COMBINATORIAL HOPF ALGEBRAS IN NONCOMMUTATIVE PROBABILILITY

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Abstract. We prove that the generalized moment-cumulant relations introduced in [arXiv:1711.00219] are given by the action of the Eulerian idempotents on the Solomon-Tits algebras, whose direct sum builds up the Hopf algebra of Word Quasi-Symmetric Functions **WQSym**. We prove t-analogues of these identities (in which the coefficient of t gives back the original version), and a similar t-analogue of Goldberg's formula for the coefficients of the Hausdorff series. This amounts to the determination of the action of all the Eulerian idempotents on a product of exponentials.

1. Introduction

The relation between moments $m_n = \mathbb{E}(X^n)$ and classical cumulants $K_n(X)$ of a random variable X, encoded in the exponential generating functions

$$\sum_{n\geq 0} m_n \frac{t^n}{n!} = \exp\left(\sum_{n\geq 1} K_n \frac{t^n}{n!}\right) \tag{1.1}$$

is, up to a rescaling by factorials, essentially the same as the relation between complete symmetric functions h_n and the power sum symmetric functions p_n

$$\sum_{n\geq 0} h_n t^n = \exp\left(\sum_{n\geq 1} \frac{p_n}{n} t^n\right). \tag{1.2}$$

Hence we can define a character on the Hopf algebra of symmetric functions Sym by setting

$$\chi_X(h_n) = \frac{1}{n!}\mathbb{E}(X^n). \tag{1.3}$$

Then, the cumulants are given by

$$K_n(X) = \chi_X\left(\frac{p_n}{(n-1)!}\right) \tag{1.4}$$

and the property that $K_n(X+Y) = K_n(X) + K_n(Y)$ whenever X and Y are independent random variables corresponds in this context to the fact that the power-sums are primitive elements for the coproduct (2.4).

In a more general setting, with a sequence $(X_i)_{i\geq 1}$ of random variables, one can define multilinear moments

$$m_n := \mathbb{E}(X_1 X_2 \cdots X_n). \tag{1.5}$$

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Stochastic independence can then be algebraically reformulated in terms of subalgebras of the algebra $\mathcal{X}$ of random variables. A family $(\mathcal{X}_i)_{i\in I}$ of subalgebras of $\mathcal{X}$ is said to be *independent* if the factorization

$$\mathbb{E}(X_1X_2\cdots X_n)=\prod_{B\in\pi}\mathbb{E}\left(\prod_{i\in B}X_i\right) \tag{1.6}$$

for any partition π of $[n]$ such that for each block B of π, the X_i for $i\in B$ are in the same subalgebra $\mathcal{X}_{j(B)}$ (and $j(B)\neq j(B')$ for $B\neq B'$). The multivariate classical cumulants are then defined as

$$K_n(X_1,X_2,\dots,X_n)=\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\cdots\frac{\partial}{\partial t_n}\bigg|_{\mathbf{t}=0}\mathbb{E}\,e^{t_1X_1+t_2X_2+\cdots+t_nX_n} \tag{1.7}$$

which coincides with (1.1) when $X_1=X_2=\cdots=X_n=X$. These cumulants are multilinear maps $K_n(X_1,\dots,X_n)$ whose fundamental property, generalizing (1.4), is that mixed cumulants vanish in the sense that $K_n(X_1,X_2,\dots,X_n)=0$ whenever at least two independent subalgebras occur.

In noncommutative probability, the X_i belong to some noncommutative algebra $\mathcal{A}$, and new notions of independence arise, for which the factorization of the moments (1.6) is replaced by other identities. Each notion of independence gives rise to appropriate version of cumulants, such that the vanishing mixed cumulants (or some weaker condition as in the case of monotone independence) characterizes independence.

In order to give a unified treatment of all these independences, the notions of exchangeability system [Leh04] and of spreadability system [HL17] have been introduced. A spreadability system for a noncommutative probability space $(\mathcal{A},\varphi)$ allows to produce independent copies $X^{(i)}$ of the random variables, and to formulate independence in terms of identities satisfied by the moments $\varphi(X_1^{(i_1)}X_2^{(i_2)}\cdots X_n^{(i_n)})$. It is in particular assumed that this quantity depends only on the *packed word* $u=\mathrm{pack}(i_1i_2\cdots i_n)$ (or *ordered set partition* [HL17]). This implies that for a spreadability system, any choice of a sequence (X_i) of random variables determines a linear form on an appropriate Hopf algebra based on packed words, namely, $\mathbf{WQSym}^*$, the graded dual of Word Quasi-Symmetric functions, also known as quasi-symmetric functions in noncommuting variables [NT06, BZ09, DHNT11]. $\mathbf{WQSym}$ is a noncommutative version of the algebra of quasi-symmetric functions. As shown by Hivert [Hiv00], the quasi-symmetric polynomials are actually the invariants of a very peculiar action of the symmetric group $\mathfrak{S}_n$ on polynomials in n variables, called the quasi-symmetrizing action. This action can be extended to polynomials in noncommuting variables [DHNT11], and letting n tend to infinity, the algebra of invariants acquires a Hopf algebra structure, just as in the case of symmetric polynomials.

It turns out that the homogenous component of degree n of its graded dual, $\mathbf{WQSym}_n^*$, can be identified with the Solomon-Tits algebra of $\mathfrak{S}_n$, and that this identification is compatible with that of $\mathbf{Sym}_n$, the space of noncommutative symmetric functions of degree n, with the ordinary Solomon descent algebra. That is, there is an embedding of Hopf algebras $\mathbf{Sym}\hookrightarrow\mathbf{WQSym}^*$ which is compatible with the internal products.

The moment-cumulant relations of [HL17] actually describe the relation between the natural basis $\mathbf{N}_u$ of $\mathbf{WQSym}^*$ and its internal product with the Eulerian idempotents of the descent algebra $\mathbf{Sym}_n$.

We provide simple conceptual proofs of most of the identities of [HL17], and give a complete description of the action of the Eulerian idempotents on $\mathbf{WQSym}^*$. As shown in [FPT16], the celebrated Goldberg formula for the coefficients of the Hausdorff series

$$H(a_1, a_2, \ldots) = \log(e^{a_1} e^{a_2} \cdots) = \sum_{w \in A^*} c_w w \tag{1.8}$$

amounts to a description of the action of the first Eulerian idempotent on $\mathbf{WQSym}^*$. Our results allow us to compute the coefficients of the expansion

$$(e^{a_1} e^{a_2} \cdots)^t = \sum_{w \in A^*} c_w(t) w \tag{1.9}$$

in which the Hausdorff series is the coefficient of t.

2. Background

In this section, we recall the basic definitions of the various Hopf algebras involved in the sequel: ordinary (Sym) and noncommutative (**Sym**) symmetric functions, word quasi-symmetric functions (**WQSym**) and word symmetric functions (**WSym**).

Given a sequence (X_i) of random variables, the moments $\varphi(X_1 \cdots X_n)$ determine a character of **Sym**. We shall see that the the structures introduced in [Leh04, HL17] allow to extend this character to a linear map of **WQSym** (in the case of spreadability systems) or **WSym** (in the case of exchangeability systems).

2.1. Ordinary symmetric functions.

Let $X = \{x_i | i \geq 1\}$ be an infinite set of commuting variables. The *complete homogeneous functions* $h_n(X)$ and the *elementary symmetric functions* $e_n(X)$ are

$$h_n(X) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{2.1}$$

that is, is the sum of all monomials of total degree n, and

$$e_n(X) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_n} x_{i_{n-1}} \ldots x_{i_1} \tag{2.2}$$

is the sum of all products of n distinct variables. These functions are invariant under permutation of the variables and both sequences freely generate the algebra $Sym(X)$ of *symmetric functions*

$$Sym = \mathbb{K}[h_1, h_2, \ldots] = \mathbb{K}[e_1, e_2, \ldots]. \tag{2.3}$$

We can thus define a coproduct

$$\Delta h_n = \sum_{k=0}^{n} h_k \otimes h_{n-k} \tag{2.4}$$

which endows it with the structure of a graded Hopf algebra. Given a second alphabet Y disjoint from X, and identifying a tensor product $f \otimes g$ with $f(X)g(Y)$, the coproduct is given by $\Delta(f) = f(X+Y)$, where by $X + Y$ we denote the (still countable) union of the alphabets X and Y. It can be shown that Sym is self-dual. Its primitive elements are the power-sums $p_n = \sum_i x_i^n$, which also generate Sym over the rationals.

For a partition $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ of n, define

$$p_\mu = \prod_{i=1}^{n} p_i^{m_i} \quad \text{and} \quad z_\mu = \prod_{i=1}^{n} i^{m_i} m_i!. \tag{2.5}$$

Each homogeneous component Sym_n is endowed with a unique *internal product* $*$ by declaring the elements $\frac{p_\mu}{z_\mu}$ to be orthogonal idempotents.

2.2. **Classical cumulants.** The classical cumulants K_n of a random variable X are related to its moments $m_n = E[X^n]$ by

$$\sum_{n\geq 0} m_n \frac{t^n}{n!} = \exp\left(\sum_{n\geq 1} K_n \frac{t^n}{n!}\right) \tag{2.6}$$

which differs from the relations between complete homogeneous symmetric functions h_n and power-sums p_n by a simple rescaling:

$$\sum_{n\geq 0} h_n t^n = \exp\left(\sum_{n\geq 1} \frac{p_n}{n} t^n\right). \tag{2.7}$$

Identifying m_n with $n!h_n$, the cumulants K_n become identified with $(n-1)!p_n$. A random variable X determines a character χ_X of the algebra of symmetric functions, by defining

$$\chi_X(h_n) = \frac{m_n}{n!} \iff \chi_X(p_n) = \frac{K_n}{(n-1)!}. \tag{2.8}$$

A convenient symbolic notation for characters of *Sym* is that of virtual alphabets: given a sequence of algebraic generators g_n of *Sym* (such as $h_n, p_n, e_n, \ldots$), one denotes $\chi(g_n)$ by $g_n(\mathbb{X})$, where the symbol $\mathbb{X}$ is called the virtual alphabet associated with χ. Then, if η is another character whose virtual alphabet is $\mathbb{Y}$, the convolution $\chi \star \eta := (\chi \otimes \eta) \circ \Delta$ is given by

$$(\chi \star \eta)(f) = f(\mathbb{X} + \mathbb{Y}). \tag{2.9}$$

On can thus associate with each random variable X a virtual alphabet $\mathbb{X}$, such that $m_n(X) = n!h_n(\mathbb{X})$. If X and Y are independent, then

$$\chi_{X+Y}(h_n) = \frac{m_n(X+Y)}{n!} = h_n(\mathbb{X} + \mathbb{Y}) = (\chi_X \otimes \chi_Y)\Delta h_n, \tag{2.10}$$

that is, χ_{X+Y} is the convolution of χ_X and χ_Y. The additivity of cumulants on independent variables corresponds to the fact that the power-sums are primitive elements. This can be seen as another incarnation of the Hopf-algebraic version of the *umbral calculus* [JR79].

