Decidability of Regularity and Related Properties of Ground Normal Form Languages*

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Abstract

We study language-theoretical properties of the set of reducible ground terms and its complement - the set of ground normal forms induced by a given rewriting system. As a tool for our analysis we introduce the property of finite irreducibility of a term with respect to a variable and prove it to be decidable. It turns out that this property generalizes numerous interesting properties of the language of ground normal forms. In particular, we show that testing regularity of this language can be reduced to verifying this property. In this way we prove the decidability of the regularity of the set of ground normal forms, the problem mentioned in the list of open problems in rewriting [Dershowitz *et al.*, 1991]. Also, the decidability of the existence of an equivalent ground term rewriting system and some other results are proved.

1 Introduction

Although the term rewriting formalism has been studied for many years, very little is known about language-theoretical properties of term sets induced by term rewriting systems. In particular, in this paper we focus our interest on the properties of the set of reducible ground terms $Red(\mathcal{R})$ and the set of ground normal forms $NF(\mathcal{R})$, where $\mathcal{R}$ is an ordinary (non-conditional, non-equational) term rewriting system. It turns out that these sets, regarded as tree languages [Gécseg and Steinby, 1984], have a very interesting structure that is worth to be studying. In particular, it is useful to relate this structure with classical notions from the formal language theory. Such results would not only be of theoretical importance but also have a practical interest in the application domains of term rewriting systems.

As a tool for our analysis in this paper we introduce a property of finite irreducibility of a term by a given term rewriting system $\mathcal{R}$ with respect to a variable. It means that if we consider the set of all irreducible ground instances of the term, the set of different ground terms substituted for the variable is finite. In this paper we show that we can effectively bound the depth of these ground terms. In other words, given a term t and a variable x, we give a bound depending on $\mathcal{R}$, t, and x such that if t is finitely irreducible with respect to x and $\delta(t)$ is an irreducible ground instance of t, then the depth of $\delta(x)$ is smaller than this bound. Using the bound we reduce the property of finite irreducibility to that of ground reducibility which is known to be decidable [Kapur *et al.*, 1987; Plaisted, 1985; Comon, 1988]. In this way we show the decidability of the finite irreducibility property.

It turns out that this property is closely related to many interesting properties of the language of ground normal forms (or its complement, the set of reducible ground terms). In particular, we show that for a rewriting system $\mathcal{R}$, if the set of reducible ground terms $Red(\mathcal{R})$ is regular then every non-linear term t in $\mathcal{R}$ is finitely irreducible by $\mathcal{R} \setminus \{t\}$ with respect to all its non-linear variables. (In this paper we identify a term rewriting system with the set of left-hand sides of the rules). On the other hand, we show that the latter condition is implied by another one, namely the existence of a left-linear term rewriting system $\mathcal{L}$ such that $Red(\mathcal{R}) = Red(\mathcal{L})$. This "linearizability property" was studied by one of the authors in [Kucherov, 1991] and proved equivalent to the regularity of $Red(\mathcal{R})$. In this paper we give a shorter proof of this result that uses a well-known Ramsey's theorem. Combining these results with the decidability of finite irreducibility, we prove the decidability of the regularity of ground normal form languages. This solves the problem 7 of the list of open problems in rewriting [Dershowitz *et al.*, 1991] (see also [Gilleron, 1990]).

Using similar ideas we show some other decidability results. In particular, we prove that it is decidable whether a term rewriting system has an equivalent [1] ground system. The difference with the previous problem is that *all* variables and not only the non-linear ones must be substituted by finite sets of ground terms. Also, we show how the results obtained imply the decidability of finiteness of the set of ground

[1]In this paper we call two rewriting systems equivalent if they have the same set of reducible ground terms

normal form, the result proved previously in [Kapur *et al.*, 1987; Plaisted, 1985].

The paper is organized as follows. Section 2 introduces some basic notions and notations. In section 3 we give the definition of the finite irreducibility property and then use it to express a necessary condition for the regularity of the ground normal form language. Using these results we show that the regularity property is equivalent to the existence of an equivalent linear rewriting system. In section 4 we give a bound that restricts the size of substitution terms in the case when the finite irreducibility property holds and we prove that the bound verifies the desired property. Then we prove the decidability result for the finite irreducibility property. In section 5 we apply these results to prove the decidability of regularity of ground normal form languages. As other applications, we obtain the decidability results for the existence of an equivalent ground term rewriting system and for the finiteness of the set of ground normal forms. Section 6 concludes the paper with some final remarks and observations.

2 Preliminaries

We use standard basic notions of term rewriting system theory. $T_\Sigma(X)$ stands for the set of (finite, first-order) terms over a finite signature Σ and an enumerable set of variables X. $\mathcal{V}ar(t) \subseteq X$ is the set of variables in $t \in T_\Sigma(X)$. T_Σ denotes the set of ground terms over Σ that will also be naturally treated as finite labeled trees. For $t \in T_\Sigma(X)$, $\mathcal{P}os(t)$ denotes the set of positions in t defined in the usual way as sequences of natural numbers and $\mathcal{VP}os(t) = \{\pi \in \mathcal{P}os(t) | t|\pi \in X\}$ is the set of variable positions in t. By ε we denote the empty sequence that corresponds to the root position. For $\pi_1, \pi_2 \in \mathcal{P}os(t)$, $\pi_1 \succeq \pi_2$ iff π_2 is a prefix of π_1, and $\pi_1 \succ \pi_2$ iff $\pi_1 \succeq \pi_2$ and $\pi_2 \not\succeq \pi_1$. $\pi \cdot \tau$ denotes the concatenation of π and τ. As usual, for $\pi \in \mathcal{P}os(t)$, $t|_\pi$ is a subterm of t at π and $t[\pi \leftarrow s]$ is the result of replacement of $t|_\pi$ by s in t.

A variable $x \in \mathcal{V}ar(t)$ is said to be linear in t if there exists only one position $\pi \in \mathcal{P}os(t)$ such that $t|_\pi = x$ and is said to be non-linear in t otherwise. A term $t \in T_\Sigma(X)$ is linear if all its variables are linear and is non-linear otherwise.

Given $\tau \in \mathcal{P}os(t)$, $|\tau|$ stands for the length of τ. For $t \in T_\Sigma(X)$ and $S \subseteq T_\Sigma(X)$, the depth of t and S, denoted by $\|t\|$ and $\|S\|$, is defined by $\|t\| = max\{|\tau| | \tau \in \mathcal{P}os(t)\}$ and $\|S\| = max\{\|t\| \, |t \in S\}$. Substitutions and ground substitutions are defined in the usual way.

We will deal with ordinary term rewriting systems defined as a finite set of rules $t \to s$, where $t, s \in T_\Sigma(X)$ and $\mathcal{V}ar(s) \subseteq \mathcal{V}ar(t)$. The only property of term rewriting systems we will be concerned with in this paper is their "reduction power", that is the set of reducible ground terms. For a rewriting system $\mathcal{R}$, a ground term $g \in T_\Sigma$ is ($\mathcal{R}$-)reducible if there exist $\pi \in \mathcal{P}os(g)$ and a rule $t \to s$ in $\mathcal{R}$ such that $g|_\pi$ is a ground instance of t, that is $g|_\pi = \delta(t)$ for some ground substitution δ. Thus, we will identify a term rewriting system $\mathcal{R}$ with the set of its left-hand sides and we will freely mix up term rewriting systems and finite term sets. For example, we will say "linear rewriting system" or simply "linear set" instead of "left-linear rewriting system".

Given a term rewriting system $\mathcal{R}$, a term $t \in T_\Sigma(X)$ is called ($\mathcal{R}$-)ground reducible (or ground reducible by $\mathcal{R}$) iff for each ground substitution δ, $\delta(t)$ is ($\mathcal{R}$-)reducible. By $Gr(\mathcal{R})$ we denote the set of ground instances of $\mathcal{R}$ and by $Red(\mathcal{R})$ the set of $\mathcal{R}$-reducible ground terms. Thus, t is $\mathcal{R}$-ground reducible iff $Gr(\{t\}) \subseteq Red(\mathcal{R})$. $NF(\mathcal{R})$ stands for the set of $\mathcal{R}$-irreducible ground terms (ground normal forms), i.e. $NF(\mathcal{R}) = T_\Sigma \setminus Red(\mathcal{R})$.

