_{ij}\}$ and $p = \mathsf{lcm}\{p_i\}$. Every α_i can be realized by a deterministic $\mathbb{K}$-automaton with $N + p$ states and a cycle of length p. A state of the ith automaton is terminal if and only if the image by α_i of words ending at this place is the image by α. The automaton is then unambiguous. □

Example 21. Let α_1 be the series of example 11. The automaton of figure *3* is an unambiguous automaton that realizes α_1.

We can see that, contrary to the case of functional transducers, the equivalent unambiguous automaton does not depend only on the structure of the $\mathbb{K}$-automaton but also on its coefficients.

6. EFFECTIVE ALGORITHMS

6.1 *Decidability of Sequentiality*

From propositions of the previous part, the sequentiality depends on the finitude of $\mathcal{L}_{\overline{R}}(\mathcal{A})$. Thus, once the set r is identified, we only deal with classical automata. We compute a (boolean) automaton that realize $\mathcal{L}_R(\mathcal{A})$, then $\mathcal{L}_{\overline{R}}(\mathcal{A})$.
Let $\mathcal{A} = (Q, A, \tilde{E}, \tilde{I}, \tilde{T})$ be the underlying automaton of $\mathcal{A}$. We build $\mathcal{A}' = (Q \times \{0;1\}, A, F, J, U)$, with

$$J = \{(p,0) \mid p \in \tilde{I}\} \cup \{(p,1) \mid p \in R \cap \tilde{I}\},$$
$$U = \{(p,1) \mid p \in \tilde{T}\},$$
$$F = \{((p,i), a, (q,i)) \mid i \in \{0;1\},\ (p,a,q) \in \tilde{E}\} \cup \{((p,0), a, (q,1)) \mid q \in R,\ (p,a,q) \in \tilde{E}\}$$

Then, the complementary of $\mathcal{L}_R(\mathcal{A})$ is computed, and the intersection with the support of α. α is

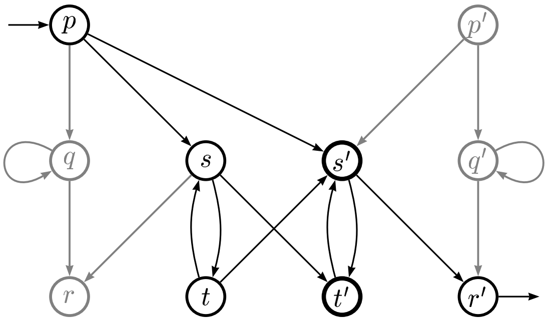

Fig. 4. The automaton that realizes $\mathcal{L}_R(\mathcal{A}_1)$.

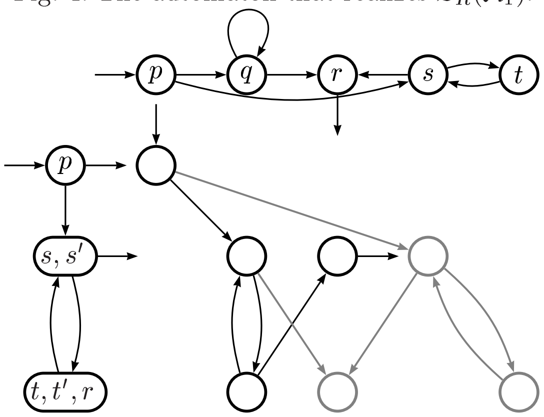

Fig. 5. The automaton that realizes $\mathcal{L}_{\overline{R}}(\mathcal{A}_1)$.

sequential if and only if this automaton, trim, is acyclic.
Remark. This construction depends only on the structure of the underlying automaton.

Example 22. Let $\mathcal{A}_1$ be the automaton of example 11. Figures *4* and *5* show the two steps of the decision algorithm (every edge is labeled by a). $\mathcal{L}_{\overline{R}}(\mathcal{A}_1)$ is infinite (the trim automaton is not acyclic), hence α_1 is not sequential.

6.2 *Determinization*

Let $\mathcal{A} = (Q, A, E, I, T)$ be a trim automaton realizing the series α which has been tested as sequential. We shall build a deterministic automaton realizing α.
We consider now I and T as vectors of $\mathbb{K}^{1\times Q}$ and $\mathbb{K}^{Q\times 1}$ respectively. μ is a matrix of size $Q \times Q$ defined by $\mu_{p,q} = E_{(p,a,q)}$.[3]
The series η in $\mathbb{K}_{\mathsf{m}}^{Q}\langle\langle a^* \rangle\rangle$ is defined by $\langle \eta, u \rangle = I.\mu^n$. We suppose Q ordered. For all u in A^*, let $D = \mathbf{Supp}(\langle \eta, u \rangle)$, and

- If $\mathbb{K} > 0$, $\langle \mathring{\eta}, u \rangle = \min\{\langle \eta, u \rangle_p \mid p \in D\}$.
- If $\mathbb{K} < 0$, $\langle \mathring{\eta}, u \rangle = \max\{\langle \eta, u \rangle_p \mid p \in D\}$.
- Else, $\langle \mathring{\eta}, u \rangle = \langle \eta, u \rangle_p$, $p = \min(Q \cap D)$.

[3] (I, μ, T) is the *linear representation* that corresponds to $\mathcal{A}$: $\langle \alpha, a^n \rangle = I.\mu^n.T$.

The translated series is then $\eta/u = \eta - \langle \overset{\circ}{\eta}, u\rangle$. Unfortunately, the set of these translated series can be infinite. We shall modify the series η/u to avoid this. We compute R, the set of states that belong to cycles of maximum weight. Q is divided into two parts :

- Elements of R or descendants from these,
- Other elements which form a set G.

Let N be the value computed in the decidability algorithm. If $p \in G$, for all $k > N$, no computation realizing $\langle \alpha, a^k\rangle$ goes through p after the N-th letter. Else, it would be in contradiction with the proof of the proposition 18. The series η is modified in order to take care of these useles states. For all $k > N$, if $p \in G$, we set $\langle \eta, a^k\rangle_p =$ -∞.
We can show that the set $\{\eta/u \mid u \in A^*\}$ is then finite.
We define $\mathcal{A}_{det} = (S, A, \mathbb{K}, F, J, U)$ with

$$S = \{\eta/u \mid u \in A^*\},$$
$$F : (\eta/u, a, \eta/(ua)) \mapsto \langle \eta/u, a\rangle - \langle \eta/ua, \mathbf{1}_{A^*}\rangle,$$
$$J : \eta/\mathbf{1}_{A^*} \mapsto \langle \overset{\circ}{\eta}, \mathbf{1}_{A^*}\rangle, \quad U : \eta/u \mapsto (\eta/u).T.$$

It is easy to prove that this deterministic automaton realizes α.

7. NON-UNARY ALPHABETS

The decision of the sequentiality of rational series over a non-unary alphabet seems to be much harder. Gaubert has proved that it gives the decision for the limitedness problem (personal communication based on results in (Gaubert, 1996)). On another hand, the decision of the ambiguity gives the decision of the sequentiality; this is a generalization due to (Mohri, 1997) of the method for transducers.

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