Example 2.1 (The formula of Good and Cartier)**.** Let $Y = \Omega = \{1, \omega, \ldots, \omega^{n-1}\}$ be the alphabet of n-th roots of unity. These are the roots of the polynomial $x^n - 1$. Using the factorization $x^n - 1 = (x-1)(1 + x + \cdots + x^{n-1})$ we see that all the roots ω^k for $1 \leq k \leq n-1$ are roots of the second factor and therefore $p_k(\Omega) = 0$ for $1 \leq k \leq n-1$, whereas $p_n(\Omega) = n$. Thus we have

$$h_n(\Omega X) = \sum_{\lambda \vdash n} \frac{p_\lambda(\Omega)\, p_\lambda(X)}{z_\lambda} = p_n(X). \tag{2.11}$$

Identifying as above m_n with $n!h_n$, K_n becomes identified with $(n-1)!p_n$, and we obtain the formula

$$nK_n = \mathbb{E}[(X^{(1)} + \omega X^{(2)} + \cdots + \omega^{(n-1)} X^{(n)})^n], \tag{2.12}$$

known as *Good's formula* in the mathematics literature [Goo75] and *Cartier's formula* for Ursell functions in the physics literature [Per75, Sim93].

2.3. **Noncommutative symmetric functions.** Let $A = \{a_i | i \geq 1\}$ be an infinite totally ordered set of noncommuting variables. We set

$$S_n(A) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} a_{i_1} a_{i_2} \ldots a_{i_n} \tag{2.13}$$

and

$$\Lambda_n(A) = \sum_{i_1 < i_2 < \cdots < i_n} a_{i_n} a_{i_{n-1}} \ldots a_{i_1} \tag{2.14}$$

and call them respectively *noncommutative complete functions* and *noncommutative elementary functions* of A.

The $S_n(A)$ freely generate a subalgebra of the formal power series over A which is denoted by $\mathbf{Sym}(A)$ and called *noncommutative symmetric functions* [GKL+95].

The $S_n(A)$ and $\Lambda_n(A)$ have the simple noncommutative generating series

$$\sigma_t(A) := \sum_{n \geq 0} t^n S_n(A) = \overrightarrow{\prod_{i \geq 1}} (1 - t a_i)^{-1} \tag{2.15}$$

and

$$\lambda_{-t}(A) := \sum_{n \geq 0} (-t)^n \Lambda_n(A) = \overleftarrow{\prod_{i \geq 1}} (1 - t a_i) = \sigma_t(A)^{-1} \tag{2.16}$$

where we have set $S_0 = \Lambda_0 = 1$, and t is an indeterminate commuting with A. We shall almost always forget about the alphabet A since the context is generally clear.

From the generators S_n, we can form a linear basis

$$S^I = S_{i_1} S_{i_2} \cdots S_{i_r} \tag{2.17}$$

of the homogeneous component $\mathbf{Sym}_n$, parametrized by integer compositions of n. The dimension of $\mathbf{Sym}_n$ is thus 2^{n-1} for $n \geq 1$.

The coproduct

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \tag{2.18}$$

endows $\mathbf{Sym}$ with the structure of a graded Hopf algebra. Since this coproduct is clearly cocommutative, $\mathbf{Sym}$ cannot be self-dual. Its graded dual is $QSym$, the algebra of quasi-symmetric functions [GKL+95]. The dual basis of S^I is the quasi-monomial basis M_I of $QSym$.

The map $S_n \mapsto h_n$ is a morphism of Hopf algebras $\mathbf{Sym} \to Sym$, dual to the natural embedding of Sym into $QSym$.

Each homogeneous component $\mathbf{Sym}_n$ is endowed with an internal product $*$ for which $\mathbf{Sym}_n$ is anti-isomophic to the Solomon descent algebra Σ_n of $\mathfrak{S}_n$. The basis element S^I is identified with the formal sum of all permutations whose descent composition is coarser than I.

Finally, the primitive elements of **Sym** span a free Lie algebra, generated by the Φ_n defined by

$$\sum_{n\geq 1} \frac{\Phi_n}{n} = \log\left(\sum_{n\geq 0} S_n\right). \tag{2.19}$$

The $\varphi_n := \frac{1}{n}\Phi_n$ are idempotents for the internal product. These are the Solomon idempotents, also called (first) Eulerian idempotents.

2.4. Cumulants in noncommutative probability.

A noncommutative probability space is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ in a unital algebra and φ a linear form, (or more generally a linear map $\mathcal{A} \to \mathcal{B}$ for some algebra $\mathcal{B}$ such that $\mathcal{A}$ is a $\mathcal{B}$-module) such that $\varphi(1) = 1$.

Elements of $\mathcal{A}$ are called random variables. To define moments in this context, we need an infinite sequence $(X_i)_{i\geq 1}$ of random variables. Then, the moments are

$$m_n := \varphi(X_1 X_2 \cdots X_n). \tag{2.20}$$

Such a sequence determines a character $\hat{\varphi}$ of **Sym** by setting

$$\hat{\varphi}(S_n) = m_n \tag{2.21}$$

(we drop the factorials which are irrelevant in this context).

Cumulants are defined with respect to a notion of independence, leading to several notions such as free, monotone, or boolean cumulants, and many others.

Attempts to give a unified treatment of all these notions have led to the introduction [Leh04] of the notion of exchangeability systems, and later [HL17] of the more general notion of spreadability systems. The interpretation of this formalism in terms of combinatorial Hopf algebras requires the introduction of **WQSym** (Word Quasi-Symmetric functions) and its dual **WQSym***. A spreadability system determines an extension of the character $\hat{\varphi}$ to **WQSym***, and the various notions of independence reflect certain symmetries of this extension.

2.5. Word quasi-symmetric functions.

The *packed word* $u = \operatorname{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. We usually represent a_i by i in the indexation of bases.

A word u is said to be *packed* if $\operatorname{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the "polynomial"

$$\mathbf{M}_u := \sum_{\operatorname{pack}(w)=u} w\,. \tag{2.22}$$

For example, restricting A to the first five integers,

$$\begin{aligned} \mathbf{M}_{13132} = \;& 13132 + 14142 + 14143 + 24243 \\ & + 15152 + 15153 + 25253 + 15154 + 25254 + 35354. \end{aligned} \tag{2.23}$$

For a word $w \in A^*$ and a letter $a \in A$ we denote by $|w|_a$ the number of occurences of the a in w. The *evaluation* $\operatorname{ev}(w)$ is then the vector obtained from the sequence $(|w|_a)_{a\in A}$ by removing all zeros.

Under the abelianization $\chi:\ \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions

$$M_I := \sum_{j_1<j_2<\cdots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r}, \tag{2.24}$$

where $\mathrm{ev}(u) = (i_1, \dots, i_r)$ is the evaluation vector of u.

The $\mathbf{M}_u$ span a subalgebra of $\mathbf{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions, consisting in the invariants of the noncommutative version of Hivert's quasi-symmetrizing action [Hiv00, DHNT11].

Packed words can be naturally identified with *ordered set partitions*, the letter a_i at the jth position meaning that j belongs to block i. For example,

$$u = 313144132 \ \leftrightarrow\ \Pi = (\{2,4,7\},\{9\},\{1,3,8\},\{5,6\})\,. \tag{2.25}$$

Let $\mathbf{N}_u \in \mathbf{WQSym}^*$ be the dual basis of $\mathbf{M}_u$. Define an internal product on $\mathbf{WQSym}_n^*$ by [NT06]

$$\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\mathrm{pack}\binom{u}{v}}, \tag{2.26}$$

where $\binom{u}{v}$ denotes the word in *biletters* $\binom{u_i}{v_i}$, lexicographically ordered with priority to the top letter. For example,

$$\mathrm{pack}\begin{pmatrix}1\,2\,1\,1\,3\,1\\ 2\,2\,1\,3\,1\,1\end{pmatrix} = 241351$$

Then,

Proposition 2.2. $(\mathbf{WQSym}^*, *)$ *is anti-isomorphic to the Solomon-Tits algebra.*

Indeed, if one writes $u = \{s'_1, \dots, s'_k\}$ and $v = \{s''_1, \dots, s''_l\}$ as ordered set partitions, then the packed word $w = \mathrm{pack}\binom{u}{v}$ corresponds to the ordered set partition obtained from

$$\{s'_1 \cap s''_1, s'_1 \cap s''_2, \dots, s'_1 \cap s''_l, s'_2 \cap s''_1, \dots, s'_k \cap s''_l\}. \tag{2.27}$$

Finally,

$$S^I = \sum_{\mathrm{ev}(u)=I} \mathbf{N}_u \tag{2.28}$$

defines an embedding of Hopf algebras compatible with the internal product $*$, that is, inducing the standard embedding of the descent algebra into the Solomon-Tits algebra.

2.6. **Spreadability systems and exchangeability systems.** A spreadability system for $(\mathcal{A}, \varphi)$ is a triple $(\mathcal{U}, \tilde\varphi, (I^{(i)}))$, where $(\mathcal{U}, \tilde\varphi)$ is a $\mathcal{B}$-valued noncommutative probablility space, $I^{(i)}$ is a morphism $\mathcal{A} \to \mathcal{U}$ such that $\varphi = \tilde\varphi \circ I^{(i)}$ for all i, and, setting $X^{(j)} := I^{(j)}(X)$,

$$\tilde\varphi(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}) = \tilde\varphi(X_1^{(j_1)} X_2^{(j_2)} \cdots X_n^{(j_n)}) \tag{2.29}$$

whenever $\mathrm{pack}(i_1 \cdots i_n) = \mathrm{pack}(j_1 \cdots j_n)$.

Thus, for each sequence $(X_i)_{i\geq 1}$ of random variables in $\mathcal{A}$, a spreadability system determines a linear map $\hat\varphi:\ \mathbf{WQSym}^* \to \mathcal{B}$ by

$$\hat\varphi(\mathbf{N}_u) := \varphi(X_1^{(u_1)} \cdots X_n^{(u_n)}). \tag{2.30}$$

In [HL17], this is denoted by $\varphi_\pi(X_1\cdots X_n)$, where π is the ordered set partition encoded by the packed word u.