3 Finite Irreducibility, Regularity, and Linearity

A well-known problem related to the properties of $Red(\mathcal{R})$ is the ground reducibility problem that was proved decidable by several authors in the mid eighties. It consists in testing whether all ground instances of a given term t are reducible by a rewriting system $\mathcal{R}$, or, formally, if $Gr(\{t\}) \subseteq Red(\mathcal{R})$ holds. From the language-theoretical point of view the ground reducibility problem is equivalent to the inclusion problem for the languages $Red(\mathcal{R})$.

Theorem 1 (generalized ground-reducibility problem) [Kapur *et al.*, 1987; Plaisted, 1985; Comon, 1988] *It is decidable whether $Red(\mathcal{R}_1) \subseteq Red(\mathcal{R}_2)$ for given arbitrary term rewriting systems $\mathcal{R}_1, \mathcal{R}_2$.*

Another known result is the decidability of finiteness of $NF(\mathcal{R})$. Note that $Red(\mathcal{R})$ is always infinite provided that $\mathcal{R}$ is not empty and T_Σ is infinite.

Theorem 2 [Kapur *et al.*, 1987; Kounalis, 1990a] *For an arbitrary rewriting system $\mathcal{R}$, it is decidable whether $NF(\mathcal{R})$ is finite.*

In this paper we are concerned with the regularity property of $Red(\mathcal{R})$ (equivalently, $NF(\mathcal{R})$), where $Red(\mathcal{R})$ is regarded as a tree language. More specifically, our objective is to prove the decidability of regularity of $Red(\mathcal{R})$. First we recall the definitions of finite tree automaton and regular tree language. The following definition of tree automaton follows [Gécseg and Steinby, 1984].

Definition 1
- *Given a signature Σ, a bottom-up tree automaton $\mathcal{A}$ is a finite Σ-algebra $A = (Q, \Sigma)$, where elements of the finite carrier Q are called* states, *together with a distinguished subset $Q_{fin} \subseteq Q$ of* final states.
- *The language of ground terms $L \subseteq T_\Sigma$* recognized by $\mathcal{A}$ *is defined by $L = \{t \in T_\Sigma | t^A \in Q_{fin}\}$, where t^A denotes the interpretation of t in the algebra A. In what follows we denote t^A by $\mathcal{A}(t)$.*
- *A language $L \subseteq T_\Sigma$ is called* regular *(or recognizable) iff there exists an automaton $\mathcal{A}$ over Σ that recognizes L.*

In this section we are going to express the regularity property of $Red(\mathcal{R})$ in terms of other "more syntactic" properties that are easier to test. In particular, we introduce a property of *finite irreducibility* of a term with respect to a variable.

Definition 2 *Let $\mathcal{R}$ be a rewriting system and $t \in T_\Sigma(X)$. t is said to be* finitely irreducible by $\mathcal{R}$ with respect to a variable $x \in \mathcal{V}ar(t)$ *iff there does not exist an infinite sequence of ground instances $\{\delta_1(t), \delta_2(t), \delta_3(t), \ldots\} \subseteq NF(\mathcal{R})$ such that*

- *for every $i \geq 1$, $\delta_i(t)$ contains no proper subterm that is an instance of t,*
- *the set $\{\delta_1(x), \delta_2(x), \delta_3(x), \ldots\}$ is infinite.*

A property complementary to finite irreducibility was called transnormality in [Kounalis, 1990b]. The first condition in the definition can be interpreted by treating t as a special rewrite rule that can be applied to every position of $\sigma(t)$ but the root position. To give a simple example, suppose $\Sigma = \{f, h, a\}$, $\mathcal{R} = \{h(x)\}$, and $t = f(y, y)$. Then t is finitely irreducible with respect to y, but this would not be the case if the first condition had been dropped. It should be noted that this condition is purely technical, and the results of this paper concerning the analysis of finite irreducibility property (cf section 4) are valid both with and without this condition. In fact, the condition is natural for some application but is not needed for the others (cf section 5). In the sequel we will assume this condition to be present and we will mention explicitly when it is not taken into account.

Now we relate the property of regularity of the set of reducible ground terms to that of finite irreducibility. It is the latter property that we will actually test afterwards. The results of the rest of the section are based on the results proved by one of the authors in [Kucherov, 1991] without using explicitly the finite irreducibility property. Furthermore, we present here a new proof of the main proposition (theorem 4 below) that uses a more general technique. In particular, the well-known theorem of Ramsey is used.

We need the following technical lemma.

Lemma 1 *Let $t \in T_\Sigma(X)$. Let $g_1, g_2, g_3 \in T_\Sigma$, and π be a position that belongs to $\mathcal{P}os(g_1), \mathcal{P}os(g_2)$, and $\mathcal{P}os(g_3)$. If $g_1[\pi \leftarrow g_2|_\pi]$, $g_1[\pi \leftarrow g_3|_\pi]$, $g_2[\pi \leftarrow g_3|_\pi]$ are instances of t, then g_2 is also an instance of t.*

Proof: We proceed by case analysis of the location of π with respect to t.

Suppose that π is "outside" t, that is there exists $\tau \in \mathcal{VP}os(t)$ such that $\tau \preceq \pi$. Assume that $t|_\tau = x$ and $\pi = \tau \cdot \nu$. If x is linear in t, then since $g_2[\pi \leftarrow g_3|_\pi]$ is an instance of t, then g_2 is also an instance of t. If x is non-linear in t, then since $g_1[\pi \leftarrow g_2|_\pi]$ and $g_1[\pi \leftarrow g_3|_\pi]$ are both instances of t, we conclude that $g_1[\pi \leftarrow g_2|_\pi]|_\tau = g_1[\pi \leftarrow g_3|_\pi]|_\tau$ and therefore $g_2|_\pi = g_3|_\pi$. As soon as $g_2[\pi \leftarrow g_3|_\pi]$ is an instance of t, g_2 is also an instance of t.

Now suppose that π is "within" t, that is $\pi \in \mathcal{P}os(t) \setminus \mathcal{VP}os(t)$. Since $g_1[\pi \leftarrow g_2|_\pi]$ is an instance of t, then $g_2|_\pi$ is an instance of $t|_\pi$. Denote $\mathcal{VP}os_\pi(t) = \{\tau | \tau \in \mathcal{VP}os(t),\ \pi \prec \tau\}$. If there is no non-linear variable $x \in \mathcal{V}ar(t)$ located at some position in $\mathcal{VP}os_\pi(t)$ as well as at some position in $\mathcal{VP}os(t) \setminus \mathcal{VP}os_\pi(t)$, then since $g_2[\pi \leftarrow g_3|_\pi]$ is an instance of t, g_2 is also an instance of t. Now assume that for some variable $x \in \mathcal{V}ar(t)$, there exist $\tau_1 \in \mathcal{VP}os_\pi(t), \tau_2 \in \mathcal{VP}os(t) \setminus \mathcal{VP}os_\pi(t)$ such that $t|_{\tau_1} = t|_{\tau_2} = x$. We prove that $g_2|_{\tau_1} = g_2|_{\tau_2}$. Since $g_1[\pi \leftarrow g_2|_\pi]$ and $g_1[\pi \leftarrow g_3|_\pi]$ both are instances of t, we conclude that $g_1|_{\tau_2} = g_2|_{\tau_1}$ and $g_1|_{\tau_2} = g_3|_{\tau_1}$. Hence, $g_2|_{\tau_1} = g_3|_{\tau_1}$.

Since $g_2[\pi \leftarrow g_3|_\pi]$ is an instance of t, then $g_2|_{\tau_2} = g_3|_{\tau_1}$ and hence $g_2|_{\tau_1} = g_2|_{\tau_2}$. Since x, τ_1, τ_2 were chosen arbitrary, we conclude that g_2 is an instance of t. $\square$

The lemma above refines lemma A.7 from [Kapur *et al.*, 1987] insofar as the additional unnecessary conditions are removed from the latter (no additional condition is imposed on the position π and the terms g_i are not assumed to be irreducible, otherwise the proof goes along the same lines). Lemma 1 plays an important role as it gives a link between reasoning on terms and combinatorial reasoning, the latter being necessary for proving the results of this paper.

