From mesh generation to associahedra and related structures

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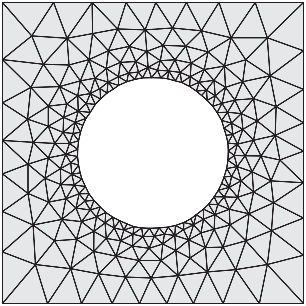


Summary

1. Mesh generation

The Delaunay criterion,

Some generation algorithms.

2. The Euclidean case

Polygons and punctured polygons.

3. Topological surfaces

Filling surfaces,

The growth rate of the diameter of flip-graphs.

4. Conclusion

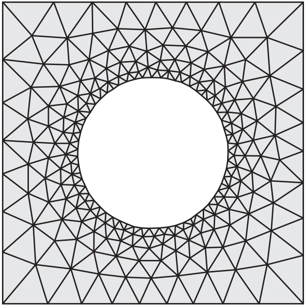
Open problems...

A mesh is....

... a decomposition of some (often Euclidean) space (or a portion of it) into elementary cells (triangles, polygons, simplices, polyhedra...)

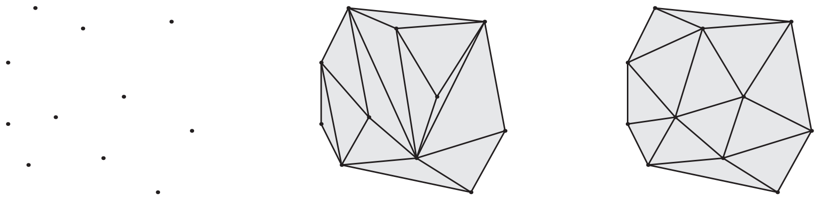
Meshes are used in a number of applications:

- Numerical Analysis,
- Engineering (all flavors),
- Computer graphics,
- ...



Meshes whose cells are triangular (or simplicial) are called **triangulations.**

A given portion of space can have several meshes, even for a same prescribed set of vertices. Not all of them are of the same quality.

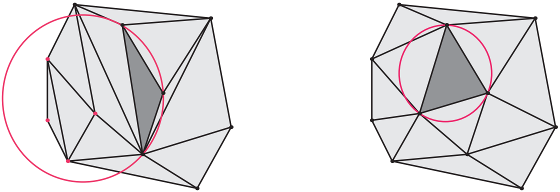


A good mesh usually...

- does not have elongated cells,
- has cells whose complexity is controlled (triangles),
- has vertices whose degree is controlled.

Definition (Delaunay, 1934).

A d-dimensional triangulation is called *Delaunay* when the spheres circumscribed to its d-simplices do not enclose any vertex.



Properties.

- Cells are triangles whose elongation is the least possible,
- Minimal angle is maximal (vertex degree is controlled),
- Unique for a set of points in general position.

Definition (Delaunay, 1934).

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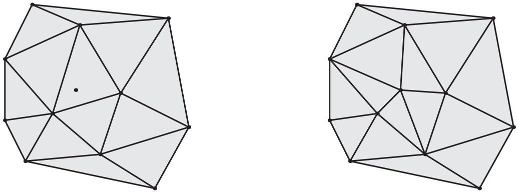

Construction of a Delaunay triangulation (Joe, 1991)

0) Consider a triangle that contains all the points,
1) Insert a point a in the triangulation,
2) If needed, modify the triangles around a and return to 1).

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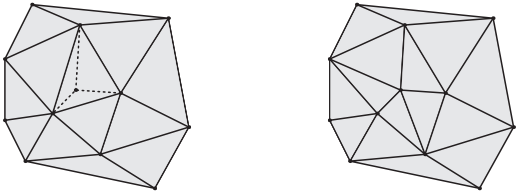

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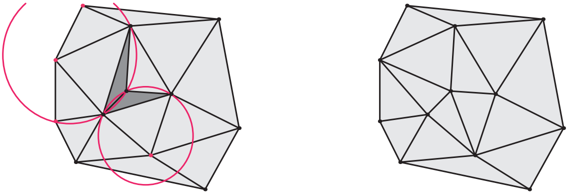

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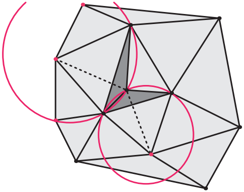

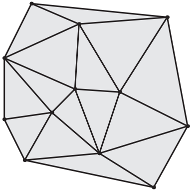

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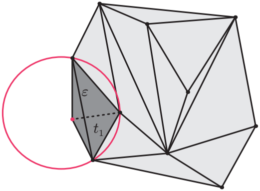