The notion of $\mathcal{S}$-independence [HL17] can be reformulated in terms of the internal product of $\mathbf{WQSym}^*$. A family of subalgebras $\mathcal{A}_i$ is $\mathcal{S}$-independent if, when $X_j \in \mathcal{A}_{v_j}$,

$$\hat{\varphi}(\mathbf{N}_u) = \hat{\varphi}(\mathbf{N}_u * \mathbf{N}_v). \tag{2.31}$$

An exchangeability system is a spreadability system satisfying

$$\hat{\varphi}(\mathbf{N}_{\sigma(u)}) = \hat{\varphi}(\mathbf{N}_u)). \tag{2.32}$$

for all permutations σ of the alphabet of u. In this case, $\hat{\varphi}$ can be interpreted as a character of $\mathbf{WSym}^*$, an algebra based on set partitions, to be defined below.

2.7. **Symmetric functions in noncommuting variables.** Let A be an alphabet, then every permutation $\sigma \in \mathfrak{S}(A)$ gives rise to an automorphism on the free algebra $\mathbf{K}\langle A\rangle$. Two words $u = u_1\cdots u_n$ and $v = v_1\cdots v_n$ are in the same orbit whenever $u_i = u_j \Leftrightarrow v_i = v_j$. Thus, the orbits are in one-to-one correspondence with set partitions into at most $|A|$ blocks. Assuming as above that A is infinite, we obtain an algebra based on all set partitions, defining the monomial basis by

$$\mathbf{m}_\pi(A) = \sum_{w\in O_\pi} w \tag{2.33}$$

where O_π is the set of words such that $w_i = w_j$ iff i and j are in the same block of π.

One can introduce a bialgebra structure by means of the coproduct

$$\Delta f(A) = f(A' + A'') \tag{2.34}$$

where $A' + A''$ denotes the disjoint union of two mutually commuting copies of A. Again, the coproduct of a monomial function is clearly

$$\Delta\mathbf{m}_\pi = \sum_{\pi'\vee\pi''=\pi} \mathbf{m}_{\mathrm{std}(\pi')} \otimes \mathbf{m}_{\mathrm{std}(\pi'')} \tag{2.35}$$

This coproduct is obviously cocommutative.

Let $\leq$ be the *reverse* refinement order on set partitions ($\pi \leq \pi'$ means that π is coarser than π', i.e. that its blocks are union of blocks of π')[1].

The basis Φ^π, defined by sums over initial intervals

$$\Phi^\pi = \sum_{\pi'\leq\pi} \mathbf{m}_{\pi'} \tag{2.36}$$

is multiplicative with respect to concatenation of set partitions and hence **WSym** is freely generated by the elements Φ^π such that π is irreducible, i.e., the coarsest interval partition dominating π has only one block.

The graded dual of **WSym** is a commutative algebra, isomorphic to the algebra $\Pi QSym$ defined in [HNT08, Sec 3.5.1]. Let N_π be the dual basis of $\mathbf{m}_\pi$ in the (commutative) graded dual $\mathbf{WSym}^* \simeq \Pi QSym$ and let ϕ_π be the dual basis of Φ^π. Then,

$$N_{\pi'} = \sum_{\pi\geq\pi'} \phi_\pi. \tag{2.37}$$

[1]We need this reverse order to be compatible with the usual conventions for symmetric functions. We reserve the notation $\preceq$ for the usual order on set partitions.

$\mathbf{WSym}^*$ is a quotient of $\mathbf{WQSym}^*$, defined by $\mathbf{N}_u \equiv \mathbf{N}_v$ iff $u = \sigma(v)$ for some permutation σ of the alphabet of u (e.g., $\mathbf{N}_{121} \equiv \mathbf{N}_{212}$). Equivalence classes are parametrized by the set partitions corresponding to any of the set compositions encoded by equivalent packed words. The internal product of $\mathbf{WQSym}^*$ descends to an internal product on $\mathbf{WSym}^*$, which is given by the meet of the lattice of set partitions:

$$N_\pi * N_\tau = N_{\pi\wedge\tau} \tag{2.38}$$

where $\pi \wedge \tau$ is the coarsest partition which is finer than π and τ.

For an echangeability system, the notion of $\mathcal{E}$-independence translates as

$$\hat{\varphi}(N_\pi) = \hat{\varphi}(N_\pi * N_\tau) \tag{2.39}$$

under the same hypotheses as in (2.31).

Endowed with this product, the homogenous component $\mathbf{WSym}_n^*$ is known as the Moebius algebra of the partition lattice Π_n [Sol67, Gre73]. It has been shown by Solomon that $\{\phi_\pi | \pi \in \Pi_n\}$ is a complete set of orthogonal idempotents of $\mathbf{WSym}_n^*$.

As a consequence, if π is not the trivial partition $\{12\dots n\}$,

$$N_\pi * \phi_{\{12\cdots n\}} = 0. \tag{2.40}$$

2.8. **Noncommutative version of Good's formula.** Although (2.12) is essentially trivial, it has a not-so-trivial noncommutative analogue in noncommutative symmetric functions. It is proved in [KLT97, Prop. 8.6] that

$$S_n(\Omega A) = K_n(\omega) \tag{2.41}$$

where $K_n(\omega)$ is Klyachko's element, a famous Lie (quasi-) idempotent.

This fact leads to an interesting interpretation of [Leh04, Definition 2.1]. To understand it, we have to follow the chain of morphisms

$$\mathbf{Sym} \hookrightarrow \mathbf{WQSym}^* \twoheadrightarrow \mathbf{WSym}^* \simeq \Pi Qsym. \tag{2.42}$$

The first embedding i is given by

$$i:\ S^I \mapsto \sum_{\mathrm{ev}(u)=I} \mathbf{N}_u \tag{2.43}$$

where $\mathbf{N}_u = \mathbf{M}_u^*$. The projection p onto $\mathbf{WSym}^*$ is dual to the inclusion of $\mathbf{WSym}$ into $\mathbf{WQSym}$, and therefore given by $\mathbf{N}_u \mapsto N_\pi$, where N_π is dual to the monomial basis of $\mathbf{WSym}$, and π is the set partition underlying the set composition encoded by u.

In [Leh04], cumulants for an exchangeability system are defined by a noncommutative analogue of Good's formula. We shall see that they can be interpreted as the image of $(n-1)!K_n(\omega)$ under the composition $\xi = p \circ i$ of these two maps.

Since ξ is valued in a commutative algebra, it factors through $Sym(X)$, and all noncommutative power sums have the same image. The choice of Klyachko's element is therefore arbitrary.

It remains to determine its image. Let $\Phi^\pi = \sum_{\pi\le\tau} \mathbf{M}_\tau$ be the power-sum basis of $\mathbf{WSym}$, and let ϕ_π be its dual basis. Under the commutative image map

$$\chi:\ \mathbf{WSym}(A) \twoheadrightarrow Sym(X) \tag{2.44}$$

given by $f(A) \mapsto f(X)$, we have

$$\mathbf{M}_\pi(A) \mapsto \prod_i m_i(\lambda)! \cdot m_\lambda(X) \tag{2.45}$$

where λ is the integer partition associated with π, $m_\lambda(X)$ is the monomial symmetric function, $m_i(\lambda)$ is the multiplicity of the part i in λ, and

$$\Phi^\pi(A) \mapsto p_\lambda(X). \tag{2.46}$$

Dually,

$$\chi^*(h_\lambda) = \prod_i m_i(\lambda)! \sum_{\Lambda(\pi)=\lambda} N_\pi, \tag{2.47}$$

and

$$\chi^*(p_\lambda) = z_\lambda \sum_{\Lambda(\pi)=\lambda} \phi_\pi. \tag{2.48}$$

The image of any Lie idempotent of **Sym** in $\mathbf{WSym}^*$ is therefore the same as that of $\varphi_n = \frac{1}{n}\Phi_n$, which is

$$\varphi_n = \sum_{I \vDash n} \frac{(-1)^{\ell(I)-1}}{\ell(I)} S^I \mapsto \sum_{|u|=n} \frac{(-1)^{\max(u)-1}}{\max(u)} \mathbf{N}_u \mapsto \sum_{\pi \vdash [n]} \frac{(-1)^{\ell(\pi)-1}}{\ell(\pi)} \ell(\pi)! N_\pi. \tag{2.49}$$

Since this must also be equal to

$$\chi^*(\frac{1}{n} p_n) = \phi_{\{1,2,\ldots,n\}}, \tag{2.50}$$

and since, by definition of the Moebius function of the lattice of partitions

$$\phi_{\{1,2,\ldots,n\}} = \sum_{\pi \vdash [n]} \mu(\pi, \hat{1}_n) N_\pi \tag{2.51}$$

we recover the classical fact that

$$\mu(\pi, \hat{1}_n) = (-1)^{\ell(\pi)-1}(\ell(\pi)-1)! \tag{2.52}$$

by merely contemplating a chain of morphisms.

2.9. **Cumulants for exchangeability systems.** An exchangeability system together with a sequence $(X_i)_{i \geq 1}$ determines a linear form $\hat{\varphi}$ on $\mathbf{WSym}^*$

$$\hat{\varphi}(N_\pi) := \varphi(X_1^{(u_1)} \cdots X_n^{(u_n)}). \tag{2.53}$$

where u is any packed word representing the set partition π.

The partitioned moments are

$$\varphi_\pi(X_1 \cdots X_n) = \hat{\varphi}(N_\pi), \tag{2.54}$$

and the cumulants defined in [Leh04] are

$$K_\pi(X_1, \ldots, X_n) = \hat{\varphi}(\phi_\pi). \tag{2.55}$$

By (2.37), we have [Leh04, Prop. 2.7]

$$\varphi_\pi(X_1, \ldots, X_n) = \sum_{\pi \leq \sigma} K_\sigma(X_1, \ldots, X_n). \tag{2.56}$$

The independence condition defined by (2.31) becomes, for exchangeability systems

$$\hat{\varphi}(N_\pi) = \hat{\varphi}(N_\pi * N_\sigma) = \hat{\varphi}(N_{\pi \wedge \sigma}). \tag{2.57}$$

This implies the vanishing of mixed cumulants: if π is a partition if $[n]$ into two nontrivial blocks b_1, b_2 such that $\{X_i | i \in b_1\}$ and $\{X_i | i \in b_2\}$ are independent, then the cumulant

$$K_n(X_1, \ldots, X_n) = \hat{\varphi}(\phi_{1^n}) = \hat{\varphi}(N_\pi * \phi_{1^n}) = 0 \tag{2.58}$$

by (2.40).

