Dissecting the square into an odd number of triangles of almost equal area

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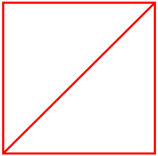

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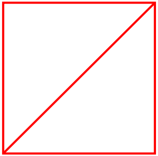

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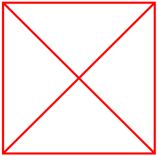

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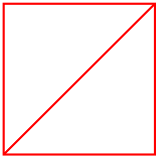

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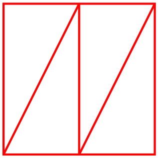

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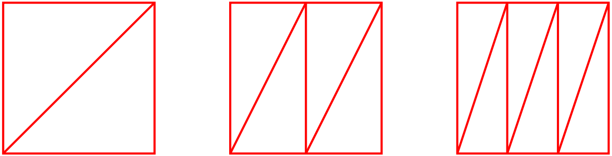

2 4 6 . . .

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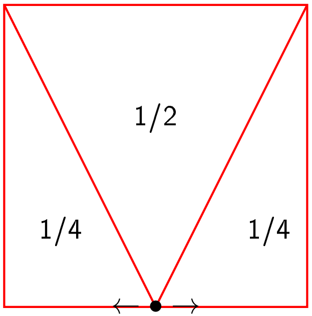

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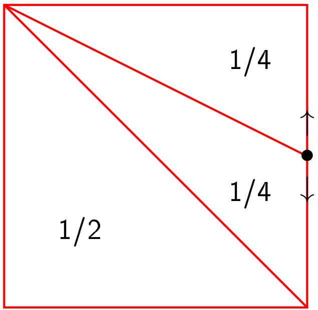

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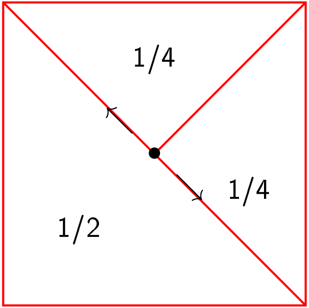

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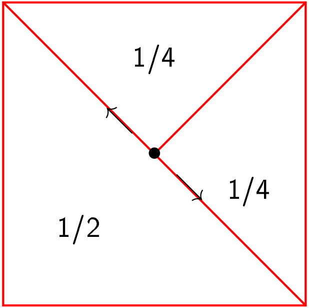

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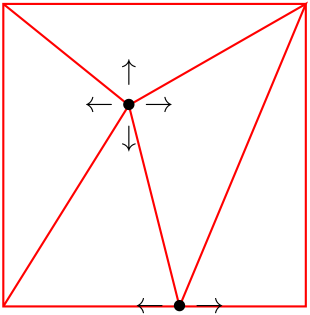
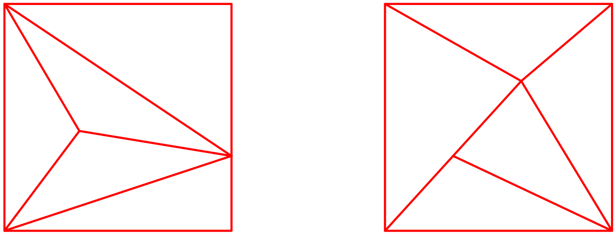


Question: Is it possible to dissect a square into an odd number of triangles of equal area?

Triangulation vs Dissection



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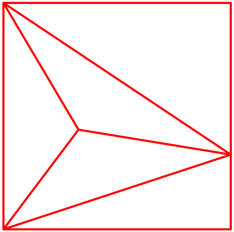
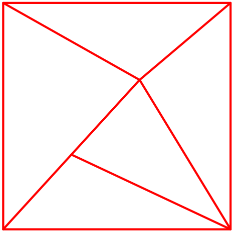


Face-to-face:

Triangulation



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Dissection

Monsky's proof *from the book*

Theorem (Richman–Thomas, Monsky (1970))

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3. Every finite dissection of the square contains an odd number of rainbow triangles. Thus at least one!