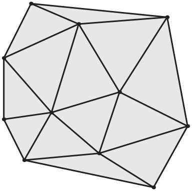

Making a triangulation Delaunay (greedy algorithm)

0) Consider the two triangles t_1 and t_2 incident to an edge ε,
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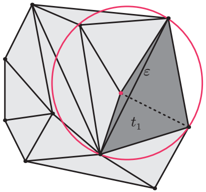

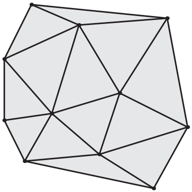

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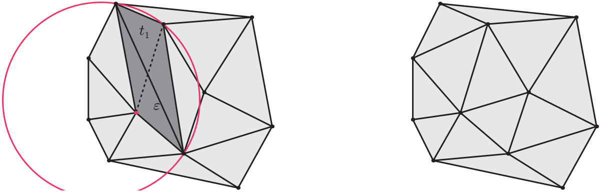

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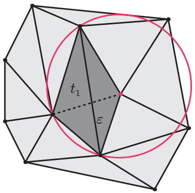

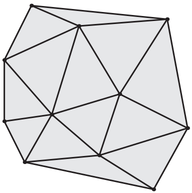

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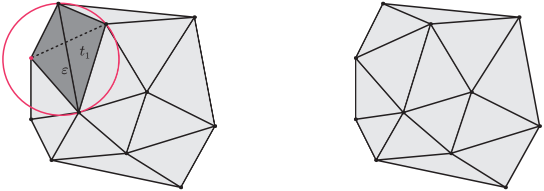

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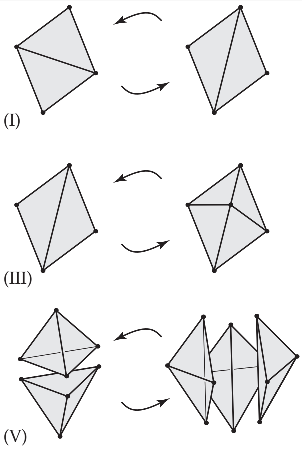
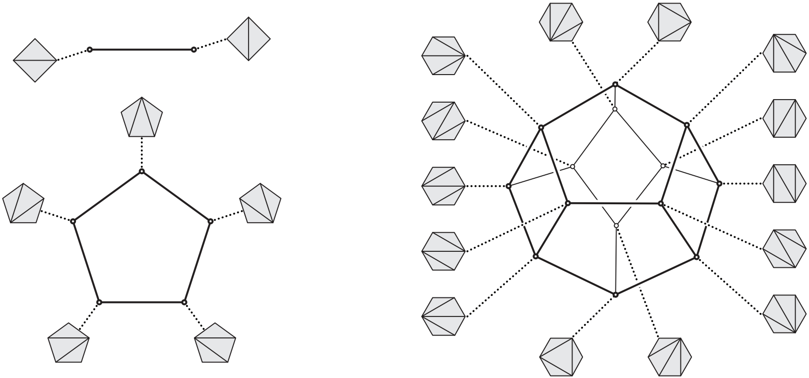
Informally, flips are...

...*local operations*, that modify the triangulations of a *fixed* point set $\mathcal{A}$.



The flip-graph of a point set $\mathcal{A}$ is the graph whose...

- ...vertices are the triangulations of $\mathcal{A}$,
- ...edges are the flips between them.



Question.

Is the flip-graph of a finite set of points $\mathcal{A}$ always connected?

Yes when

- $\mathcal{A}$ has dimension at most 2 (Lawson, 1972),
- $|\mathcal{A}| - \dim(\mathcal{A}) \leq 4$ (Azaola and Santos, 2000).

No...

- ... for particular sets $\mathcal{A}$ of dimension 6 and above (Santos, 2000),
- ... or of dimension 5 (Santos, 2005).

Open Question.

Is the flip-graph of a finite, 3- or 4-dimensional set of points always connected?

One can enumerate the triangulations of a finite set of points efficiently by exploring its flip-graph *provided this graph is connected!*

Examples.