Monotone independence is not a special case of $\mathcal{E}$-independence, but can be recovered from the notion of $\mathcal{S}$-independence. In this case, the requirement that mixed cumulants vanish is too strong, and is replaced by the condition that they should be eigenfunctions of Rota's dot multiplication by and integer:

$$K_u(N.X_1, \ldots, N.X_n) = N^{\max(u)} K_u(X_1, \ldots, X_n) \tag{2.59}$$

where u is a packed word, and $N.X = X^{(1)} + \cdots + X^{(N)}$ is the sum of N independent copies of X.

We shall see that the dot operation translates as multiplication of the alphabet by N in the relevant Hopf algebras. The operator $f(A) \mapsto f(NA)$ is semisimple, and its spectral projectors are know as the Eulerian idempotents.

3. Review of the Eulerian algebra

3.1. **Basics.** The Eulerian algebra is a commutative subalgebra of dimension n of the group algebra of the symmetric group $\mathfrak{S}_n$, and in fact of its descent algebra Σ_n. It was apparently first introduced in [BBMP69] under the name *algebra of permutors*[2]. It is spanned by the Eulerian idempotents, or, equivalently, by the sums of permutations having the same number of descents.

It is easier to work with all symmetric groups at the same time, with the help of generating functions. Recall that the algebra of noncommutative symmetric functions **Sym** is endowed with an internal product $*$, for which each homogeneous component $\mathbf{Sym}_n$ is anti-isomorphic to Σ_n [GKL+95, Section 5.1].

The *Eulerian idempotents* $E_n^{[k]}$ are the homogenous components of degree n in the series $E^{[k]}$ defined by

$$\sigma_t(A)^x = \sum_{k\geq 0} x^k E^{[k]}(A), \tag{3.1}$$

(see [GKL+95, Section 5.3]). We have

$$E_n^{[k]} * E_n^{[l]} = \delta_{kl} E_n^{[k]}\,, \quad \text{and} \quad \sum_{k=1}^{n} E_n^{[k]} = S_n, \tag{3.2}$$

so that the $E_n^{[k]}$ span a commutative n-dimensional $*$-subalgebra of $\mathbf{Sym}_n$, denoted by $\mathcal{E}_n$ and called the Eulerian subalgebra.

3.2. **Noncommutative Eulerian polynomials.** The *noncommutative Eulerian polynomials* are defined by [GKL+95, Section 5.4]

$$\mathcal{A}_n(t) = \sum_{k=1}^{n} t^k \Big(\sum_{\substack{|I|=n \\ \ell(I)=k}} R_I \Big) = \sum_{k=1}^{n} \mathbf{A}(n,k)\, t^k\,, \tag{3.3}$$

[2] A self-contained and elementary presentation of the main results of [BBMP69] can be found in [GKL+95].

where R_I is the ribbon basis [GKL+95, Section 3.2] The following facts can be found (up to a few misprints[3]) in [GKL+95]. The generating series of the $\mathcal{A}_n(t)$ is

$$\mathcal{A}(t) := \sum_{n\geq 0} \mathcal{A}_n(t) = (1-t)\ (1-t\,\sigma_{1-t})^{-1}\ , \tag{3.4}$$

where $\sigma_{1-t} = \sum (1-t)^n S_n$.

Let $\mathcal{A}_n^*(t) = (1-t)^{-n}\,\mathcal{A}_n(t)$. Then,

$$\mathcal{A}^*(t) := \sum_{n\geq 0} \mathcal{A}_n^*(t) = \sum_I \left(\frac{t}{1-t}\right)^{\ell(I)} S^I\ . \tag{3.5}$$

This last formula can also be written in the form

$$\mathcal{A}^*(t) = \sum_{k\geq 0} \left(\frac{t}{1-t}\right)^k (S_1+S_2+S_3+\cdots)^k \tag{3.6}$$

or

$$\frac{1}{1-t\,\sigma_1(A)} = \sum_{n\geq 0} \frac{\mathcal{A}_n(t)}{(1-t)^{n+1}}\ . \tag{3.7}$$

Let $S^{[k]} = \sigma_1(A)^k$ be the coefficient of t^k in this series. In degree n,

$$S_n^{[k]} = \sum_{I\vDash n,\ell(I)\leq k} \binom{k}{\ell(I)} S^I = \sum_{i=1}^n k^i E_n^{[i]}\ . \tag{3.8}$$

This is another basis of $\mathcal{E}_n$. Expanding the factors $(1-t)^{-(n+1)}$ in the right-hand side of (3.7) by the binomial theorem, and taking the coefficient of t^k in the term of weight n in both sides, we get

$$S_n^{[k]} = \sum_{i=0}^k \binom{n+i}{i} \mathbf{A}(n,k-i)\ . \tag{3.9}$$

Conversely,

$$\frac{\mathcal{A}_n(t)}{(1-t)^{n+1}} = \sum_{k\geq 0} t^k\, S_n^{[k]}\ , \tag{3.10}$$

so that

$$\mathbf{A}(n,p) = \sum_{i=0}^p (-1)^i \binom{n+1}{i} S_n^{[p-i]}\ . \tag{3.11}$$

The expansion of the $E_n^{[k]}$ on the basis $\mathbf{A}(n,i)$, which is a noncommutative analog of Worpitzky's identity (see [Gar90] or [Lod89]) is

$$\sum_{k=1}^n x^k\, E_n^{[k]} = \sum_{i=1}^n \binom{x+n-i}{n} \mathbf{A}(n,i)\ . \tag{3.12}$$

Indeed, when x is a positive integer N,

$$\sum_{k=1}^n N^k\, E_n^{[k]} = S_n(NA) = \sum_{I\vDash n} F_I(N) R_I(A) \tag{3.13}$$

[3]Eqs. (93) and (97) of [GKL+95] should be read as (3.11) and (3.12) of the present paper.

where F_I are the fundamental quasi-symmetric functions, and for a composition $I = (i_1, \dots, i_r)$ of n,

$$F_I(N) = \binom{N+n-r}{n}. \tag{3.14}$$

4. Adams operations of Sym and their substitutes

On any bialgebra $\mathcal{H}$ with multiplication μ and comultiplication Δ, one can define the *Adams operations*

$$\Psi^k(x) = \mu_k \circ \Delta^k(x) \tag{4.1}$$

where Δ^k is the iterated coproduct with values in $\mathcal{H}^{\otimes k}$ and μ_k the multiplication map $\mathcal{H}^{\otimes k} \to \mathcal{H}$. In other terms, Ψ^k is the kth convolution power of the identity.

On ordinary symmetric functions, these operations act by multiplication by k of the alphabet, and are therefore algebra morphisms. However, on noncommutative symmetric functions, the Ψ^k are not anymore multiplicative, and therefore of lesser interest.

One can replace them by the algebra morphism $\psi_k :\ f(A) \mapsto f(kA)$, which is diagonalized by the Eulerian idempotents.

Recall that, by definition,

$$\sigma_1(kA) = \sigma_1(A)^k, \tag{4.2}$$

and that for any $f \in \mathbf{Sym}$,

$$f(kA) = f(A) * \sigma_1(kA) = f(A) * \sum_{r\geq 1} k^r E^{[r]}. \tag{4.3}$$

Thus, a simultaneous eigenbasis of the ψ_k is for example

$$K_I := S^I * E^{[\ell(I)]} \tag{4.4}$$

which satisfy

$$K_I(kA) = k^{\ell(I)} K_I(A). \tag{4.5}$$

The basis K_I is actually multiplicative: if $I = (i_1, \dots, i_r)$,

$$K_I = [k^r] S^I * \sigma_1(A)^k = [k^r]\mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) *_r (\sigma_1(kA) \otimes \cdots \otimes \sigma_1(kA)] \tag{4.6}$$

$$= (S_{i_1} * E^{[1]}) \cdots (S_{i_r} * E^{[1]}) = K_{i_1} \cdots K_{i_r}. \tag{4.7}$$

Of course, $S_i * E_i^{[1]} = E_i^{[1]} = \frac{1}{i}\Phi_i$, so that K_I is just a scaled version of the classical basis Φ^I.

Since

$$\sigma_1(kA) = [1 + (S_1 + S_2 + S_3 + \cdots)]^k = \sum_I \binom{k}{\ell(I)} S^I, \tag{4.8}$$

we have the simple closed form

$$S^I(kA) = \sum_{J \geq I} \beta_k(J, I) S^J, \tag{4.9}$$

where

$$\beta_k(J, I) := \prod_p \binom{k}{\ell(J_p)}, \tag{4.10}$$

where $J = (J_1 J_2 \cdots J_r)$ is a concatenation of compositions J_p of weight i_p (by definition of the refinement order).

Applying [GKL+95, Prop. 4.9], we get the expression

$$K_I = \sum_{J \geq I} \frac{(-1)^{\ell(J)-\ell(I)}}{\ell(J,I)} S^J, \tag{4.11}$$

where $\ell(J,I) = \prod_{p=1}^{r} \ell(J_p)$.

5. Extension to $\mathbf{WQSym}^*$

The identification of $\mathbf{Sym}_n$ with the (opposite) descent algebra of $\mathfrak{S}_n$ can be refined as follows.

We have seen that $\mathbf{WQSym}^*$ can be identified with the (opposite) Solomon-Tits algebra. The dual basis $\mathbf{N}_u = \mathbf{M}_u^*$ of the monomial basis of $\mathbf{WQSym}$ has the internal product rule [NT06]

$$\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\operatorname{pack}\binom{u}{v}}, \tag{5.1}$$

and the embedding of the Solomon algebra into the Solomon-Tits algebra is given by the Hopf embedding of $\mathbf{Sym}$ into $\mathbf{WQSym}^*$

$$S^I \mapsto \sum_{\operatorname{ev}(u)=I} \mathbf{N}_u. \tag{5.2}$$

For example, $S^{21} = \mathbf{N}_{112} + \mathbf{N}_{121} + \mathbf{N}_{211}$.

In particular, $S_n(kA)$ and the Eulerian idempotents can be interpreted as elements of $\mathbf{WQSym}_n^*$, and one can define a new basis

$$\mathbf{K}_u := \mathbf{N}_u * E_n^{[r]} \quad (r = \ell(\operatorname{ev}(u)) = \max(u)) \tag{5.3}$$

which will be a simultaneous eigenbasis for the modified Adams operations $\psi_k(F) := F * (\sigma_1)^k$.

The closed expressions given in $\mathbf{Sym}$ can be readily extended to $\mathbf{WQSym}^*$ thanks to the following lemma.