□

Ok... but how *close* can the areas be?

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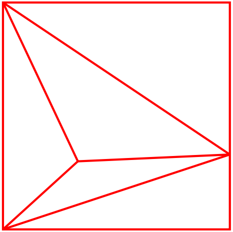
"A difference between two things that should be the same."

Intuition of low discrepancy



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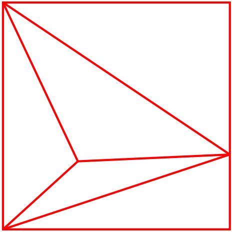

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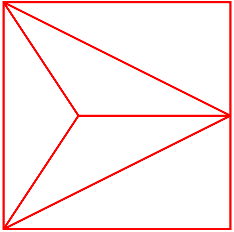

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Measuring area deviation

D: dissection with triangle areas $A_1, \ldots, A_n$

- Root-mean-square error (RMS, standard deviation):

$$\mathrm{RMS}(D) := \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(A_i - \frac{1}{n}\right)^2}$$

- Range:

$$\mathrm{R}(D) = \max_{i,j\in[n]} |A_i - A_j|$$

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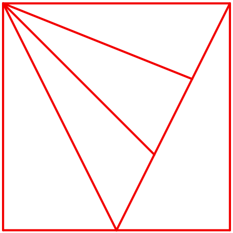
$$\boxed{\frac{\mathrm{R}(D)}{2\sqrt{n}} \le \mathrm{RMS}(D) \le \mathrm{R}(D)}$$

Graph of a dissection

Definition (Graph Γ of a dissection)

Nodes : corners of triangles

Edge : between corners of a triangle not containing side nodes



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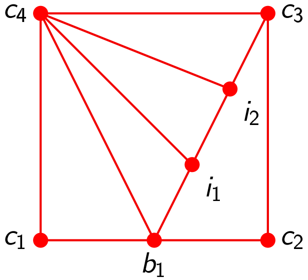

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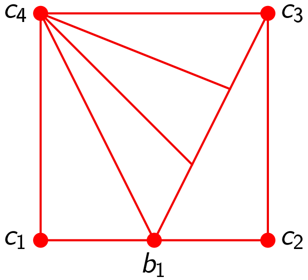

Boundary nodes

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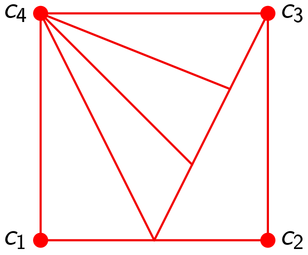

Corner nodes

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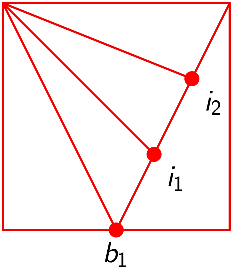

Side nodes

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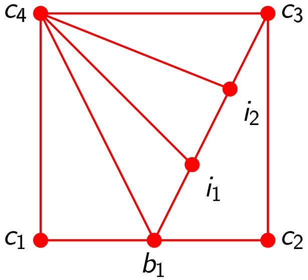
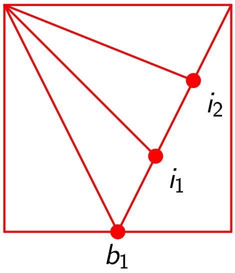


Edges

Simplicial graph of a dissection

Side nodes $\longrightarrow$ linear constraints



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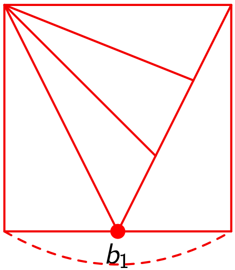

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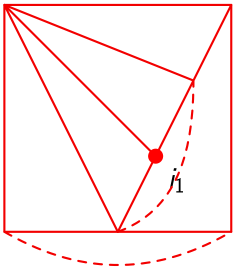