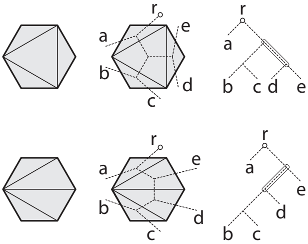
- If the set of points is made up of the vertices of a n-gon, the number of triangulations is Catalan number C_{n-2},
- The square has 2 symmetrical triangulations,
- The (3-dimensional) cube has 74 triangulations, partitioned into 6 symmetry classes (De Loera, 1996),
- The 4-dimensional cube has 92 487 256 triangulations, partitioned into 247 451 symmetry classes (P, 2013).

Open Question.

How many triangulations does the d-dimensional cube have when $d > 4$?

Remark (Sleator, Tarjan, and Thurston, 1988).

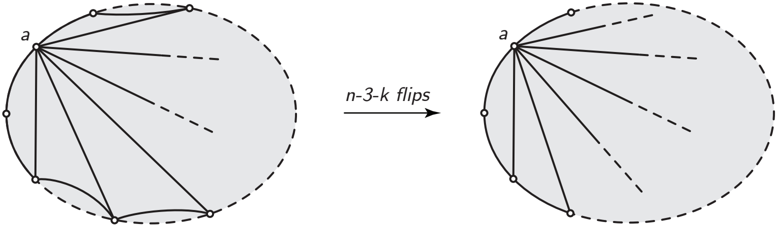
The flip-graph of a convex polygon can be alternatively obtained by replacing triangulations by binary trees and flips by rotations.



Theorem (Sleator, Tarjan, and Thurston, 1988).

The diameter of the flip-graph of a convex n-gon is at most $2n - 10$ when $n > 12$ and this bound is sharp when n is *large enough*.

In order to obtain the upper bound of $2n - 10$, flip a triangulation U (left) to a canonical triangulation (right) as follows:



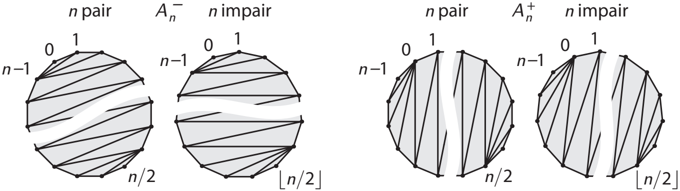
Here, k is the number of interior edges of U incident to vertex a. Call l the number of interior edges incident to a in another trianulation V. One can transform U into V with $2n - 6 - (k + l)$ flips.

When $n > 12$, a counting argument provides a vertex a so that $k + l \geq 4$.

Hence $d(U, V) \leq 2n - 10$ when $n > 12$.

Theorem (P, 2014).

The diameter of the flip-graph of a convex n-gon is $2n - 10$ when $n > 12$.

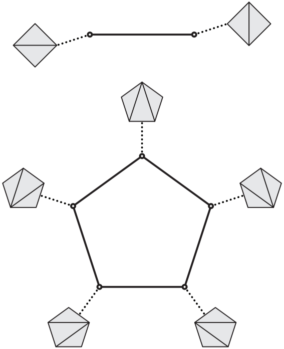


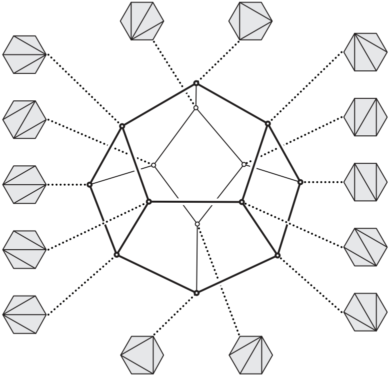
Open Questions.

- Other pairs of triangulations with this distance?
- Polynomial algorithm to compute flip-distance?
- Other (2-dimensional) sets of points?

Theorem (Lee, 1989).

The flip-graph of a convex *n*-gon is the 1-skeleton of the $(n-3)$-dimensional associahedron.



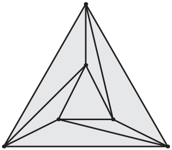


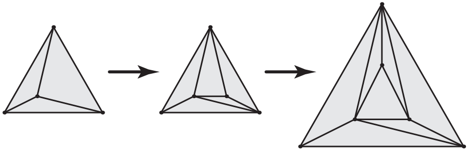
Theorem (Lee, 1989).

The flip-graph of a convex n-gon is the 1-skeleton of the $(n-3)$-dimensional associahedron.

Definition.

A d-dimensional triangulation is *regular* if it can be obtained by projecting the *lower* faces of a $(d+1)$-dimensional polytope.





Theorem (Lee, 1989).