In the combinatorial part of the proof below we use the following "infinite version" of the well-known Ramsey theorem (see [Graham *et al.*, 1980, page 16]).

Theorem 3 (Ramsey's theorem, infinite version) *Let I be an infinite set and n a natural number. Denote by $\mathcal{P}_n(I)$ the set of all n-element subsets of I and assume that $\mathcal{P}_n(I) = P_1 \uplus P_2$. Then there exists an infinite subset $J \subseteq I$ such that either $\mathcal{P}_n(J) \subseteq P_1$ or $\mathcal{P}_n(J) \subseteq P_2$.*

Now we are in position to prove the main result of this section.

Theorem 4 *For a rewriting system $\mathcal{R}$, if the set $Red(\mathcal{R})$ is a regular tree language, then every non-linear term $t \in \mathcal{R}$ is finitely irreducible by $\mathcal{R}\backslash\{t\}$ with respect to all its non-linear variables $x \in \mathcal{V}ar(t)$.*

Proof: By contradiction, assume that there exists a non-linear term $t \in \mathcal{R}$ and a non-linear variable $x \in \mathcal{V}ar(t)$ such that t is not finitely irreducible by $\mathcal{R}\backslash\{t\}$ with respect to x. By definition, there exists an infinite set of ground substitutions $\{\delta_1, \delta_2, \ldots\}$ such that for every $i,\ i \geq 1,\ \delta_i(t)$ is irreducible by $\mathcal{R}\backslash\{t\}$ and does not contain an instance of t as a proper subterm, and the set $\{\delta_1(x), \delta_2(x), \ldots\}$ is infinite. Without loss of generality we assume that for $i, j \geq 1,\ i \neq j,\ \delta_i(x) \neq \delta_j(x)$.

Assume now that $\mathcal{A}$ is an automaton that recognizes $Red(\mathcal{R})$. Since the set of states Q of $\mathcal{A}$ is finite, and $\{\delta_1(x), \delta_2(x), \ldots\}$ is infinite, assume without loss of generality that there exists $q_0 \in Q$ such that $\mathcal{A}(\delta_i(x)) = q_0$ for all $i,\ i \geq 1$.

Let π be a position of x in t. Denote $g_{ij} = \delta_i(t)[\pi \leftarrow \delta_j(x)]$. Since $\mathcal{A}(\delta_i(x)) = \mathcal{A}(\delta_j(x))$, then $\mathcal{A}(\delta_i(t)) = \mathcal{A}(g_{ij})$ and thus g_{ij} must be reducible. We now contradict this by proving that there exist $i, j \geq 1,\ i \neq j$ such that g_{ij} is not reducible. Actually, the statement we will prove is much stronger. We show that there exists an infinite subset of indexes $\bar{J}$ such that g_{ij} is not reducible for every $i, j \in \bar{J},\ i < j$.

We observe that since $\delta_i(x) \neq \delta_j(x)$ for $i \neq j$, and x is non-linear in t, then g_{ij} is not an instance of t. Also, g_{ij} can be potentially reducible only at a position above π. Consider a position $\tau,\ \tau \prec \pi$ and a rule $s \in \mathcal{R}$. We assume that $s \neq t$ whenever $\tau = \varepsilon$. It follows from lemma 1 that if for distinct i_1, i_2, i_3 the terms $g_{i_1 i_2}, g_{i_1 i_3}, g_{i_2 i_3}$ are reducible by s at τ, then $g_{i_2 i_2} = \delta_{i_2}(t)$ is also reducible by s at τ and this would contradict the assumption that every $\delta_i(t)$ is reducible only by t at root. Therefore, given s and τ as above, among every three pairs $(i_1, i_2), (i_1, i_3), (i_2, i_3)$ there exists at least one such for which the corresponding term g_{ij} is not reducible by s at τ.

Now we apply theorem 3 with $n = 2$ and I being the set of natural numbers. We identify uniquely every pair (i, j), $i < j$ with the 2-element subset $\{i, j\}$, and we split the set of all such pairs into two subsets P_1, P_2 in the following way. A pair (i, j) belongs to P_2 iff g_{ij} is reducible by s at τ, otherwise it belongs to P_1. By theorem 3 there exists an infinite set of indexes J such that either $\mathcal{P}_2(J) \subseteq P_1$ or $\mathcal{P}_2(J) \subseteq P_2$. By the above remark, the latter alternative is impossible even for a 3-element set J. Thus, we get an infinite set of indexes J such that g_{ij} is irreducible by s at τ for all $i, j \in J, i < j$. Since the number of rules in $\mathcal{R}$ and the number of positions in t are finite, by applying theorem 3 iteratively for every $s \in \mathcal{R}, \tau \in \mathcal{P}os(t)$ (except for $s = t, \tau = \varepsilon$), we finally get an infinite set of indexes $\bar{J}$ such that g_{ij} is $\mathcal{R}$-irreducible for all $i, j \in \bar{J}$, $i < j$. Thus, we get a contradiction with the fact that every g_{ij} belongs to the language recognized by $\mathcal{A}$. □

Now we show that for a given rewrite system $\mathcal{R}$, the language $Red(\mathcal{R})$ is regular if and only if there exists a linear rewriting system $\mathcal{L}$ such that $Red(\mathcal{R}) = Red(\mathcal{L})$. The "if part" follows immediately from the following theorem proved in [Gallier and Book, 1985].

Theorem 5 ([Gallier and Book, 1985]) *If $\mathcal{L}$ is a linear rewriting system, then $Red(\mathcal{L})$ is a regular tree language.*

Thus, we concentrate on proving the existence of a linear system $\mathcal{L}$ equivalent to $\mathcal{R}$ provided that $Red(\mathcal{R})$ is regular. Moreover, we show that $\mathcal{L}$ may be constructed by instantiating the non-linear variables of $\mathcal{R}$.

Definition 3 *Given finite sets $\mathcal{R} \subseteq T_\Sigma(X)$, $\mathcal{L} \subseteq T_\Sigma(X)$, $\mathcal{L}$ is said to be an* instantiation *of $\mathcal{R}$ iff every term of $\mathcal{L}$ is an instance of some term of $\mathcal{R}$. We call $\mathcal{L}$* linear (resp. ground,...) instantiation *of $\mathcal{R}$ iff $\mathcal{L}$ is a linear (resp. ground,...) set in addition.*

In other words, $\mathcal{L}$ is an instantiation of $\mathcal{R}$ iff $\mathcal{L}$ can be obtained by instantiation or deletion of terms of $\mathcal{R}$. Furthermore, if $\mathcal{L}$ is a linear instantiation of $\mathcal{R}$ then every non-linear term in $\mathcal{R}$ either is deleted or has its non-linear variables substituted by a finite number of ground terms. The following lemma relates the condition of finite irreducibility and the existence of an equivalent linear instantiation.