Lemma 5.1. *Define a right action of $\mathfrak{S}_n$ on $\mathbf{WQSym}_n^*$ by*

$$\mathbf{N}_u \cdot \sigma := \mathbf{N}_{u\sigma}, \ \textit{where } u\sigma = u_{\sigma(1)} u_{\sigma(2)} \cdots u_{\sigma(n)}. \tag{5.4}$$

Then, for any $I \vDash n$,

$$\mathbf{N}_{u\sigma} * S^I = (\mathbf{N}_u * S^I) \cdot \sigma. \tag{5.5}$$

Proof – If S^I contains $\mathbf{N}_v$, it contains the $\mathbf{N}_{v\tau}$ for all permutations τ, and

$$\operatorname{pack}\begin{pmatrix} u\tau \\ v \end{pmatrix} = \operatorname{pack}\begin{pmatrix} u \\ v\tau^{-1} \end{pmatrix} \cdot \tau. \tag{5.6}$$

∎

For example, with $u = 111122$, $v = 212211$, $\tau = 451623$, we have $u\tau = 121211$, $v\tau^{-1} = 211212$, $\operatorname{pack}\binom{121211}{212211} = 232411$, $\operatorname{pack}\binom{111122}{211212} = 211234$, and $211234\tau = 232411$.

This implies[4] that $(f \cdot \sigma) * g = (f * g) \cdot \sigma$ for all $f \in \mathbf{WQSym}_n^*, g \in \mathbf{Sym}_n$ and $\sigma \in \mathfrak{S}_n$.

Hence,

$$\psi_k(\mathbf{N}_u) = \sum_{v \geq u} \beta_k(v,u)\mathbf{N}_v, \tag{5.7}$$

where $\beta_k(v,u) := \beta_k(\mathrm{ev}(v), \mathrm{ev}(u))$, and

$$\mathbf{K}_u = \sum_{v \geq u} \frac{(-1)^{\ell(\mathrm{ev}(v)) - \ell(\mathrm{ev}(u))}}{\ell(\mathrm{ev}(v), \mathrm{ev}(u))} \mathbf{N}_v, \tag{5.8}$$

where the order on packed words u, v is the *reverse* refinement order on the corresponding set compositions σ, π.

Given a spreadability system and a sequence (X_i) of random variables, and defining the linear map $\hat{\varphi}$ as above by $\hat{\varphi}(\mathbf{N}_u) = \varphi_u(X_1, \ldots, X_n)$, the cumulants are given by

$$K_u(X_1, \ldots, X_n) = \hat{\varphi}(\mathbf{K}_u) \tag{5.9}$$

and we recover the relations between moments and cumulants of [HL17, Th. 4.7 and 4.8].

By construction,

$$\mathbf{K}_u * \mathbf{S}_n(NA) = N^{\ell(\mathrm{ev}(u))}\mathbf{K}_u \tag{5.10}$$

which is [HL17, Th. 4.14]. Indeed, according to [HL17, Definition 4.1], $\varphi_\pi(N.X_1, \ldots, N.X_n) = \hat{\varphi}(\mathbf{N}_u * S_n(NA))$.

Adapting the argument given at the end of [DHNT11], one can prove the following extension of the "splitting formula" of [GKL+95]:

Proposition 5.2. *If $f_1, f_2, \ldots, f_r \in \mathbf{WQSym}^*$ and $g \in \mathbf{Sym}$, then*

$$(f_1 f_2 \cdots f_r) * g = \mu_r[(f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g]. \tag{5.11}$$

The same argument as in (4.6) proves then the following product rule for the $\mathbf{K}_u$:

$$\mathbf{N}_u\mathbf{N}_v = \sum_w c_{u,v}^w \mathbf{N}_w \ \Rightarrow\ \mathbf{K}_u\mathbf{K}_v = \sum_w c_{u,v}^w \mathbf{K}_w. \tag{5.12}$$

That is, $\mathbf{N}_u \mapsto \mathbf{K}_u$ is an algebra automorphism.

6. Partial cumulants

The defining formula (2.56) of the cumulants can be inverted using the Möbius function (2.52), but neither this nor the formula of Cartier and Good (2.12) are suitable for the efficient calculation of cumulants of higher orders. In the case of exchangeability systems recursive formulas are available which are more adequate for this purpose; see [Leh04, Proposition 3.9]. In the classical case, the recursion reads as follows:

$$K(X_1, X_2, \ldots, X_n) = \mathbb{E}\, X_1 X_2 \cdots X_n - \sum_{\substack{A \subsetneq [n] \\ 1 \in A}} K_{|A|}(X_i : i \in A)\, \mathbb{E} \prod_{j \in A^c} X_j \tag{6.1}$$

[4] Incidentally, this also proves the existence of the descent algebra. If one denotes by ϕ the dual of the inclusion map $\mathbf{FQSym} \to \mathbf{WQSym}$, which is given by $\phi(\mathbf{N}_u) = \mathbf{F}_{\mathrm{std}(u)}$, then $\phi(S^I * \mathbf{N}_v) = \phi(S^I) \cdot \mathrm{std}(v)$, so that $\phi(S^I * \mathbf{N}_v) = \phi(S^I) * \phi(\mathbf{N}_v)$.

and in the univariate case it specifies to the familiar formula [RS00]

$$\kappa_n = m_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k m_{n-k}. \tag{6.2}$$

In the free case it specifies to the free Schwinger-Dyson equation [MS13] and from the point of view of combinatorial Hopf algebras this has been recently considered under the name of "splitting process" [EFP16].

Turning to our general setting we note that already in the case of monotone probability we lack a simple recursive formula; however Hasebe and Saigo [HS11a, HS11b] found a good replacement in terms of differential equations. This was generalized to spreadability systems in terms of partial cumulants introduced in [HL17, Section 6] which are the images by $\hat{\varphi}$ of some interesting elements of $\mathbf{WQSym}^*$ which we shall study in this section.

We start with the analogous questions in $\mathbf{Sym}$, which are simpler and imply easily the general results in $\mathbf{WQSym}^*$.

Let $T = (t_1, \ldots, t_r)$ be a sequence of binomial elements (scalars), so that the noncommutative symmetric functions of $t_j A$ are defined by

$$\sigma_1(t_j A) := \sigma_1(A)^{t_j} \tag{6.3}$$

and the analogs of the formal multivariate moments [HL17, Eq. (6.2)] are

$$S^I(T; A) := S_{i_1}(t_1 A) \cdots S_{i_r}(t_r A). \tag{6.4}$$

The (generic) cumulants are thus

$$K_I = \left. \frac{\partial^r}{\partial t_1 \partial t_2 \cdots \partial t_r} \right|_{T=(0,\ldots,0)} S^I(T; A) \tag{6.5}$$

Imitating [HL17, Def. 6.1], we define the partial cumulants as

$$K_{I;j}^{(t_1,\ldots,t_{j-1},1,t_{j+1},\ldots,t_r)} := \left. \frac{\partial}{\partial t_j} \right|_{t_j=0} S^I(T; A). \tag{6.6}$$

Recall that $\sigma_1 = \exp(\phi)$, where $\phi = \sum_{n\geq 1} \frac{1}{n} \Phi_n = E^{[1]}$. Therefore,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \sigma_1(A)^t = \phi, \tag{6.7}$$

so that

$$\left. \frac{\partial}{\partial t_j} \right|_{t_j=0} S^I(T; A) = S_{i_1}(t_1 A) \cdots S_{i_{j-1}}(t_{j_1} A) \phi_{i_j} S_{i_{j+1}}(t_{j+1} A) \cdots S_{i_r}(t_r A), \tag{6.8}$$

and inserting

$$\phi_n = \sum_{H \vDash n} \frac{(-1)^{\ell(H)-1}}{\ell(H)} S^H =: \sum_{H \vDash = n} \tilde{\mu}(H, (n)) S^H, \tag{6.9}$$

we obtain the analog of [HL17, Prop. 6.2]

$$K_{I;j}^{(t_1,\ldots,t_{j-1},1,t_{j+1},\ldots,t_r)} = \sum_{K \vDash i_j} S^{I'HI''}((T', 1^{\ell(H)}, T''); A) \tilde{\mu}(H, (n)), \tag{6.10}$$

where $T' = (t_1, \ldots, t_{j-1})$, $T'' = (t_{j+1}, \ldots, t_r)$ and 1^p means $(1, \ldots, 1)$ (p times).

Now,

$$\frac{\partial}{\partial t_j} S^I(T;A) \tag{6.11}$$

is the homogeneous part of degree n in

$$S_{i_1}(t_1A)\cdots S_{i_{j-1}}(t_{j-1}A)\frac{\partial}{\partial t_j} e^{t_j\phi} S_{i_{j+1}}(t_{j+1}A)\cdots S_{i_r}(t_rA) \tag{6.12}$$

$$= S_{i_1}(t_1A)\cdots S_{i_{j-1}}(t_{j-1}A)\phi\sigma_1^{t_j} S_{i_{j+1}}(t_{j+1}A)\cdots S_{i_r}(t_rA) \tag{6.13}$$

$$= S_{i_1}(t_1A)\cdots S_{i_{j-1}}(t_{j-1}A)\sigma_1^{t_j}\phi S_{i_{j+1}}(t_{j+1}A)\cdots S_{i_r}(t_rA), \tag{6.14}$$

the last two expressions being respectively equal to

$$\sum_{a=1}^{i_j} K^{(t_1,\dots,t_{j-1},1,t_j,\dots,t_r)}_{(i_1,\dots,i_{j-1},a,i_j-a,i_{j+1},\dots,i_r);j} \tag{6.15}$$

and

$$\sum_{a=1}^{i_j} K^{(t_1,\dots,t_{j-1},t_j,1,\dots,t_r)}_{(i_1,\dots,i_{j-1},i_j-a,a,i_{j+1},\dots,i_r);j+1} \tag{6.16}$$

These relations are then extended to $\mathbf{WQSym}^*$ by defining $\mathbf{N}_u(T)$ in such a way that

$$S^I(T;A) = \sum_{\mathrm{ev}(u)=I} \mathbf{N}_u(T), \tag{6.17}$$

which implies that (cf. [HL17, Th. 4.5])

$$\mathbf{N}_u(T) = \sum_{v\geq u} \beta_T(v,u)\mathbf{N}_v. \tag{6.18}$$

Example 6.1. With $I=(2,2,1)$ we have

(6.19)

$$S^{221}(T;A) = \left(t_1S_2 + \binom{t_1}{2}S^{11}\right)\left(t_2S_2 + \binom{t_2}{2}S^{11}\right)t_3S_1$$

$$= t_1t_2t_3S^{221} + t_1\binom{t_2}{2}t_3S^{2111} + \binom{t_1}{2}t_2t_3S^{1121} + \binom{t_1}{2}\binom{t_2}{2}t_3S^{11111}, \tag{6.20}$$

so that

$$K^{(t_1,1,t_3)}_{(221);2} = \left(t_1S_2 + \binom{t_1}{2}S^{11}\right)(S_2 - \frac{1}{2}S^{11})t_3S_1 \tag{6.21}$$

$$= t_1t_3S^{221} - \frac{1}{2}t_1t_3s^{2111} + \binom{t_1}{2}t_3S^{1121} - \frac{1}{2}\binom{t_1}{2}t_3S^{11111}. \tag{6.22}$$