The flip-graph of a convex n-gon is the 1-skeleton of the $(n-3)$-dimensional associahedron.

Theorem (Gel'fand, Kapranov, and Zelevinskiĭ, 1990).

The sub-graph induced by regular triangulations in the flip-graph of a d-dimensional set of n points is the 1-skeleton of a $(n-d-1)$-dimensional polytope, called the *secondary polytope*.

Open Questions.

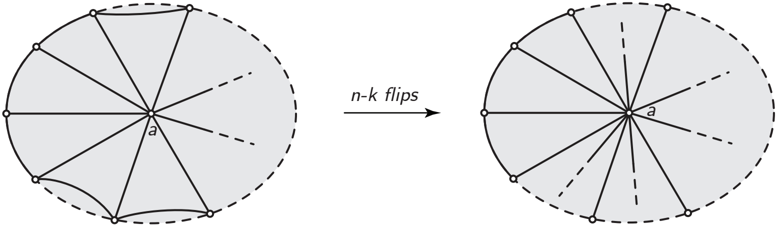
- Diameter (of the 1-skeleton) of secondary polytopes?
- Do secondary polytopes satisfy the Hirsch bound ($\Delta \leq f - v$)?
- Polytopal realizations of other flip-graphs?

Remark.

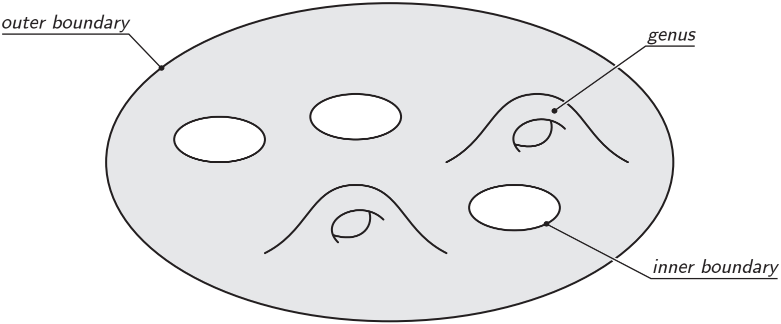
All the triangulations of a *punctured* polygon (i.e. with a unique interior point), or of a twice punctured polygon are regular.

Theorem (Parlier and P, 2016^{+})

The maximal distance between two triangulations of a punctured n-gon is at most $2n-7$ and, for some placements of the puncture, at least $2n-8$.

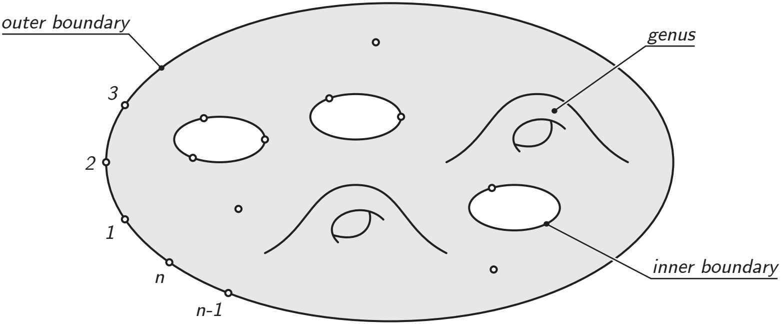


Consider an orientable surface Σ of genus g with $r > 0$ boundaries. One of the boundaries, the *outer boundary*, will be privileged.



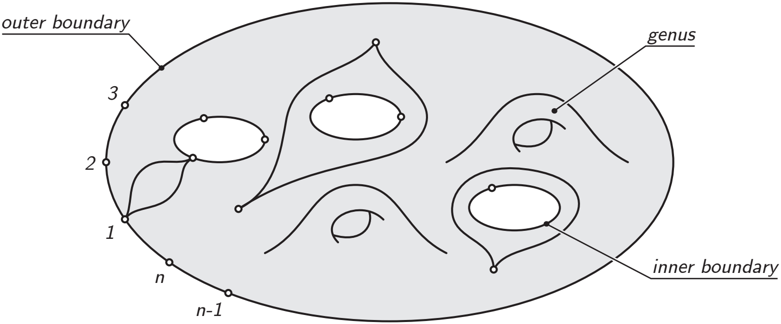
(This is an orientable surface with 4 boundaries and genus 2)

Place n on the outer boundary, labeled from 1 to n clockwise, as well as other vertices on the surface (with at least one vertex on each boundary). Let Σ_n denote the resulting surface.