Lemma 2 *If for a rewriting system $\mathcal{R}$, there exists a linear instantiation $\mathcal{L}$ of $\mathcal{R}$ such that $Red(\mathcal{R}) = Red(\mathcal{L})$, then every non-linear term $t \in \mathcal{R}$ is finitely irreducible by $\mathcal{R} \backslash \{t\}$ with respect to all its non-linear variables $x \in \mathcal{V}ar(t)$.*

Proof: By contradiction, if a non-linear term $t \in \mathcal{R}$ is not finitely irreducible by $\mathcal{R} \backslash \{t\}$ with respect to some non-linear variable $x \in \mathcal{V}ar(t)$, then there is an infinite sequence $\{\delta_1(t), \delta_2(t), \delta_3(t), \ldots\}$ of ground terms reducible only by t and only at the root position, and the set $\{\delta_1(x), \delta_2(x), \delta_3(x), \ldots\}$ is infinite. Clearly, if we replaced x by any finite set of ground terms, infinitely many terms from $\{\delta_1(t), \delta_2(t), \delta_3(t), \ldots\}$ would become irreducible, and thus the set of reducible ground terms would be changed. Also, t cannot be deleted from $\mathcal{R}$. Thus, an equivalent linear instantiation does not exist and we have a contradiction. □

We note that the converse of the lemma above does not hold [Hofbauer and Huber, 1992b]. The reason is that the linearizability condition is a property of the whole rewriting system that cannot be decomposed into a sum of local conditions imposed on each of its terms. For example, suppose $\mathcal{R} = \{h(f(x,y)), f(f(x,y),z), f(x, f(y,z)), f(x,x), f(h(x),h(x))\}$, and the signature consists of f, h, and a constant a. It is easy to see that $Red(\mathcal{R}) = T_\Sigma \setminus NF(\mathcal{R})$, where $NF(\mathcal{R}) = \{h^i(a) | i \geq 0\} \cup \{f(h^i(a), h^j(a)) | i,j \geq 0,\ i \neq j\}$. The term $f(x,x)$ can be replaced by $f(a,a)$ without changing $Red(\mathcal{R})$. Also, $f(h(x),h(x))$ can be simply dropped since it is subsumed by $f(x,x)$. However, both terms cannot be instantiated simultaneously, and thus the system cannot be linearized as a whole.

Thus, linearizability of a system is a stronger condition than finite irreducibility of its elements with respect to the non-linear variables. According to theorem 4, the latter condition is also implied by the regularity of the language of reducible ground terms. The following theorem proves equivalent the regularity and the linearizability.

Theorem 6 ([Kucherov, 1991]) *For a rewriting system $\mathcal{R}$, $Red(\mathcal{R})$ is a regular tree language iff there exists a linear instantiation $\mathcal{L}$ of $\mathcal{R}$ such that $Red(\mathcal{R}) = Red(\mathcal{L})$.*

Proof: The "if part" follows immediately from theorem 5. Now suppose that $Red(\mathcal{R})$ is regular. Take a non-linear term $t \in \mathcal{R}$ and a non-linear variable $x \in \mathcal{V}ar(t)$. By theorem 4, t is finitely irreducible by $\mathcal{R} \setminus \{t\}$ with respect to x. By definition 2 we can transform t into $\sigma_1(t), \ldots, \sigma_n(t)$ by replacing x by a finite number of ground terms such that if $\delta(t)$ is a ground instance of t and no instance of t is a proper subterm of $\delta(t)$, then $\delta(t)$ is either reducible by $\mathcal{R} \setminus \{t\}$ or is an instance of $\sigma_i(t)$ for some $i, 1 \leq i \leq n$. Hence, it is easy to see that replacing t by $\sigma_1(t), \ldots, \sigma_n(t)$ does not affect the set of reducible ground terms. On the other hand, the transformation eliminates one non-linear variable. By iterating this transformation for each non-linear term and each non-linear variable we obtain a linear system $\mathcal{L}$ that satisfies the theorem. □

Finally, we note that if we have a procedure for testing finite irreducibility and computing a corresponding set of replacement terms, then the above theorem gives an effective procedure for testing the existence of $\mathcal{L}$ and computing it. Checking finite irreducibility is the subject of the following section.

4 Substitution Bound for Deciding Finite Irreducibility

In this section we prove the decidability of the property of finite irreducibility with respect to a variable. Given a rewriting system $\mathcal{R}$, a term t, and a variable $x \in \mathcal{V}ar(t)$, we give a bound on the depth of $\delta(x)$, where $\delta(t)$ is irreducible and t is finitely irreducible with respect to x. Using this bound we reduce the property of finite irreducibility to that of ground reducibility that is known to be decidable.

The technique that we use to construct the bound is similar to that of [Kapur *et al.*, 1987]. The idea is to give a bound such that if $\delta(x)$ exceeds the bound, then a larger substitution σ can be constructed such that $\sigma(t)$ is irreducible. Obviously, in this way we get an infinite number of such substitutions. In [Kapur *et al.*, 1987] the attention is focused on constructing another bound that is in a sense complementary to ours. Its meaning is exactly the opposite: if $\delta(x)$ exceeds the bound then a *smaller* substitution σ exists such that $\sigma(t)$ is irreducible. However, the possibility of constructing a bound in the sense of this paper was indicated in [Kapur *et al.*, 1987] too, and the idea of the construction was sketched. In section 6 we will make further remarks on the relation between the two bounds.

Assume that we are given a term rewriting system $\mathcal{R}$, a term $t \in T_\Sigma(X)$ and a variable $x \in \mathcal{V}ar(t)$. Now we give a number $B(\mathcal{R}, t, x)$ that we use in the proof of the main theorem below. Actually, we will show that it bounds the depth of $\delta(x)$ where $\delta(t)$ is an $\mathcal{R}$-irreducible instance of t, and t is finitely irreducible by $\mathcal{R}$ with respect to x.

Let $card(\mathcal{R})$ denote the number of terms in $\mathcal{R}$, $maxarity(\Sigma)$ the maximal arity of function symbols in Σ, $nocc(x,t)$ and $depth(x,t)$ respectively the number and the maximal depth of positions of x in t. Suppose $C(\mathcal{R}) = 3\times(card(\mathcal{R})\times\|\mathcal{R}\|)!$, $D(\mathcal{R}) = C(\mathcal{R}) \times maxarity(\Sigma)^{\|\mathcal{R}\|}$, $A(\mathcal{R}, t, x) = D(\mathcal{R}) \times (card(\mathcal{R}) \times nocc(t,x) \times depth(x,t))$, $B(\mathcal{R}, t, x) = \|\mathcal{R}\| \times A(\mathcal{R}, t, x)$. Now we prove the following main theorem.

Theorem 7 *Let a term rewriting system $\mathcal{R}$, a term $t \in T_\Sigma(X)$ and a variable $x \in \mathcal{V}ar(t)$ be given. A number $B(\mathcal{R}, t, x)$ can be computed that verifies the following condition. If there exists a substitution δ such that $\delta(t)$ is $\mathcal{R}$-irreducible and $\|\delta(x)\| > B(\mathcal{R}, t, x)$, then there exists a substitution σ such that $\sigma(t)$ is irreducible and $\|\sigma(x)\| > \|\delta(x)\|$.*

We prove that the number $B(\mathcal{R}, t, x)$ defined in the above formula satisfies the theorem. Before giving the proof we give the following technical proposition.

Proposition 1 *Let $g \in T_\Sigma$, $t \in T_\Sigma(X)$, and g is not an instance of t.*

(i) Let $\pi \in \mathcal{P}os(g)$ and $|\pi| \geq \|t\|$. If $g_1 \in T_\Sigma$, $g' = g[\pi \leftarrow g_1]$, and $\|g'\| \geq \|g\| + \|t\|$, then g' is not an instance of t.

(ii) Let $\pi_1, \ldots, \pi_n \in \mathcal{P}os(g)$ and $|\pi_i| \geq \|t\|$ for every $i, 1 \leq i \leq n$. Assume that $g|_{\pi_1} = \ldots = g|_{\pi_n}$. Assume that $g_1, g_2 \in T_\Sigma$, $g' = g[\pi_1 \leftarrow g_1, \ldots, \pi_n \leftarrow g_1], g'' = g[\pi_1 \leftarrow g_2, \ldots, \pi_n \leftarrow g_2]$. If both g' and g'' are instances of t, then $g_1 = g_2$.

Proof: (i) We suppose that g and t have the same function symbol at every position from $\mathcal{P}os(t) \setminus \mathcal{VP}os(t)$. Otherwise the statement is trivial. Consider $\tau \in \mathcal{VP}os(t)$ such that $\tau \prec \pi$. Let $x = t|_\tau$. If x is linear in t, then the result is obvious. Assume that x is non-linear and $\nu \in \mathcal{VP}os(t)$ is another position of x. From $\|g'\| \geq \|g\| + \|t\|$ we conclude that τ belongs to a longest path in g'. Hence, $\|g'|_\tau\| \geq \|g'\| - \|t\| \geq \|g\|$. On the other hand, $\|g'|_\nu\| = \|g|_\nu\| < \|g\|$. Therefore, $g'|_\tau \neq g'|_\nu$, and g' is not an instance of t.