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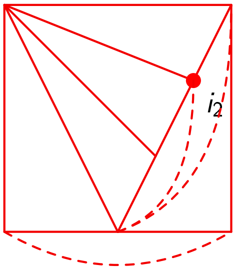

$i_2 \longrightarrow (i_2, (b_1, c_3))$

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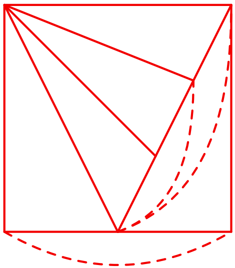

Another planar graph with more triangles:
a simplicial graph of the dissection

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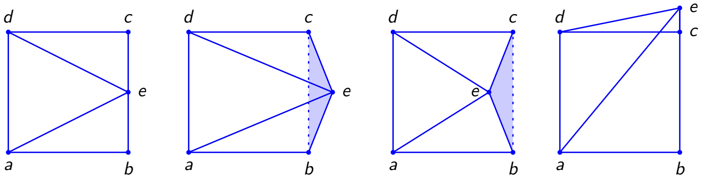

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The area difference polynomial $\pi_D \in \mathbb{R}[X_D]$ of D is the polynomial

$$\pi_D(X_D) = \sum_{i \in [n]} \left(A(t_i) - \tfrac{1}{n}\right)^2 + \sum_{\ell \in L} A(\ell)^2 + \sum_{v \in C} \left((x_v - p_v)^2 + (y_v - q_v)^2\right).$$

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where (p_c, q_c) are the coordinates of the corners of the square and $A(t_i)$ denotes the area of triangle t_i, i.e. a determinant of size 3

$$A(t_i) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Example

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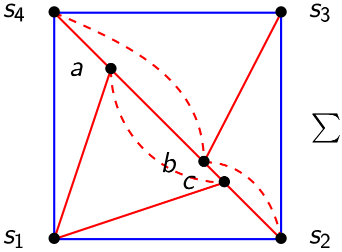
Consider the graph of the dissection:



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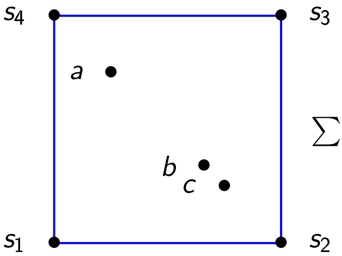

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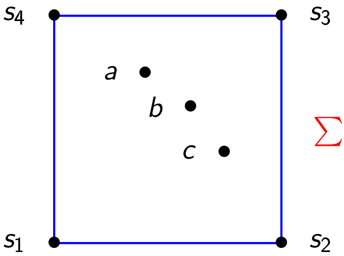

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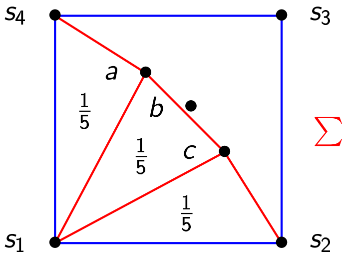

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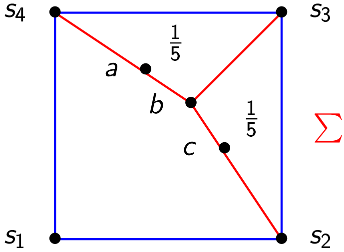

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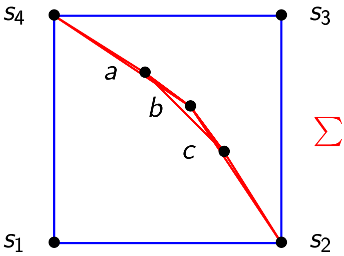

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In the discrepancy polynomial:

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$\mathcal{D}_n :=$ { all dissections of the square with n triangles }.
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How fast does

$$\Delta(n) \overset{n\to\infty}{\longrightarrow} 0 \quad ?$$

Previous results

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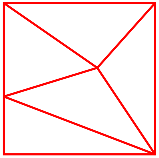