(This is an orientable surface with 4 boundaries and genus 2)

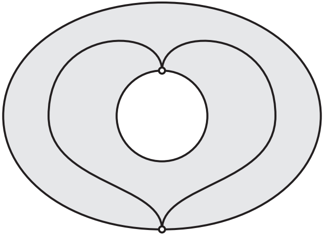
Two arcs are *isotopic* if they can be continuously deformed into one another. They are non-isotopic when some obstacle lies between them (a boundary, a genus, or a vertex).



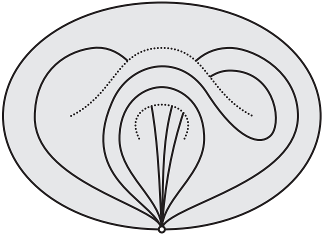
(This is an orientable surface with 4 boundaries and genus 2)

Definition.

A triangulation of Σ_n is a maximal set of pairwise non-crossing, non-isotopic arcs whose vertices are the ones we have placed on Σ in order to obtain Σ_n.



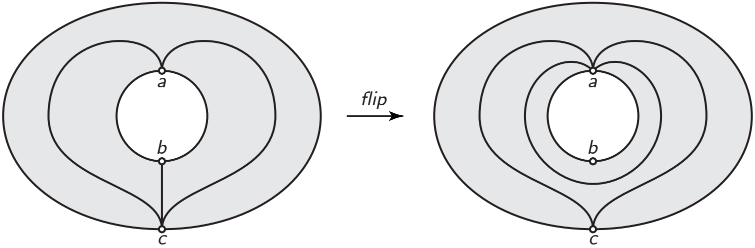
Triangulation of a cylinder with a vertex on each boundary.



Triangulation of a bordered torus with a vertex on the boundary.

Definition.

A flip consists in exchanging the diagonals of a quadrilateral.



Lemma (Švarc, 1955, Milnor, 1968).

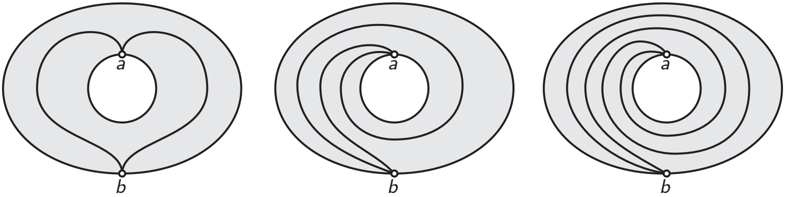
The flip-graph whose vertices are the triangulations of Σ_n is quasi-isometric to any Cayley graph of the mapping class group of Σ_n.

Question.

What can one tell about the flip-graph of Σ_n?

Some answers.

- This graph is connected (Mosher, 1988),
- This graph can be infinite...



A natural way to solve this problem is to consider the triangulations of Σ up to homeomorphism.

Question.

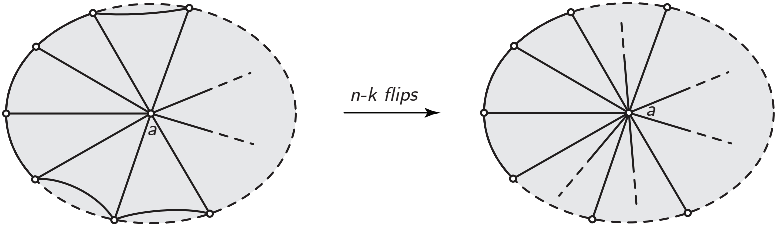
Call $\mathcal{MF}(\Sigma_n)$ the quotient of the flip-graph of Σ_n by its homeomorphisms. What is the diameter Δ of $\mathcal{MF}(\Sigma_n)$?

What we know (so far!)

- Disk: $\Delta = 2n - 10$ when $n > 12$ (P, 2014),
- Cylinder with a unique point on the inner boundary:
$$\Delta = \lfloor 5n/2 \rfloor - 2 \text{ (Parlier and P, } 2014^{+}),$$
- Surface with three boundaries: $\Delta = 3n + O(1)$ (Parlier and P, 2014^{+}),
- Torus with a boundary containing all the vertices:
$$5n/2 + O(1) \leq \Delta \leq 23n/8 + O(1) \text{ (Parlier and P, } 2015^{+}),$$
- Punctured disk: $\Delta = 2n - 2$ (Parlier and P, 2016^{+}).