(ii) This part generalizes lemma A.1 from [Kapur *et al.*, 1987]. Assume that g' and g'' are instances of t. Consider $\tau_1, \ldots, \tau_n \in \mathcal{VP}os(t)$ such that $\tau_i \prec \pi_i$ for every $i, 1 \leq i \leq n$, and let $x_i = t|_{\tau_i}$. Clearly, if each of $x_1, \ldots, x_n$ is linear in t, then

g must be also an instance of t which is a contradiction. Hence, among $x_1, \ldots, x_n$ there is a non-linear variable. Moreover, among $x_1, \ldots, x_n$ there is a non-linear variable that occurs also at a position different from $\tau_1, \ldots, \tau_n$. This follows from the fact that equal subterms in g' correspond to equal subterms in g, and if such a variable does not exist, then g must be also an instance of t. Assume that x_i is a non-linear variable which occurs at a position $\nu \in \mathcal{VP}os(t)$, and $\nu \notin \{\tau_1, \ldots, \tau_n\}$. Obvously, $g'|_\nu = g''|_\nu = g|_\nu$. On the other hand, $g'|_\nu = g'|_{\tau_i}$, and $g''|_\nu = g''|_{\tau_i}$. Hence, $g'|_{\tau_i} = g''|_{\tau_i}$, and $g_1 = g'|_{\pi_i} = g''|_{\pi_i} = g_2$. □

Now we are ready to give the proof.

Proof of theorem 7: Consider the term $\delta(x)$ and a path of maximal length in it. Since it is longer than $B(\mathcal{R}, t, x)$, find on this path $A(\mathcal{R}, t, x) + 1$ positions $\tau_0, \ldots, \tau_{A(\mathcal{R},t,x)}$ such that $\tau_{i-1} \prec \tau_i$ and $|\tau_i| - |\tau_{i-1}| = \|\mathcal{R}\|$ for every $i, 1 \leq i \leq A(\mathcal{R}, t, x)$. For $i, j, 1 \leq i < j \leq A(\mathcal{R}, t, x)$, denote by δ_{ij} the substitution defined by $\delta_{ij}(x) = \delta(x)[\tau_j \leftarrow \delta(x)|_{\tau_i}]$ and $\delta_{ij}(y) = \delta(y)$ for $y \neq x$. Since τ_i, τ_j belong to this longest path, $\|\delta_{ij}(x)\| > \|\delta(x)\|$. We show that a substitution σ verifying the theorem can be chosen among δ_{ij}. Similar to [Kapur *et al.*, 1987] we distinguish global and local reducibility.

Definition 4 *A term $g \in T_\Sigma$ is said to be locally (respectively globally) reducible with respect to a position $\tau \in \mathcal{P}os(g)$ iff it is reducible at some position $\pi \prec \tau$ such that $|\tau| - |\pi| \leq \|\mathcal{R}\|$ (respectively $|\tau| - |\pi| > \|\mathcal{R}\|$).*

The proof consists of three parts. In the first part we show that each $\delta_{ij}(x)$ is not globally reducible with respect to τ_j. In the second part we select a subset of pairs (i, j) such that $\delta_{ij}(t)$ is not reducible at any position preceding a position of x in t. Finally, we prove that among the remaining pairs there exists a pair (i, j) such that $\delta_{ij}(x)$ is not locally reducible with respect to τ_j. Clearly, these three parts cover all possibilities for $\delta_{ij}(t)$ to be reducible and prove the theorem.

Part 1. Consider a pair (i, j), $1 \leq i < j \leq A(\mathcal{R}, t, x)$ and a position $\nu \prec \tau_j$ such that $|\tau_j| - |\nu| > \|\mathcal{R}\|$. Denote $g = \delta(x)|_\nu$ and $g' = \delta_{ij}(x)|_\nu$. Since ν, τ_j are both on the longest path and $|\tau_j| - |\tau_i| \geq \|\mathcal{R}\|$, then $\|g'\| - \|g\| \geq \|\mathcal{R}\|$. Since g is irreducible, then $g \notin Gr(\mathcal{R})$ and by proposition 1(i), $g' \notin Gr(\mathcal{R})$. Since ν and (i, j) were chosen arbitrary, we conclude that for each (i, j), $1 \leq i < j \leq A(\mathcal{R}, t, x)$, $\delta_{ij}(x)$ is not reducible at any $\nu \prec \tau_j$, $|\tau_j| - |\nu| > \|\mathcal{R}\|$, i.e. is not globally reducible with respect to τ_j.

Part 2. Take a position $\nu \in \mathcal{VP}os(t)$ of x in t and consider any position $\tau \prec \nu$. Note that the subterm $t|_\tau$ may have several positions of x. Take $s \in \mathcal{R}$. Let $\pi = \nu \cdot \tau_1$. Since $\delta_{ij}(x)|_\pi \neq \delta_{ik}(x)|_\pi$ for $j \neq k$, by proposition 1(ii) for every $i \geq 1$, there exists at most one $j > i$ such that $\delta_{ij}(t)$ is reducible by s at τ. Since the number of positions of x in t is $nocc(t, x)$ and the depth of any of them is bounded by $depth(x, t)$, there are at most $nocc(t, x) \times depth(x, t)$ possible values of τ. Consequently, given i, $i \geq 1$, there are at most $card(\mathcal{R}) \times nocc(t, x) \times depth(x, t)$ indexes j, $j > i$ such that $\delta_{ij}(t)$ is reducible at some $\tau \in \mathcal{P}os(t)$ by some $s \in \mathcal{R}$.

Now we construct a subsequence $\{l_1,\ldots,l_{D(\mathcal{R})}\} \subset \{1,\ldots,A(\mathcal{R},t,x)\}$ through the following "diagonalization procedure". Take $l_1 = 1$. Delete from the sequence $\{2,\ldots,A(\mathcal{R},t,x)\}$ those j for which $\delta_{l_1 j}(t)$ is reducible at some $\tau \in \mathcal{P}os(t)$. By the above remark, we have deleted at most $card(\mathcal{R}) \times nocc(t,x) \times depth(x,t)$ numbers. Take l_2 to be the smallest element in the resulting sequence, and apply the same deleting procedure to the rest of it. By iterating this procedure $D(\mathcal{R})$ times we construct a subsequence $\{l_1,\ldots,l_{D(\mathcal{R})}\} \subset \{1,\ldots,A(\mathcal{R},t,x)\}$. Note that since at every step we delete at most $card(\mathcal{R}) \times nocc(t,x) \times depth(x,t)$ indexes, and $A(\mathcal{R},t,x) = D(\mathcal{R}) \times (card(\mathcal{R}) \times nocc(t,x) \times depth(x,t))$, the procedure can be applied $D(\mathcal{R})$ times and therefore is correctly defined. Note finally that by construction for every $l', l'' \in \{l_1,\ldots,l_{D(\mathcal{R})}\},\ l' < l''$, the term $\delta_{l'l''}(t)$ is not reducible at any $\tau \in \mathcal{P}os(t)$ by any $s \in \mathcal{R}$.

Part 3. Now we observe that the positions $\tau_{l_1},\ldots,\tau_{l_{D(\mathcal{R})}}$ have at most $maxarity(\Sigma)^{\|\mathcal{R}\|}$ different suffixes of the length $\|\mathcal{R}\|$, and hence among $l_1,\ldots,l_{D(\mathcal{R})}$ there are at least $C(\mathcal{R})$ indexes $k_1,\ldots,k_{C(\mathcal{R})}$ such that for some $\rho,\ |\rho| = \|\mathcal{R}\|,\ \tau_{k_i} = \tau'_{k_i} \cdot \rho$ for every $i,\ 1 \le i \le C(\mathcal{R})$. Let $\pi_n^{(m)},\ n \in \{k_1,\ldots,k_{C(\mathcal{R})}\},\ 1 \le m \le \|\mathcal{R}\|$, be the positions defined by $\pi_n^{(m)} \prec \tau_n,\ |\tau_n| - |\pi_n^{(m)}| = m$. We prove that there exists a pair $k', k'' \in \{k_1,\ldots,k_{C(\mathcal{R})}\}, k' < k''$ such that $\delta_{k'k''}(x)$ is not reducible at any $\pi_{k''}^m,\ 1 \le m \le \|\mathcal{R}\|$ by any $s \in \mathcal{R}$, which also means that $\delta_{k'k''}(x)$ is not locally reducible with respect to $\tau_{k''}$. The proof of this is similar to that of theorem 4 but we use the "finite version" of Ramsey's theorem.