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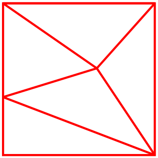
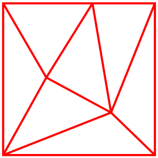

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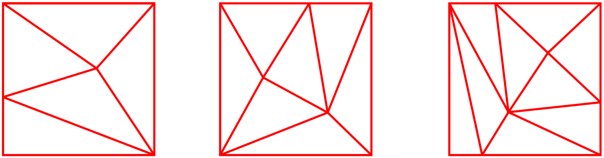

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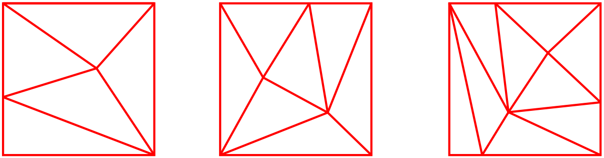
Mansow (2003) used `Matlab` to study the range of triangulations

$$M(n) = \min_{T \in \mathcal{D}_n} \left\{ \max_{t_i, t_j \in T} |A(t_i) - A(t_j)| \right\}.$$

$M(5) \leq 0.0225 \quad M(7) \leq 0.0031 \quad M(9) \leq 0.00014$



and $M(11) \leq 4.2 \times 10^{-6}$, (weakly) suggesting an exponential decrease.

Previous results

Upper bound for triangulations

Easy constructions: an upper bound of the form $O(1/n^2)$.

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Proof technique: used the theory of continued fractions.

New results – Lower bound

Because $\mathrm{R}(D) \geq \mathrm{RMS}(D)$ and $n\mathrm{RMS}(D)^2 = \pi_D(X_D)$,
it suffices to get a lower bound for $\pi_D(X_D)$ to bound $\mathrm{R}(D)$.
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Proof technique: Gap theorems from real algebraic geometry.

"An algebraic number $\alpha \neq 0$ can not be arbitrarily close to 0."

...depending on the degree and the size of the coefficients of its minimal polynomial.

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Using intuitions from exhaustive generation and exceptional properties of the Thue–Morse sequence,

we provide a family of dissections Z_n for every odd n with

$$\mathrm{R}(Z_n) \leq \frac{1}{n^{\log_2 n - 5}} = \frac{1}{2^{\Omega(\log^2 n)}} \text{ (superpolynomial)}$$

... back to the area difference polynomial.

How to get a lower bound on $\pi_D(X_D)$?

Ansatz: gap theorem in real algebraic geometry

Theorem (Emiris–Mourrain–Tsigaridas, 2010)

If $f \in \mathbb{Z}[x_1, \ldots, x_k]$ *is strictly positive on the* k*-simplex:*

$$\left\{ x \in \mathbb{R}^k_{\geq 0} \ : \ \sum_{i=1}^{k} x_i \leq 1 \right\},$$

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and f is of degree d, with coefficients bounded by 2^τ,

then the minimum m_{DMM} of f on the k-simplex satisfies

$$-\log m_{DMM} \quad < \quad (k^2 + k) \log \sqrt{d} + \left[k^2 \log d + k(3 + 3 \log d + \tau + d \log k) + d(\log k + 1) + \log d + \tau + 2\right] d(d-1)^{k-1}.$$

m_{DMM} is the Davenport–Mahler–Mignotte bound.

Ansatz: gap theorem in real algebraic geometry

Corollary

The minimum M for the discrepancy polynomial $\pi_D(X_D)$ satisfies

$$-\log M = O(n^2 9^n).$$

In other words,

$$\boxed{\Delta(n) = \frac{1}{2^{O(n^2 9^n)}} = \frac{1}{2^{2^{O(n)}}}.}$$

Open Question: How to get a better lower bound?