Similarly,

$$S^{2111}(T;A) = \left(t_1S_2 + \binom{t_1}{2}S^{11}\right)t_2t_3t_4S^{111} \tag{6.23}$$

and

$$K^{(t_1,t_2,1,t_4)}_{(2111);3} = \left(t_1S_2 + \binom{t_1}{2}S^{11}\right)t_2t_4S^{111}. \tag{6.24}$$

We have indeed

$$K^{(t_1,1,t_3)}_{(221);2} + K^{(t_1,t_2,1,t_3)}_{(2111);3} = t_1t_3S^{221} + t_1t_3(t_2 - \frac{1}{2})S^{2111} + \binom{t_1}{2}t_3S^{1121} + \binom{t_1}{2}(t_2 - \frac{1}{2})t_3S^{11111}$$

$$= \frac{\partial}{\partial t_2}S^{221}(T;A). \tag{6.25}$$

In $\mathbf{WQSym}^*$, this would translate for the set composition $\pi = 24|15|3 \leftrightarrow u = 21312$ as

$$\mathbf{N}_{21312}(T;A) = t_1t_2t_3\mathbf{N}_{21312} + t_1\binom{t_2}{2}t_3\mathbf{N}_{21413} + \binom{t_1}{2}t_2t_3\mathbf{N}_{31423} + \binom{t_1}{2}\binom{t_2}{2}t_3\mathbf{N}_{31524}, \tag{6.26}$$

and

$$\mathbf{N}_{21413}(T;A) = t_1t_2t_3t_4\mathbf{N}_{21413} + \binom{t_1}{2}t_2t_3t_4\mathbf{N}_{31524}, \tag{6.27}$$

and the partial cumulants $K^{(t_1,1,t_3)}_{24|15|3;15}$, $K^{(t_1,t_2,1,t_4)}_{24|1|5|3;5}$ would be respectively

$$K^{(t_1,1,t_3)}_{21312;2} = t_1t_3\mathbf{N}_{21312} - \frac{1}{2}t_1t_3\mathbf{N}_{21413} + \binom{t_1}{2}t_3\mathbf{N}_{31423} - \frac{1}{2}\binom{t_1}{2}t_3\mathbf{N}_{31524} \tag{6.28}$$

$$K^{(t_1,t_2,1,t_4)}_{21413;3} = t_1t_2t_4\mathbf{N}_{21413} + \binom{t_1}{2}t_2t_4\mathbf{N}_{31524} \tag{6.29}$$

7. Left and right products with the Eulerian idempotents

We have seen that the cumulant basis is given by the internal products $\mathbf{N}_u * E^{[r]}$ where $r = \ell(\mathrm{ev}(u)) = \max(u)$.

The aim of this section is to compute the internal products

$$\mathbf{N}_u * E^{[k]} \quad \text{and} \quad E^{[k]} * \mathbf{N}_u \tag{7.1}$$

for arbitrary u and k.

Let u be a packed word. Recall that v is said to refine u if for all $i < j$, $v_i > v_j \iff u_i \geq u_j$ and $v_i = v_j \implies u_i = u_j$. In this case, we write $v \geq u$ or $v \in \mathrm{Ref}(u)$. This is the usual notion of refinement on set compositions: each block of u is a union of *consecutive* blocks of v.

We shall say that v is a *weak refinement* of u, and write $v \succeq u$ or $v \in \mathrm{WRef}(u)$, if for all $i < j$, $v_i = v_j \implies u_i = u_j$. On set compositions, this means that each block of u is a union of blocks of v.

For example,

$$\begin{aligned} \mathrm{Ref}(122) &= \{122, 123, 132\}, \\ \mathrm{WRef}(122) &= \{122, 211, 123, 132, 213, 231, 312, 321\}. \end{aligned} \tag{7.2}$$

Thus, the word $w = \mathrm{pack}\binom{u}{v}$ is finer than u and is such that $w_i = w_j \implies v_i = v_j$.

We shall compute the generating functions

$$U_v(t) = \sigma_1^t * \mathbf{N}_v = \sum_r t^r E^{[r]} * \mathbf{N}_v = \sum_u \binom{t}{\max(u)} \mathbf{N}_u * \mathbf{N}_v \tag{7.3}$$

$$= \sum_{w \in \mathrm{WRef}(v)} \left(\sum_{u \in U(v,w)} \binom{t}{\max(u)} \right) \mathbf{N}_w, \tag{7.4}$$

where

$$U(v,w) = \left\{ u |\ \mathrm{pack} \binom{u}{v} = w \right\} \tag{7.5}$$

and

$$V_u(t) = \mathbf{N}_u * \sigma_1^t = \sum_r t^r \mathbf{N}_u * E^{[r]} = \sum_v \binom{t}{\max(v)} \mathbf{N}_u * \mathbf{N}_v \tag{7.6}$$

$$= \sum_{w \geq u} \left(\sum_{v \in V(u,w)} \binom{t}{\max(v)} \right) \mathbf{N}_w, \tag{7.7}$$

where

$$V(u,w) = \left\{ v |\ \mathrm{pack} \binom{u}{v} = w \right\}. \tag{7.8}$$

When convenient, we shall freely identify packed words with their corresponding set compositions without further notice.

7.1. **A closed formula for $U_v(t)$.** The pairs (v, w) such that $U(v, w)$ is nonempty are those such that $w \in \mathrm{WRef}(v)$.

Proposition 7.1. *Define a set composition $u_0 = u_0(v, w)$ as the one obtained by merging two consecutive blocks p', p'' of w if the blocks of v containing p' are strictly to the left of those containing p'', and iterating the process until no further blocks can be merged.*

Then, $U(v, w)$ is the interval

$$U(v,w) = [u_0, w] \quad \textit{(compositions finer than } u_0 \textit{ and coarser than } w\textit{)}. \tag{7.9}$$

For example, let us compute $U(13211, 15342)$. The corresponding set compositions are $145|3|2$ and $1|5|3|4|2$. To construct u_0, one can merge the second and third blocks of w, and also the fourth and the fifth ones, which yields $u_0 = 1|35|24$. Therefore,

$$U(13211, 15342) = [13232, 15342] = \{13232, 14232, 14342, 15342\}. \tag{7.10}$$

To compute $U(13211, 24315)$, the corresponding set compositions are $145|3|2$ et $4|1|3|2|5$, we obtain $u_0 = 4|123|5$, so that

$$U(13211, 24315) = [22213, 24315] = \{22213, 23214, 23314, 24315\}. \tag{7.11}$$

Finally, with $U(13211, 13214)$, the set compositions are $145|3|2$ et $14|3|2|5$, $u_0 = 1234|5$, and

$$U(13211, 13214) = [11112, 13214] = \{11112, 12113, 12213, 13214\}. \tag{7.12}$$

Proof – It follows from the definition of pack $\binom{u}{v}$ that if two elements i, j are in the same block of w, then, they must also be in the same block in u and in v, and otherwise i is in a block strictly left to the block of j in v if and only if either i is in a block left to the block of j in u, or i and j are in the same block of u, and the block of i if left to the block of j in v.

Thus, any $u \in U(v,w)$ must be obtained by merging some consecutive parts in w, so that $u \le w$. Moreover, to get w as pack $\binom{u}{v}$, one can only merge blocks satisfying the constraints mentioned in the definition of u_0, so that $u \ge u_0$. ∎

So, $U(v,w)$ is the interval $[u_0, w]$. This interval is clearly a boolean lattice. Moreover, if we set $a_0 = \max(u_0)$ and $a = \max(w)$, the number of elements of this lattice such that $\max(u) = k$ is $\binom{a-a_0}{a-k}$, so that

$$\sum_{u\in U(v,w)} \binom{t}{\max(u)} = \sum_{k=a_0}^{a} \binom{a-a_0}{a-k}\binom{t}{k} = \binom{t+a-a_0}{a}, \tag{7.13}$$

and finally

$$U_v(t) = \sum_{w} \binom{t+\max(w)-a_0(v,w)}{\max(w)} \mathbf{N}_w \tag{7.14}$$

where $a_0(v,w) = \max(u_0(v,w))$.

In particular, the coefficient of $\mathbf{N}_w$ in $E^{[1]} * \mathbf{N}_v$ is the coefficient of $t\mathbf{N}_w$ in $U_v(t)$, which is

$$(-1)^{a_0-1}\frac{(a-a_0)!(a_0-1)!}{a!} = (-1)^{a_0-1}\mathrm{B}(a-a_0+1, a_0) \tag{7.15}$$

where B is the Beta function. This is formula (7.3) of [HL17] for the Weisner coefficients.

7.2. **A closed formula for $V_u(t)$.** Let us now describe $V(u,w)$. Since the packing process commutes with the right action of the symmetric group (see (5.6)), we can apply to u the smallest permutation σ such that $u\sigma$ is nondecreasing (i.e., $\sigma = \mathrm{std}(u)^{-1}$), so that pack $\binom{u\sigma}{v\sigma} = w\sigma$. We can therefore assume that u is nondecreasing.

Proposition 7.2. *Let $w^{(i)} = \mathrm{pack}(w_{j_1}w_{j_2}\cdots w_{j_p})$, where $\{j_1,\dots,j_p\} = \{j | u_j = i\}$. Then,*

$$\sum_{v\in V(u,w)} \mathbf{M}_v = \mathbf{M}_{w^{(1)}}\mathbf{M}_{w^{(2)}}\cdots\mathbf{M}_{w^{(\max(u))}}. \tag{7.16}$$

Proof – Note first that no relation is imposed between the letters of v corresponding to different letters of u. The only order constraints are among places where u has identical letters, and these are the same as in the corresponding letters of w. This is precisely the definition of the convolution on packed words, describing the product of the $\mathbf{M}$ basis. ∎

Since the map $\mathbf{M}_u \mapsto \binom{t}{u}$ is a character of **WQSym**,

$$\sum_{v\in V(u,w)} \binom{t}{\max(v)} = \prod_{i=1}^{\max(u)} \binom{t}{\max(w^{(i)})}. \tag{7.17}$$

We have therefore

$$V_u(t) = \sum_{w\ge u} \prod_{i=1}^{\max(u)} \binom{t}{\max(w^{(i)})} \mathbf{N}_w. \tag{7.18}$$

For example, take $u = 1122$ and $w = 2133$. We have $w^{(1)} = 21$ et $w^{(2)} = \mathrm{pack}(33) = 11$. The product is

$$\mathbf{M}_{21}\mathbf{M}_{11} = \mathbf{M}_{2111} + \mathbf{M}_{2122} + \mathbf{M}_{2133} + \mathbf{M}_{3122} + \mathbf{M}_{3211}, \tag{7.19}$$

and the set of v is

$$V(1122, 2133) = \{2111, 2122, 2133, 3122, 3211\}. \tag{7.20}$$

We can then see that

$$2\binom{t}{2} + 3\binom{t}{3} = \frac{t^2(t-1)}{2} = \binom{t}{2}\binom{t}{1}, \tag{7.21}$$

as claimed.