Theorem 8 (Ramsey's theorem, finite version) *Let a finite set I and natural numbers N, n be given. Let $A_1,\ldots,A_N$ be natural numbers, and $A_i \ge 2,\ 1 \le i \le N$. Denote by $\mathcal{P}_n(I)$ the set of all n-element subsets of I and assume that $\mathcal{P}_n(I) = P_1 \uplus \ldots \uplus P_N$. Then there exists a number $R(A_1,\ldots,A_N;n)$ such that if I contains at least $R(A_1,\ldots,A_N;n)$ objects, then there exist $i,\ 1 \le i \le N$ and a subset $J \subseteq I$ such that J contains at least A_i objects, and $\mathcal{P}_n(J) \subseteq P_i$.*

$R(A_1,\ldots,A_N;n)$ are called Ramsey numbers. We are going to apply the theorem with $n = 2, A_1 = \ldots = A_N = 3$. It is known (see, for example, [Constantine, 1987]) that the numbers $R_N = R(\underbrace{3,\ldots,3}_{N};2),\ N \ge 2$ satisfy the recurrence relation $R_2 = 6,\ R_N \le N \times (R_{N-1} - 1) + 2$. Hence, $R_N \le 3 \times N!$.

Now we observe that for a given $s \in \mathcal{R}$ and $m,\ 1 \le m \le \|\mathcal{R}\|$, if $k', k'', k''' \in \{k_1,\ldots,k_{C(\mathcal{R})}\}$ and $k' < k'' < k'''$, then either $\delta_{k'k''}(x)$ is not reducible at $\pi_{k''}^m$ by s, or $\delta_{k'k'''}(x)$ is not reducible at $\pi_{k'''}^m$ by s, or $\delta_{k''k'''}(x)$ is not reducible at $\pi_{k'''}^m$ by s. Otherwise by assuming $g_1 = \delta(x)|_{\pi_{k'''}^m}$, $g_2 = \delta(x)|_{\pi_{k''}^m}$, $g_3 = \delta(x)|_{\pi_{k'}^m}$, and applying lemma 1, we would conclude that $\delta(x)$ is reducible at $\pi_{k''}^m$ by s which is a contradiction.

Now we apply Ramsey's theorem. We identify uniquely every pair (k', k''), $k', k'' \in \{k_1,\ldots,k_{C(\mathcal{R})}\},\ k' < k''$ with the 2-element subset $\{k', k''\}$, and we split the set of all such pairs into $(card(\mathcal{R}) \times \|\mathcal{R}\| + 1)$ subsets $P_0, P_1, \ldots, P_{card(\mathcal{R}) \times \|\mathcal{R}\|}$ in the following way. Each $P_i,\ 1 \le i \le card(\mathcal{R}) \times \|\mathcal{R}\|$ is one-to-one associated with a pair $s, m,\ s \in \mathcal{R}, 1 \leq m \leq \|\mathcal{R}\|$. The way we distribute the pairs among P_i is the following. If P_i corresponds to a pair s, m, then a pair (k', k'') belongs to P_i iff $\delta_{k'k''}(x)$ is reducible by s at $\pi^m_{k''}$. If there are several possibilities to place (k', k''), we choose any of them. If there are no s, m as above such that $\delta_{k'k''}$ is reducible by s at $\pi^m_{k''}$, we place (k', k'') into P_0. If no pair is finally placed in P_0, then since $C(\mathcal{R}) = 3 \times (card(\mathcal{R}) \times \|\mathcal{R}\|)!$, by theorem 8 there exists a 3-element subset $J \subseteq \{k_1, \ldots, k_{C(\mathcal{R})}\}$ such that $\mathcal{P}_2(J) \subseteq P_i$ for some $i,\ 1 \leq i \leq card(\mathcal{R}) \times \|\mathcal{R}\|$. But this contradicts the above remark. Therefore, there exists at least one pair k', k'' such that $\delta_{k'k''}(x)$ is not reducible by any $s \in \mathcal{R}$ at any $\pi^m_{k''}$, $1 \leq m \leq \|\mathcal{R}\|$. Thus, $\delta_{k'k''}(x)$ is not locally reducible with respect to $\tau_{k''}$. This completes the proof.

□

We remark that many ideas in the proof above were borrowed from [Kapur *et al.*, 1987]. Moreover, a slightly simpler proof of part 3 can be given which uses a minor modification of lemma 5.4 from [Kapur *et al.*, 1987]. However, we have preferred to give a longer proof not only in order to make the paper self-contained, but also because it uses Ramsey's theorem which embodies complex combinatorial reasoning of [Kapur *et al.*, 1987]. Also, it was interesting for us to discover that the same "Ramsey's theorem technique" is applicable for proving both principal results of this paper - theorem 4 and theorem 7. We believe that Ramsey's theorem, being a very powerful combinatorial result, can be very fruitful in proving this kind of properties of term sets.

The following corollary adapts theorem 7 to the first condition in the definition of finite irreducibility.

Corollary 1 *Let a term rewriting system $\mathcal{R}$, a term $t \in T_\Sigma(X)$ and a variable $x \in \mathcal{V}ar(t)$ be given. A number $B(\mathcal{R}, t, x)$ can be computed that verifies the following condition. If there exists a substitution δ such that $\delta(t)$ is $\mathcal{R}$-irreducible, $\delta(t)$ has no proper subterm that is an instance of t, and $\|\delta(x)\| > B(\mathcal{R}, t, x)$, then there exists a substitution σ such that $\sigma(t)$ is irreducible, $\|\sigma(x)\| > \|\delta(x)\|$, and $\sigma(t)$ has no proper subterm that is an instance of t.*

Proof: The proof of theorem 7 remains valid but we have to insert t into $\mathcal{R}$ and to treat it as a special rewrite rule that cannot be applied at the root position. This particularity is relevant only to part 2. It is easy to see that the proof of part 2 still works. The only modification is that if $\tau = \varepsilon$, then every rule of $\mathcal{R}$ but t is potentially applicable. Thus, we have one less possibility of reduction and even more freedom in choosing a suitable subsequence of positions.

We have to correct obviously the bound B by taking $card(\mathcal{R}) + 1$ instead of $card(\mathcal{R})$ and $\|\mathcal{R} \cup \{t\}\|$ instead of $\|\mathcal{R}\|$. □

Corollary 2 *Let $B(\mathcal{R}, t, x)$ be a bound verifying the condition of theorem 4 (resp. corollary 1). Assume that $\delta(t)$ is $\mathcal{R}$-irreducible (resp. $\mathcal{R}$-irreducible and has no proper* *subterm that is an instance of t) and $\|\delta(x)\| > B(\mathcal{R}, t, x)$. Then there exists an infinite number of substitutions $\sigma_1, \sigma_2, \ldots$ such that for every i, $i \geq 1$, $\sigma_i(t)$ is $\mathcal{R}$-irreducible (resp. $\mathcal{R}$-irreducible and has no proper subterm that is an instance of t), and the set $\{\sigma_1(x), \sigma_2(x), \ldots\}$ is infinite.*

Proof: By iterating the procedure of constructing a larger substitution described in the proof of theorem 7 (resp. corollary 1), we obtain the required infinite sequence of substitutions. □

In the rest of the paper we will assume that $B(\mathcal{R}, t, x)$ denotes the bound modified according to the proof of corollary 1 unless the contrary is explicitly stated.

Theorem 7 allows us to prove the decidability of finite irreducibility of a term with respect to a variable. Note that the proof uses the ground reducibility property that is known to be decidable [Plaisted, 1985; Kapur *et al.*, 1987; Comon, 1988].