Improving the upper bound

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Generate and optimize the dissections with 9 triangles and 8 vertices took 3 days

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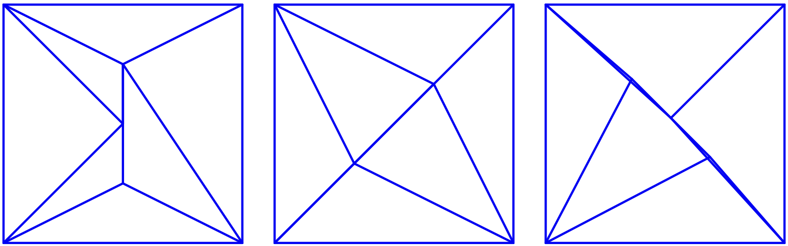
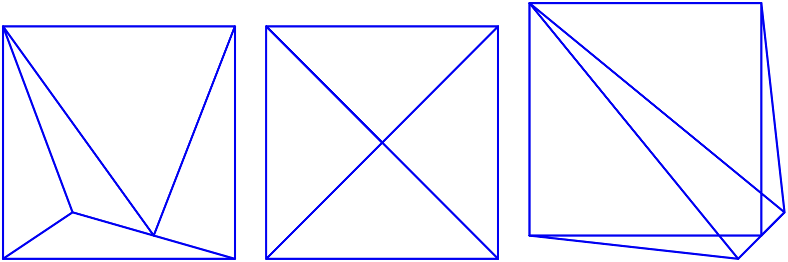


Computational evidences

We now know more on the gradient variety:

- Can have dimension > 0
- Some dissections degenerate or flip-over
- Some dissections have minima outside the square

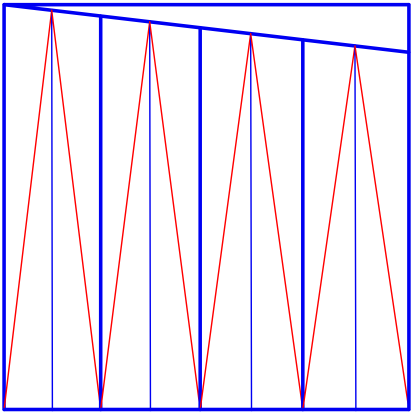


Dissections achieve better bounds

7 triangles		Triangulations	Dissections
7 vertices	$\pi_D(X_D)$	0.0000114433268	0.000183330891
	Range	0.00400810	0.0127879
8 vertices	$\pi_D(X_D)$	0.0000753290	$4.23566898 \times 10^{-6}$
	Range	0.0102149	0.00232068

n	RMS
3	1.17851×10^{-1}
5	1.0295×10^{-2}
7	7.778788×10^{-4}
9*	2.736839×10^{-4}

A nice family of dissections



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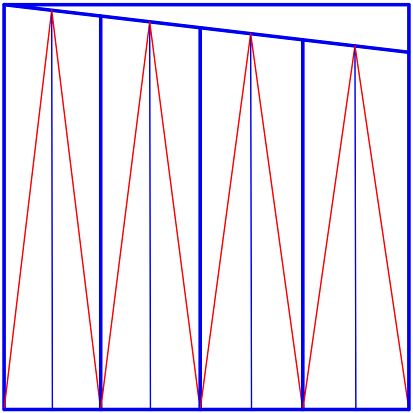
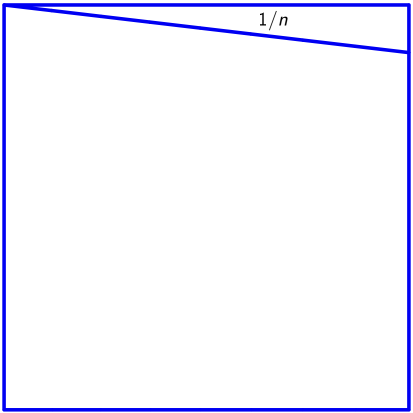


Theorem (L.-Rote-Ziegler)

This family of dissections has a range order of $O(1/n^5)$.