Theorem (Parlier and P, 2016^+)

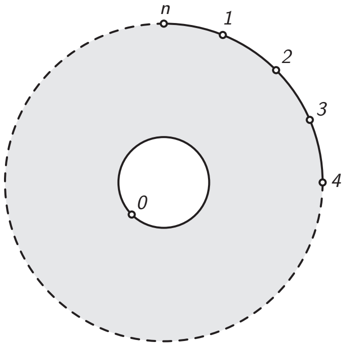
The maximal distance between two triangulations of the punctured disk is exactly $2n-2$.



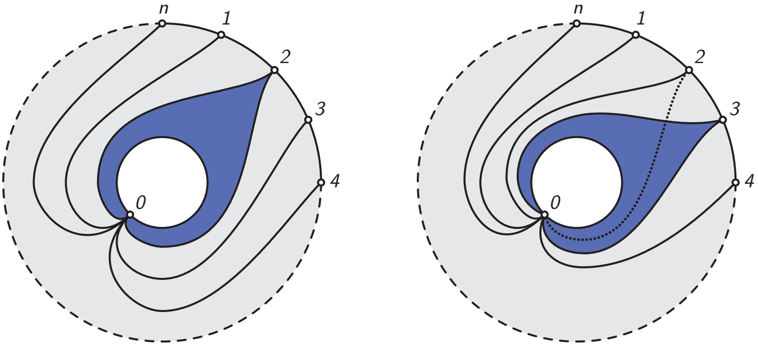
Remarks.

i. The puncture cannot disappear and is incident to at least one edge.

ii. The puncture can be incident to exactly one edge of the triangulation, Consider a cylinder with n vertices in the outer boundary labeled 1 to n clockwise and one vertex labeled 0 in the inner boundary.



There is a (blue) triangle in any triangulation, whose vertices are 0 and a boundary vertex x. A triangulation has exactly $n+2$ interior edges.



Hence at most $n-1$ flips are required to insert all the edges incident to 0.

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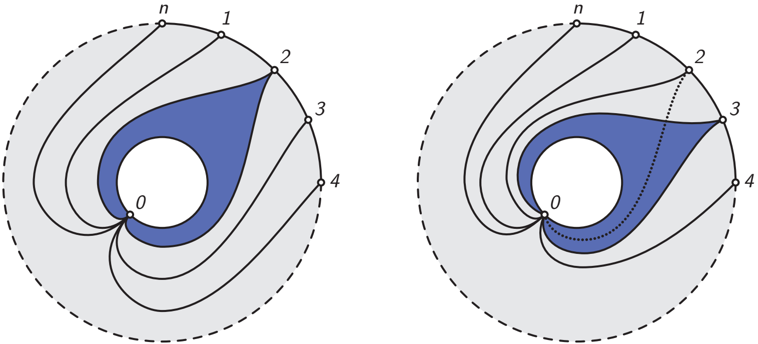
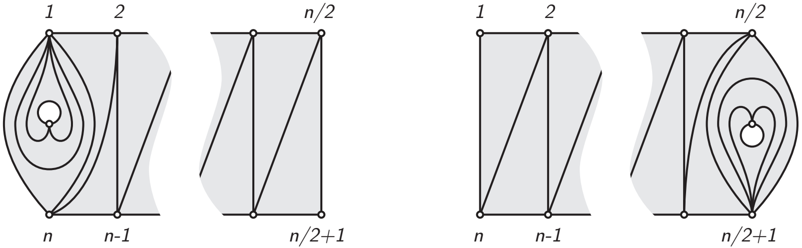


Boundary vertices are separated by at most $n/2-1$ vertices. Bringing the blue triangles together thus requires at most $n/2$ flips.

Lemma

The distance of two triangulations of $\mathcal{A}$ is not greater than $\lfloor 5n/2 \rfloor - 2$.

This upper bound is sharp for all $n \geq 1$.



These two triangulations have flip distance exactly $\lfloor 5n/2 \rfloor - 2$.

Remarks.

- We have observed growth rates of 2 (disk, punctured disk), 5/2 (cylinder), 3 (surface with 3 boundaries).
- Growth rates cannot exceed 4 (Parlier and P, 2014^+)...
- ...or be between 2 and 5/2 (Parlier and P, 2016^+).

Open Questions.

- What are all the possible growth rates?
- What is the *exact* growth rate for the bordered torus?
- What diameters can we compute when we allow the topology to vary (and not only n)?
- ...and when vertices are not marked?
- Polynomial algorithm to compute the distance?