Theorem 9 *It is decidable whether given a rewriting system $\mathcal{R}$, a term $t \in T_\Sigma(X)$ is finitely $\mathcal{R}$-irreducible with respect to a variable $x \in \mathcal{V}ar(t)$.*

Proof: Compute $B(\mathcal{R}, t, x)$ and compute all instances $\sigma_1(t), \ldots, \sigma_K(t)$ such that for every i, $1 \leq i \leq K$,

- $\sigma_i(x) \in T_\Sigma$, and $\sigma_i(y) = y$ for every $y \neq x$,
- $\sigma_i(t)$ is $\mathcal{R}$-irreducible,
- $\|\sigma_i(x)\| \leq B(\mathcal{R}, t, x)$.

We show now that t is finitely $\mathcal{R}$-irreducible with respect to x iff t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\}$. We use an easy observation that t is ground reducible if and only if for every ground instance $\delta(t)$ in which no proper subterm is an instance of t, $\delta(t)$ is reducible.

Assume that t is finitely $\mathcal{R}$-irreducible with respect to x. Let $\delta(t)$ be a ground instance of t that has no instance of t as its proper subterm. If $\|\delta(x)\| \leq B(\mathcal{R}, t, x)$, then by construction of σ_i, $\delta(t)$ is either reducible by $\mathcal{R}$, or is an instance of $\sigma_i(t)$ for some i, $0 \leq i \leq K$. If $\|\delta(x)\| > B(\mathcal{R}, t, x)$, then $\delta(t)$ is reducible since otherwise, by corollary 2, t cannot be finitely irreducible. Thus, $\delta(t)$ is reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\}$ and therefore t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\}$.

Conversely, assume that t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\}$. Consider a ground instance $\delta(t)$ and suppose it contains no proper subterm that is an instance of t. Assume that $\|\delta(x)\| > B(\mathcal{R}, t, x)$. $\delta(t)$ is not reducible by $\sigma_1(t), \ldots, \sigma_K(t)$ at any position different from ε since $\delta(t)$ has no proper subterm that is an instance of t. On the other hand, since $\|\sigma_i(x)\| \leq B(\mathcal{R}, t, x)$, $\delta(t)$ cannot be an instance of $\sigma_1(t), \ldots, \sigma_K(t)$. Therefore, $\delta(t)$ is reducible by $\mathcal{R}$. This proves that t is finitely irreducible by $\mathcal{R}$ with respect to x. □

The proof of theorem 9 gives a decision procedure for testing finite irreducibility of t with respect to x. It consists in computing all substitutions σ with $\sigma(x) \in T_\Sigma$,

$\sigma(x) \leq B(\mathcal{R}, t, x)$, and $\sigma(y) = y$ for $y \neq x$, then selecting out those for which $\sigma(t)$ is $\mathcal{R}$-reducible, and then checking if t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\}$, where $\sigma_1, \ldots, \sigma_K$ are the remaining substitutions.

It should be noted that theorem 9 is still valid if the first condition in the definition 2 is dropped. The decision procedure is now the following. At first, we compute $B(\mathcal{R}, t, x)$ according to theorem 7. Then we compute all instances $\sigma_1(t), \ldots, \sigma_K(t)$ such that for every i, $1 \leq i \leq K$,

- $\sigma_i(y) = y$ for every $y \neq x$,
- $\sigma_i(t)$ is $\mathcal{R}$-irreducible,
- $\sigma_i(x)$ contains variables, $\|\sigma_i(x)\| = B(\mathcal{R}, t, x)$, and the height of each variable in $\sigma_i(x)$ is exactly $B(\mathcal{R}, t, x)$.

The following statement is a trivial consequence from theorem 7. t is finitely $\mathcal{R}$-irreducible with respect to x (where definition 2 of finite irreducibility is taken without the first condition) iff for every i, $1 \leq i \leq K$, $\sigma_i(t)$ is ground reducible by $\mathcal{R}$. This gives the decision procedure.

5 Decidability Results

In this section we apply the above results to the analysis of several interesting properties of the set of ground normal forms.

5.1 Regularity of $Red(\mathcal{R})$

The decidability of the regularity of the set of reducible ground terms follows naturally from the results of the previous sections.

Theorem 10 *It is decidable whether given a rewriting system $\mathcal{R}$, the set $Red(\mathcal{R})$ is a regular tree language.*

Proof: The result follows from theorem 9. A decision procedure implied by the proof of theorem 6 is the following. Starting from the initial system $\mathcal{R}$, transform it by iterating the following procedure. Take a non-linear term $t \in \mathcal{R}$ and a non-linear variable $x \in \mathcal{V}ar(t)$. Compute the bound $B(\mathcal{R} \setminus \{t\}, t, x)$. Substitute x by all ground terms not deeper than $B(\mathcal{R} \setminus \{t\}, t, x)$ and select out those instances that are $\mathcal{R}$-irreducible. If $\sigma_1(t), \ldots, \sigma_K(t)$ are the resulting terms, check if t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\} \setminus \{t\}$. If this is the case, proceed with the system $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\} \setminus \{t\}$.

If all ground reducibility tests succeed, then $Red(\mathcal{R})$ is regular, otherwise $Red(\mathcal{R})$ is not regular. Note that in the first case the system is finally transformed into a linear one. □

It should be noted that the decision procedure defined in the proof of theorem 6 treats *subsequently* each non-linear term and each non-linear variable in it. The straightforward way of applying the procedure makes the size of the system enormous. The reason is that every time after instantiating a variable, the depth of the rewriting system increases and we have to modify the bound for instantiating the next variable. However, this modification turns out to be unnecessary and this follows from the following general consideration [Hofbauer and Huber, 1992b]. Assume that we are given a rewriting system $\mathcal{R}$, a term t, and a variable $x \in \mathcal{V}ar(t)$. Let N be a number that can be taken to verify the finite irreducibility of t by $\mathcal{R}$ with respect to x. It means that N can be taken as a value of $B(\mathcal{R}, t, x)$ in theorem 7. If we consider now another rewriting system $\mathcal{R}'$ such that $Red(\mathcal{R}) = Red(\mathcal{R}')$, then N can be also taken as a value of $B(\mathcal{R}', t, x)$. On the other hand, we remark that at every step the algorithm instantiates one variable and does not affect the others. In particular, the depth of the variables to be substituted is always bounded by the depth of the initial system. Taking these two arguments into account, we conclude that if we take the bound to be $max\{B(\mathcal{R}\backslash\{t\}, t, x) | t \in \mathcal{R}, x \in \mathcal{V}ar(t)\}$, then it can be used throughout the whole run of the algorithm. Obviously, such a bound can be computed by the formula given before theorem 4 where $depth(x, t)$ and $nocc(x, t)$ are replaced respectively by $\|\mathcal{R}\|$ and the maximal number of positions of a variable in a term in $\mathcal{R}$.

Furthermore, from the possibility of using a single bound it follows that we can also instantiate all the non-linear variables *simultaneously*. In this way we construct a system $\mathcal{L}$ by replacing the non-linear variables by ground terms of depth smaller than the bound and then check if each term from $\mathcal{R}$ is $\mathcal{L}$-ground reducible. This is equivalent to $Red(\mathcal{R}) \subseteq Red(\mathcal{L})$ (cf theorem 1). Note that $\mathcal{L}$ is always linear.

Obviously, since $NF(\mathcal{R}) = T_\Sigma \setminus Red(\mathcal{R})$, it is also decidable if the set of ground normal forms is regular.

5.2 Existence of an equivalent ground system

The results of the previous sections allow us to prove decidable the problem of whether given a rewriting system, there exists an equivalent ground rewriting system.