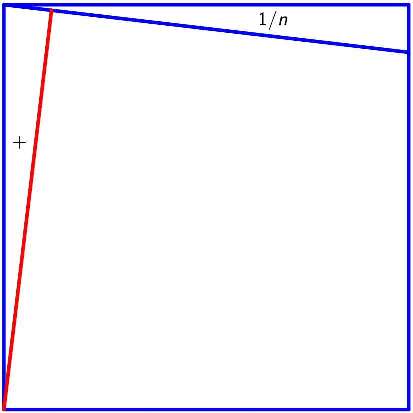
An even nicer family of dissections

Thue–Morse sequence:



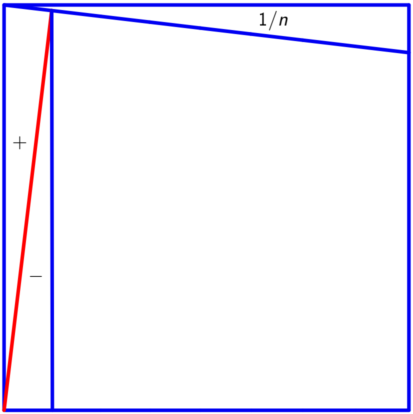
An even nicer family of dissections

Thue–Morse sequence: $+$



An even nicer family of dissections

Thue–Morse sequence: $+, -$



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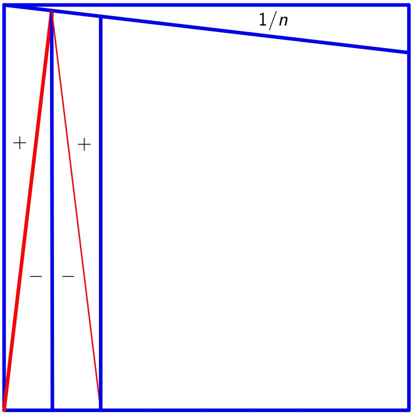
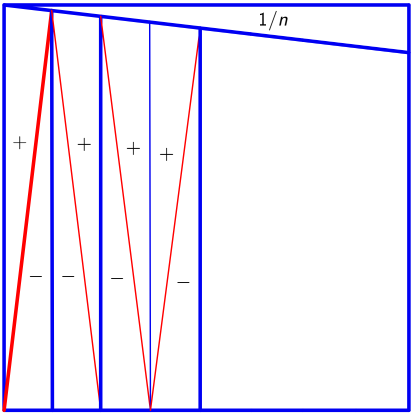


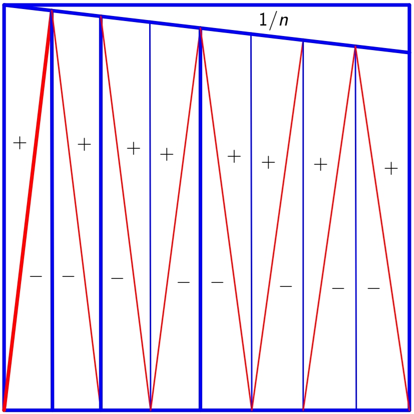
An even nicer family of dissections

Thue–Morse sequence: $+, -, -, +, -, +, +, -,$



An even nicer family of dissections

Thue–Morse sequence: $+, -, -, +, -, +, +, -,$ etc.