7.3. **Mixed cumulants.** The mixed cumulants described in Section 7 of [HL17] correspond to the elements $\mathbf{K}_u * \mathbf{N}_v = \mathbf{N}_u * E^{[\max(u)]} * \mathbf{N}_v$ of $\mathbf{WQSym}^*$.

We shall compute the generating series

$$\mathbf{N}_u * \sigma_1^t * \mathbf{N}_v = \sum_r \mathbf{N}_u * E^{[r]} * \mathbf{N}_v. \tag{7.22}$$

This amounts to computing

$$\sum_{u' \geq u} \prod_{i=1}^{\max(u)} \binom{t}{\max(u'^{(i)})} \mathbf{N}_{u'} * \mathbf{N}_v. \tag{7.23}$$

Let us compute the coefficient of $\mathbf{N}_w$ in (7.23). First, the packed words having a nonzero coefficient in this expansion are those finer than u, which are weakly finer than v. Let w be such a word. The coefficient of $\mathbf{N}_w$ is obtained by summing the coefficients of all $u' \geq u$ such that pack $\binom{u'}{v} = w$. The set of those u' is therefore the set of packed words which are finer than u and than $u_0(v, w)$, and that are coarser than w.

This set is an intersection of boolean lattices, hence also a boolean lattice. Moreover, the subwords of u consisting of identical letters cannot interfere with each other, so that this lattice is in fact the product of the lattices obtained by restricting v and w to positions where u has identical letters.

Thus, the result is obtained by applying (7.14) on each set of positions where u has identical letters since these pieces are independent and $\mathbf{N}_{1^k} = S_n$ is neutral for $*$. ■

The result is therefore

$$\mathbf{N}_u * \sigma_1^t * \mathbf{N}_v = \sum_{w \in W(u,v)} \prod_{i=1}^{\max(u)} \binom{t + \max(w^{(i)}) - a(v^{(i)}, w^{(i)})}{\max(w^{(i)})} \mathbf{N}_w, \tag{7.24}$$

where a is as previously defined, and $W(u, v)$ is the set of packed words finer than u which have equal letters only at positions where u has equal letters, and $v^{(i)}$ et $w^{(i)}$ are the subwords of v and w corresponding to the positions of the letter i in u.

Example 7.3. Let $u = 11122$ and $v = 12234$.

Take $w = 23145$. We form the packed words of the restrictions of v and w to the positions where u has equal letters, which gives for the first block $v^{(1)} = 122$ and

$w^{(1)} = 231$, with a contribution of $\binom{t+1}{3}$ for this factor (since $u_0^{(1)} = 221$) and for the second, $v^{(2)} = 12$ et $w^{(2)} = 12$, hence a contribution of $\binom{t+1}{2}$ (since $u_0^{(2)} = 11$).

For $w = 21354$, we have respective contributions of $\binom{t+1}{3}$ and $\binom{t}{2}$ since $u_0^{(1)} = 212$ and $u_0^{(2)} = 21$.

For $w = 31245$, we have respective contributions of $\binom{t}{3}$ and $\binom{t+1}{2}$ since $u_0^{(1)} = 312$ and $u_0^{(2)} = 11$.

For $w = 31254$, we have respective contributions of $\binom{t}{3}$ and $\binom{t}{2}$ since $u_0^{(1)} = 312$ and $u_0^{(2)} = 21$.

Finally, for $w = 12243$, we have respective contributions of $\binom{t+1}{2}$ and $\binom{t}{2}$ since $u_0^{(1)} = 111$ and $u_0^{(2)} = 21$.

8. Miscellaneous remarks

We have already noticed

$$\mathrm{pack}\begin{pmatrix} u\cdot\sigma \\ v\cdot\sigma \end{pmatrix} = \mathrm{pack}\begin{pmatrix} u \\ v \end{pmatrix}\cdot\sigma \tag{8.1}$$

for any permutation σ.

Taking the smallest permutations sorting u ($\sigma = \mathrm{std}(u)^{-1}$), we can restrict to the case where u is nondecreasing. With a second permutation, we can restrict to the case where v is nondecreasing on positions where u has equal letters. Moreover, if

$$\mathrm{pack}\begin{pmatrix} u \\ v_1 \end{pmatrix} = \mathrm{pack}\begin{pmatrix} u \\ v_2 \end{pmatrix}, \tag{8.2}$$

then

$$\mathrm{pack}\begin{pmatrix} u' \\ v_1 \end{pmatrix} = \mathrm{pack}\begin{pmatrix} u' \\ v_2 \end{pmatrix} \tag{8.3}$$

for all u' finer than u. In the latter argument, it is thus possible to restrict to the case where u is a shifted concatenation of packed words, one per block of equal letters of u.

9. A generalization of Goldberg's formula

The product of exponentials

$$g = e^{a_1}e^{a_2}\cdots = \sum_{u\ \text{packed}} \frac{1}{u!}\mathbf{M}_u(A), \quad (u! := \prod_i |u|_i!) \tag{9.1}$$

is naturally an element of the completion $\widehat{\mathbf{WQSym}}$ of $\mathbf{WQSym}$. It is grouplike for the coproduct of $\mathbf{WQSym}$, which in this case coincides with the coproduct of $\mathbb{K}\langle\langle A\rangle\rangle$. The Hausdorff series

$$H(a_1, a_2, \ldots) = \log g = \sum_u c_u\mathbf{M}_u \tag{9.2}$$

is thus also an element of $\widehat{\mathbf{WQSym}}$. The coefficient c_u can be computed by Goldberg's formula, of which new proofs respectively based on arguments from combinatorial Hopf algebras and of noncommutative probability have been recently given in [FPT16] and [HL17].

The homogeneous component H_n of H is the image of g_n by the first Eulerian idempotent $e_n^{[1]}$, *aka* Solomon's idempotent. More generally, the image of g_n by $e_n^{[k]}$ is the coefficient of t^k in g^t. Define polynomials $c_u(t)$ by

$$g^t = (e^{a_1}e^{a_2}\cdots)^t =: \sum_u c_u(t)\mathbf{M}_u(A). \tag{9.3}$$

Since g is grouplike, we can, as in [FPT16], define an injective morphism of Hopf algebras

$$\varphi : \mathbf{Sym} \longrightarrow \mathbf{WQSym} \tag{9.4}$$

$$S_n \longmapsto g_n \tag{9.5}$$

Then, $g^t = \varphi(\sigma_1^t)$ and

$$c_u(t) = \langle \mathbf{N}_u, \varphi(\sigma_1^t)\rangle = \langle \varphi^\dagger(\mathbf{N}_u), \sigma_1^t\rangle \tag{9.6}$$

$$= \sum_I \binom{t}{\ell(I)} \langle \varphi^\dagger(\mathbf{N}_u), S^I\rangle \tag{9.7}$$

where $\varphi^\dagger : \mathbf{WQSym}^* \to QSym$ is the adjoint map, and $\langle\cdot,\cdot\rangle$ is the duality bracket.

Now, recall that the coproduct of $\mathbf{N}_u$ is given by [NT06]

$$\Delta\mathbf{N}_u = \sum_{u=u_1\cdot u_2} \mathbf{N}_{\mathrm{pack}(u_1)} \otimes \mathbf{N}_{\mathrm{pack}(u_2)} \tag{9.8}$$

(deconcatenation). We can omit the packing operation in this formula if we make the convention that $\mathbf{N}_w = \mathbf{N}_u$ if $u = \mathrm{pack}(w)$. Then, since φ, and hence also $\varphi^\dagger$ are morphisms of Hopf algebras, for a composition $I = (i_1, \dots, i_r)$,

$$\langle \varphi^\dagger(\mathbf{N}_u), S^I\rangle = \prod_{k=1}^r \langle \varphi^\dagger(\mathbf{N}_{u_k}), S^{i_k}\rangle \tag{9.9}$$

where $u = u_1u_2\cdots u_r$ with $|u_k| = i_k$ for all k. Moreover, this is nonzero only if all the u_k are nondecreasing, in which case the result is $1/(u_1!\cdots u_r!)$.

Let $u = u_1\cdots u_r$ be the minimal factorization of u into nondecreasing words (*i.e.*, such that the last letter of each u_i is strictly greater than the first one of u_{i+1}), and let $I = (|u_1|, \dots, |u_r|)$.

Let also $J = (j_1, \dots, j_s)$ be the composition obtained by factoring u into maximal blocks of identical letters,

$$u = b_1^{j_1} b_2^{j_2} \cdots b_s^{j_s}. \tag{9.10}$$

Then, it is worth noticing that $c_u(t)$ is the $(t, \mathbb{E})$ specialization of

(9.11)

$$c_u(X; A) = \sum_{K\geq I} M_K(X) S^{K\vee J} = \sum_{H\geq J} \left(\sum_{K\vee J=H} M_K(X)\right) S^H(A) \quad \in QSym \otimes \mathbf{Sym},$$

where $\vee$ denotes the join in the lattice of compositions of n. For a binomial element t, $M_K(t) = \binom{t}{\ell(K)}$, and the (virtual) exponential alphabet $\mathbb{E}$ is defined by $S_k(\mathbb{E}) = \frac{1}{k!}$. The set $\{K | K \vee J = H\}$ is a boolean lattice: it is the interval $[I \vee L, H]$ where $\mathrm{Des}(L) = \mathrm{Des}(H)\setminus \mathrm{Des}(J)$. Therefore,

$$c_u(t) = \sum_{H\geq J} \binom{t + \ell(J) - \ell(I)}{\ell(H)} S^H(\mathbb{E}). \tag{9.12}$$

For example, with $u = 113223$, we have $I = 33$, $J = 2121$,

$$\begin{aligned} c_u(X,A) =& (M_{33} + M_{213} + M_{321} + M_{2121})S^{2121} \\ &+ (M_{123} + M_{1113} + M_{1221} + M_{11121})S^{11121} \\ &+ (M_{312} + M_{3111} + M_{2112} + M_{21111})S^{21111} \\ &+ (M_{1212} + M_{12111} + M_{11112} + M_{111111})S^{111111} \end{aligned} \tag{9.13}$$

whose $(t, \mathbb{E})$-specialization reduces, by binomial convolution, to

$$\binom{t+2}{4}\frac{1}{2!1!2!1!} + \binom{t+2}{5}\frac{1}{1!1!1!2!1!} + \binom{t+2}{5}\frac{1}{2!1!1!1!1!} + \binom{t+2}{6}\frac{1}{1!1!1!1!1!1!} \tag{9.14}$$

This is the image of the polynomial

$$\left[\frac{t(2t^2+t)}{2}\right]^2 =: E^{2121}(t) \tag{9.15}$$

under the linear substitution $t^k \mapsto \binom{t+2}{k}$.