Theorem 11 *It is decidable whether given a rewriting system $\mathcal{R}$, there exists a finite ground rewriting system $\mathcal{G} \subseteq T_\Sigma$ such that $Red(\mathcal{R}) = Red(\mathcal{G})$.*

Proof: Assume that a finite system $\mathcal{G} \subseteq T_\Sigma$ exists such that $Red(\mathcal{R}) = Red(\mathcal{G})$. Consider the set $\mathcal{G}'$ of subterms of terms in $\mathcal{G}$ that are reducible by $\mathcal{R}$ and have no proper subterms reducible by $\mathcal{R}$. Clearly, $\mathcal{G}' \subseteq Gr(\mathcal{R})$. On the other hand, $Red(\mathcal{G}') = Red(\mathcal{G})$, and hence, $Red(\mathcal{G}') = Red(\mathcal{R})$. Thus, we can always assume that $\mathcal{G} \subseteq Gr(\mathcal{R})$, that is, every term of $\mathcal{G}$ is a ground instance of a term of $\mathcal{R}$.

It is easy to see now that the existence of a finite set $\mathcal{G} \subseteq Gr(\mathcal{R})$ such that $Red(\mathcal{G}) = Red(\mathcal{R})$ implies the finite irreducibility of every $t \in \mathcal{R}$ by $\mathcal{R}\backslash\{t\}$ with respect to *every* variable $x \in \mathcal{V}ar(t)$. By analogy with the previous subsection, we can test the existence of $\mathcal{G}$ by iterating the following procedure while the system contains non-ground terms. Take $t \in \mathcal{R}$ and $x \in \mathcal{V}ar(t)$. Compute the bound $B(\mathcal{R}\backslash\{t\}, t, x)$. Substitute x by all ground terms not deeper than $B(\mathcal{R}\backslash\{t\}, t, x)$ and select out those instances that are $\mathcal{R}$-irreducible. If $\sigma_1(t), \ldots, \sigma_K(t)$ are the resulting terms, check if

t is ground reducible by $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\} \setminus \{t\}$. If this is the case, iterate the procedure with the system $\mathcal{R} \cup \{\sigma_1(t), \ldots, \sigma_K(t)\} \setminus \{t\}$. □

Thus, testing the existence of a finite ground system equivalent to $\mathcal{R}$ is equivalent to testing the existence of an equivalent finite ground instantiation of $\mathcal{R}$, and is done by checking the finite irreducibility of the terms in $\mathcal{R}$ with respect to *all* variables. Note the only difference with the previous case: testing the regularity of $Red(\mathcal{R})$ is testing the existence of an equivalent finite *linear* instantiation of $\mathcal{R}$, and is done by checking the finite irreducibility of the terms in $\mathcal{R}$ with respect to the *non-linear* variables. All the comments from the previous subsection concerning the bound and the strategy of applying the decision procedure are valid for this case too.

5.3 Finiteness of $NF(\mathcal{R})$

It is known that the finiteness of the set of ground normal forms is a decidable property [Kapur *et al.*, 1987; Plaisted, 1985]. However, it is interesting to see that this is a very particular case of the above results.

Theorem 12 *It is decidable whether given a rewriting system $\mathcal{R}$, the set of ground normal forms $NF(\mathcal{R})$ is finite.*

Proof: The finiteness of $NF(\mathcal{R})$ can be expressed as the finite irreducibility of the degenerate term $t = x$ by $\mathcal{R}$ with respect to x, where the definition of finite irreducibility (definition 2) is taken without the first condition. By the remark at the end of the previous section, this property is decidable. □

From the construction of B it follows that $\|NF(\mathcal{R})\|$ is bounded by $3 \times \|\mathcal{R}\|^2 \times maxarity(\Sigma) \times (card(\mathcal{R}) \times \|\mathcal{R}\|)!$ in the case when $NF(\mathcal{R})$ is finite.

Obviously, if $NF(\mathcal{R})$ is finite, then $Red(\mathcal{R})$ is regular. Moreover, if $NF(\mathcal{R})$ is finite, then there exists a finite ground system $\mathcal{G}$ such that $Red(\mathcal{R}) = Red(\mathcal{G})$ ($\|\mathcal{G}\|$ can be bounded by $\|NF(\mathcal{R})\| + 1$). Consequently, the decidable properties we have considered in this section induce the following classification of rewriting systems. Note that every class is strictly embedded into the one below.

$$\{\mathcal{R} | NF(\mathcal{R}) \text{ is finite}\}$$
$$\bigcap$$
$$\{\mathcal{R} | \text{ there exists a finite } \mathcal{G} \subseteq T_\Sigma \text{ such that } Red(\mathcal{G}) = Red(\mathcal{R})\}$$
$$\bigcap$$
$$\{\mathcal{R} | Red(\mathcal{R}) \text{ is regular}\}$$
$$\bigcap$$
$$\text{all term rewriting systems}$$

6 Concluding Remarks

We have introduced a property of finite irreducibility of a term with respect to a variable and have proved it to be decidable. We have shown that various interesting properties can be expressed in terms of finite irreducibility, including the property of regularity of the language of ground normal forms. Using these relations we proved decidable the latter property as well as the property of existence of an equivalent ground term rewriting system. The decision algorithms for all these properties use the ground reducibility test.

Let us give some additional comments on the relation between the property of finite reducibility and that of ground reducibility. Suppose that t is *not* ground reducible. Suppose that we know *a priori* that the set of irreducible instances of t is finite, which obviously means that t is finitely irreducible with respect to each of its variables. In this case the bound that we gave in the paper guarantees that if $\delta(t)$ is irreducible, then $\|\delta(x)\| \leq B(\mathcal{R}, t, x)$ for every $x \in \mathcal{V}ar(t)$. In other words, the bound B allows us to compute in this case the set of irreducible ground instances and to check if it is empty. If t has a potentially infinite set of irreducible ground instances, then to test the ground reducibility of t we have to test in addition that there exists no irreducible instance $\delta(t)$ such that $\|\delta(x)\| > B(\mathcal{R}, t, x)$ for some $x \in \mathcal{V}ar(t)$. In the proposed algorithm this is done by the ground reducibility test. Thus, if we try to test ground reducibility using directly the proposed technique, we enter into a vicious circle. However, it is important to note that this circle can be broken up by giving another bound, say $I(\mathcal{R}, t, x)$, for the smallest depth of $\delta(x)$ in the case when t is infinitely irreducible with respect to x. Given such a bound, we could replace the ground reducibility test by checking whether there exists an irreducible instance $\delta(t)$ such that $\delta(x) \leq I(\mathcal{R}, t, x)$ for every $x \in \mathcal{V}ar(t)$, but $\delta(y) > B(\mathcal{R}, t, y)$ for some $y \in \mathcal{V}ar(t)$. Thus, the two bounds would provide the complete information for testing finite irreducibility and we could apply this algorithm to testing all interesting properties relative to the set of ground normal forms including that of ground reducibility. In fact, the meaning of the bound I is precisely the one of the bound of [Kapur *et al.*, 1987], but to use it we have to prove that the condition above holds which requires some additional analysis.

Throughout the paper we identified term rewriting systems with the sets of left-hand sides, and we considered two rewriting system equivalent if they had the same set of reducible ground terms. The results of section 5 allow us to transform, if this is at all possible, a rewriting system $\mathcal{R}$ into an equivalent "good" (linear or ground) rewriting system $\mathcal{L}$ by substituting some variables by ground terms. We remark that this instantiation can be extended to the right-hand sides of $\mathcal{R}$. Moreover, if $\mathcal{R}$ is convergent, the system we obtain is equivalent to $\mathcal{R}$ in the classical sense, i.e. it generates the same equivalence relation on T_Σ. More precisely, if $\mathcal{R}$ is convergent and $\mathcal{L}$ is an instantiation of $\mathcal{R}$ such that $Red(\mathcal{R}) = Red(\mathcal{L})$, then $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{L}}$ on T_Σ.

During the work on this paper we came to know of the work of D.Hofbauer and M.Huber [Hofbauer and Huber, 1992a]. Using the approach of test sets, they proved independently that the existence of an equivalent linear rewriting system (and therefore the regularity of the ground normal form language) can be effectively tested. Also, after this work had been finished we became aware that theorem 10 was obtained independently by S.Vágvôlgyi and R.Gilleron [Vágvölgyi and Gilleron, 1992] using a very similar approach combining the results of [Kapur *et al.*, 1987] and [Kucherov, 1991].

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