An even nicer family of dissections

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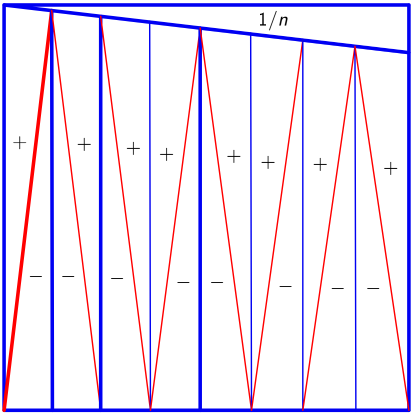
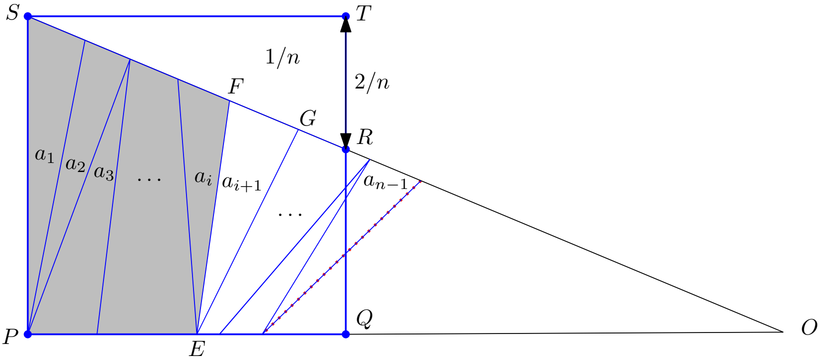


Theorem (L.-Rote-Ziegler)

This family of dissections has minimal range at most $\frac{1}{n^{\Omega(\log_2 n)}}$.

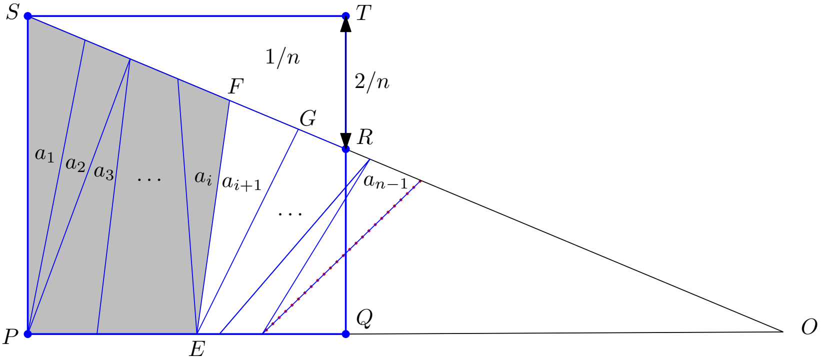
Estimating the error



Set $A_i := a_1 + \cdots + a_i$, we have

$$\frac{\triangle EGO}{\triangle EFO} = \frac{n/4 - A_{i+1}}{n/4 - A_i} \text{ and } \overline{RO}/\overline{SO} = \overline{QO}/\overline{PO}$$

Estimating the error



To end with a vertical segment, the product of the ratios of "+" and "-" should equal $\overline{RO}/\overline{SO}$ and $\overline{QO}/\overline{PO}$:

$$\prod_{i=1}^{n-1} \left(\frac{n/4 - A_{i+1}}{n/4 - A_i} \right)^{\tau_i} \stackrel{!}{=} 1.$$

The key property

The Thue–Morse sequence $\{s_i\}_{i\geq 1}$ annihilates powers:

Lemma (Prouet (1851))

Let $k \geq 0$, $b \neq 0$, and let $f(x)$ be a polynomial of degree d. If $d \geq k$, then there is a polynomial $F(x)$ of degree $d-k$ such that the following identity holds for all x_0:

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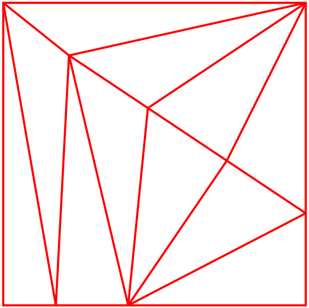
- Set $u := 4/n^2$ and write $\Phi := \prod_{i=1}^{n-1} \left(\frac{1-iu}{1-(i-1)u}\right)^{s_i}$
- Take the logarithm of Φ and express it as a Taylor expansion around $1/n$
- Use the lemma to make the areas a_i's be close to $1/n$ to a "high degree"

Open Question

- Can a family of triangulation with exponentially decreasing discrepancy be constructed?
- That is, is the smallest discrepancy *really* exponential?

Merci!



A small discrepancy triangulation with 11 triangles