Define a linear map $\mathcal{F}_s : \ t^k \mapsto \binom{t+s}{k}$, and associate to a composition J the product of normalized Eulerian polynomials

$$E^J(t) := \prod_{k=1}^{\ell(J)} \frac{tE_{j_k}(t, t+1)}{j_k!} \tag{9.16}$$

as in Goldberg's formula [Reu93, Theorem 3.11], where

$$E_n(x,y) = \sum_{\sigma \in \mathfrak{S}_n} x^{d(\sigma)} y^{n-d(\sigma)}, \tag{9.17}$$

and $d(\sigma)$ is the number of descents of σ.

Theorem 9.1. *The t-Goldberg coefficient $c_u(t)$ is*

$$c_u(t) = \mathcal{F}_{s-r}(E^J(t)), \tag{9.18}$$

where $r = \ell(I)$, $s = \ell(J)$, and I, J are the compositions recording respectively the lengths of the nondecreasing runs and of the maximal blocks of identical letters of u.

The coefficient of t in $\mathcal{F}_{s-r}(t^k)$ is

$$\begin{aligned} (-1)^{k-s+r-1}\frac{(s-r)!(k-s+r-1)!}{k!} &= (-1))^{k-s+r-1}\mathrm{B}(s-r+1, k-s+r) \\ &= \int_{-1}^{0} t^k \cdot t^{r-1}(1+t)^{s-r}\frac{dt}{t^s} \end{aligned} \tag{9.19}$$

(since $k \geq s$ and $r \geq 1$), which applied to (9.18) yields the classical form of Goldberg's formula:

$$c_u = [t]c_u(t) = \int_{-1}^{0} t^{r-1}(1+t)^{s-r}\prod_{k=1}^{s}\frac{E_{j_k}(t,t+1)}{j_k!}dt. \tag{9.20}$$

Note that $r-1$ is the number of descents of u, and $s-r$ its number of rises.

10. Mixed cumulants and Goldberg coefficients

It remains to explain the occurrence of Goldberg coefficients in the expansion of mixed cumulants on the cumulant basis.

As before, we shall work with the t-analogues.

Let u be a packed word. Define as above two compositions $I(u)$ and $J(u)$ recording the lengths of the maximal nondecreasing factors and of the maximal blocks of identical letters of u. For example,

$$I(31121) = (1,3,1) \quad \text{and} \quad J(31121) = (1,2,1,1). \tag{10.1}$$

Note that $J(u) \geq I(u)$.

Following [HL17], with a pair of words such that $v \succeq u$, we associate their *refinement word* $m(u,v)$ defined as follows: $|m(u,v)| = \max(v)$, and $m_i = u_j$ whenever $v_j = i$. For example,

$$m(12113, 41223) = 2131, \;\; m(111123, 353241) = 31121. \tag{10.2}$$

We can now reformulate (7.14) as

$$U_v(t) = \sigma_1^t * \mathbf{N}_v = \sum_{\substack{w\in \mathrm{WRef}(v)\\ m=m(v,w)}} \binom{t+\ell(J(m))-\ell(I(m))}{|J(m)|} \mathbf{N}_w. \tag{10.3}$$

For example,

$$\begin{aligned} U_{112}(t) = &\binom{t+1}{2}\mathbf{N}_{112} + \binom{t}{2}\mathbf{N}_{221} + \binom{t+1}{3}\mathbf{N}_{123} + \binom{t+1}{3}\mathbf{N}_{132} \\ &+ \binom{t+1}{3}\mathbf{N}_{213} + \binom{t}{3}\mathbf{N}_{231} + \binom{t+1}{3}\mathbf{N}_{312} + \binom{t}{3}\mathbf{N}_{321}. \end{aligned} \tag{10.4}$$

Computing $U_{1122}(t)$, one finds a sum of 38 terms, corresponding to the packed words of which the first two or the last two letters can be identical. There are 7 different coefficients in this expansion, given below together with their associated words:

$$\begin{aligned} \binom{t+2}{4} &: \{1324, 1342, 3124, 3142\} \\ \binom{t+1}{4} &: \{1234, 1243, 1423, 1432, 2134, 2143, 2314, 2341, 2413, \\ &\quad\; 2431, 3214, 3241, 4123, 4132, 4213, 4231\} \\ \binom{t+1}{3} &: [1123, 1132, 1233, 1322, 2133, 2213, 2231, 3122] \\ \binom{t+1}{2} &: \{1122\} \\ \binom{t}{4} &: \{3412, 3421, 4312, 4321\} \\ \binom{t}{3} &: \{2311, 3211, 3312, 3321\} \\ \binom{t}{2} &: \{2211\} \end{aligned} \tag{10.5}$$

Let us now establish the equivalence of (7.14) and (10.3). We have to prove that for all v and w such that $w \in \mathrm{WRef}(v)$:

$$\binom{t+\max(w)-a_0(v,w)}{\max(w)} = \binom{t+\ell(J(m))-\ell(I(m))}{|J(m)|}. \tag{10.6}$$

First of all, it is clear, by definition, that $|J(m)| = \max(w)$. Moreover, $\max(w) - a_0(v,w)$ is the number of blocks of w (regarded as a set composition) whose all elements are all strictly to the left in v of all elements of the next block. This amounts precisely to merging two parts of $J(m)$ whenever the corresponding values in m values are in increasing order. Thus, $\max(w) - a_0(v,w) = \ell(J(m)) - \ell(I(m))$.

Consider now the expansion of $U_v(t)$ on the basis $\mathbf{K}_w$. Recall the transition matrices between $\mathbf{N}$ and $\mathbf{K}$.

$$\mathbf{K}_v = \sum_{w\in\mathrm{Ref}(v)} \frac{(-1)^{\max(w)-\max(v)}}{\pi_{\mathrm{ev}}(m(u,v))}\mathbf{N}_w, \tag{10.7}$$

where $\pi_{\mathrm{ev}}(m) = \prod_i |m|_i$.

Notice that by definition

$$\mathbf{K}_v = [t^{\max(v)}]\mathbf{N}_v * \sigma_1^t. \tag{10.8}$$

Indeed, to obtain the correct power of t, one must select in each $\binom{t}{M}$ the coefficient of t, which is $\frac{(-1)^{M-1}}{M}$.

In the other direction,

$$\mathbf{N}_v = \sum_{w\in\mathrm{Ref}(v)} \frac{1}{\pi_{\mathrm{ev}}(m(u,v))!}\mathbf{K}_w, \tag{10.9}$$

where $\pi_{\mathrm{ev}}(m)! = \prod_i |m|_i!$.

Ths yields for the expansion of $U_v(t)$ on the basis $\mathbf{K}$

$$U_v(t) = \sum_{x\in\mathrm{WRef}(v)} c_{m(v,x)}(t)\mathbf{K}_x, \tag{10.10}$$

where $c_{m(v,x)(t)}$ is defined in Eq. (9.12).

Indeed, to go from the expression of U_v on the $\mathbf{N}_w$ to its expression on the $\mathbf{K}_x$, notice that the $\mathbf{N}_w$ contributing to a given $\mathbf{K}_x$ are those such that $w \le x$ and $w \succeq v$. This is a boolean lattice, an interval $[x_0, x]$ where x_0 is obtained by merging consecutive blocks of x whenever these blocks are contained in a block of v.

We must therefore evaluate

$$\sum_{w\in[x_0,x]} \frac{1}{\pi_{\mathrm{ev}}(m(w,x))!}\binom{t+\ell(J(m(v,w)))-\ell(I(m(v,w)))}{|J(m(v,w))|}. \tag{10.11}$$

For a $w \in [x_0, x]$, it is easy to compute $m(v,w)$. The set of those $m(v,w)$ is the set of words obtained from $m(v,x)$ by replacing blocks i^r of consecutive identical letters by smaller blocks i^p, $1 \le p \le r$. On this set, $\ell(J(m)) - \ell(I(m))$ is constant, $H = \mathrm{ev}(m(w,x))$ runs over the set of compositions finer than $\mathrm{ev}(m(x_0,x))$, $1/\mathrm{ev}(m(w,x))! = S^H(\mathbb{E})$ and $|J(m(v,w))| = \max(w) = \ell(H)$.

For example, let $v = 111123$ and $x = 356241$. Then, $x_0 = 244231$. The contribution to $\mathbf{K}_x$ in $U_v(t)$ come from the four words

$$\{244231, 245231, 355241, 356241\}. \tag{10.12}$$

Their words $m(v,w)$ are respectively 3121, 31211, 31121 and 311211. The $m(w,x)$ are 122344, 122345, 123455 and 123456. The sum of all contributions is

$$\frac{1}{4}\binom{t+1}{4}+\frac{1}{2}\binom{t+1}{5}+\frac{1}{2}\binom{t+1}{5}+\binom{t+1}{6}. \tag{10.13}$$

This is the same as $c_u = c_{311211}(t)$. Indeed, $I(u) = (1,3,2)$ et $J(u) = (1,2,1,2)$, so that we get

$$\sum_{H\geq 1212}\binom{t+1}{\ell(H)}S^H(\mathbb{E}) \tag{10.14}$$

which yields the same expression as (10.13).

Finally, one can give the general formula for $\mathbf{N}_u * \sigma_1^t * \mathbf{N}_v$ on the $\mathbf{K}_w$. This is immediate, since the transition from $\mathbf{N}$ to $\mathbf{K}$ does not change the structure of the product. We get

$$\mathbf{N}_u * \sigma_1^t * \mathbf{N}_v = \sum_{w\in W(u,v)}\prod_{i=1}^{\max(u)} c_{m^{(i)}}(t)\mathbf{K}_w, \tag{10.15}$$

where $c_u(t)$ are the t-Goldberg coefficients, and $m^{(i)} = m(v^{(i)}, w^{(i)})$.

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