Cop and Robber Game and Hyperbolicity

J. Chalopin[1] V. Chepoi[1] P. Papasoglu[2] T. Pecatte[3]

[1]LIF, CNRS & Aix-Marseille Université

[2]Mathematical Institute, University of Oxford

[3]ÉNS de Lyon

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Cop & Robber Game

A game between one cop **C** and one robber **R** on a graph G

Initialization:

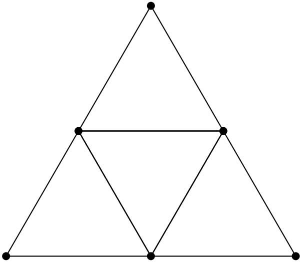
- **C** chooses a vertex
- **R** chooses a vertex

Step-by-step:

- **C** traverses at most 1 edge
- **R** traverses at most 1 edge

Winning Condition:

- **C** wins if it is on the same vertex as **R**
- **R** wins if it can avoid **C** forever



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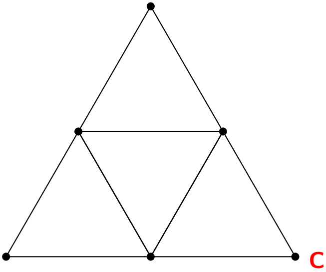

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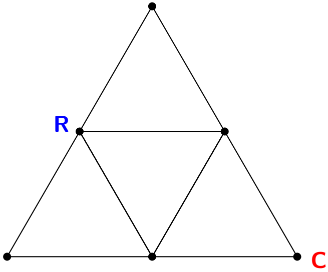

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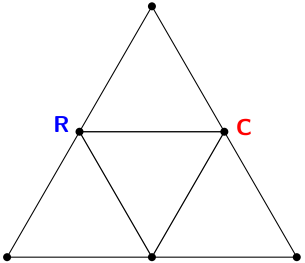

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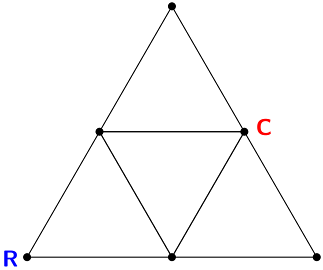

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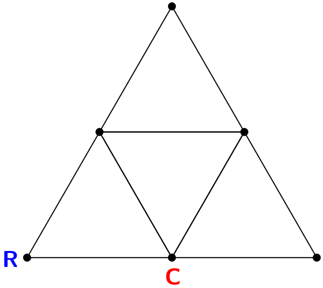

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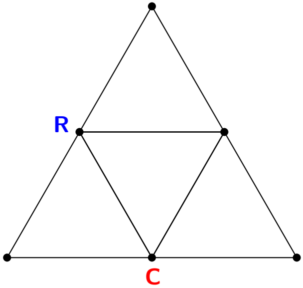

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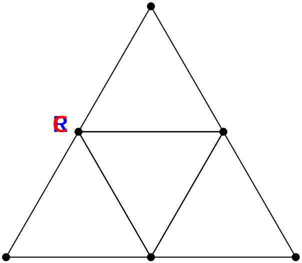


Cop-win graphs are dismantlable graphs

A graph G is cop-win if **C** can win whatever **R** does

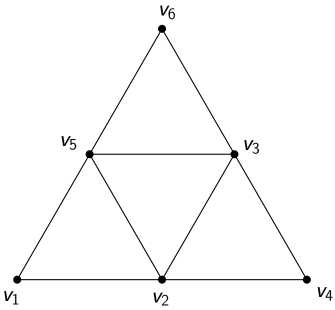
Theorem (Nowakowski and Winkler; Quilliot '83)

A graph G is cop-win iff there exists a dismantling order $v_1, v_2, \ldots, v_n$ such that

$$\forall i > 1, \exists j < i, N[v_i, G_i] \subseteq N[v_j]$$

G_i: graph induced by $X_i = \{v_1, v_2, \ldots, v_i\}$

Examples of cop-win graphs: trees, cliques, chordal graphs, bridged graphs



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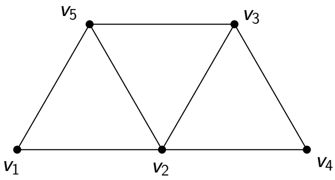

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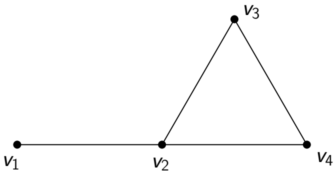

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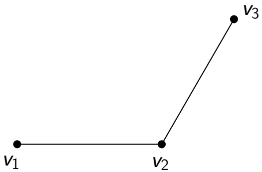

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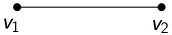

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Cop & Robber Game with Speeds

A game between one cop **C** moving at speed $\mathbf{s}'$ and one robber **R** moving at speed $\mathbf{s}$

Same game as before except that at each step

- **C** traverses at most $\mathbf{s}'$ edges
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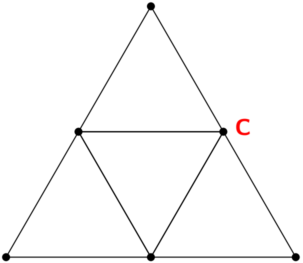

- **C** has speed $\mathbf{s}' = 1$
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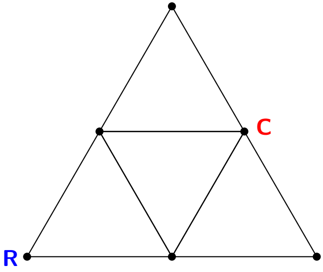

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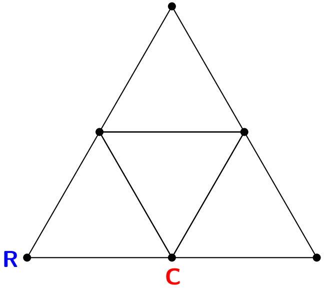

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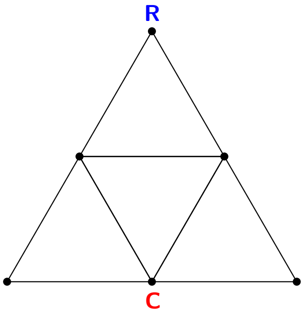

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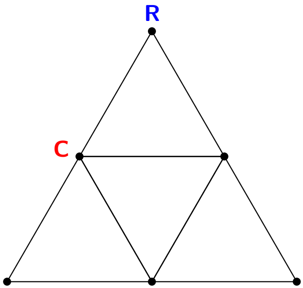

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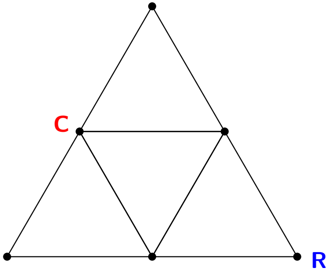

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(s, s')-Cop-win Graphs and (s, s')-dismantlability

A graph G is $(\mathbf{s},\mathbf{s'})$-cop-win if **C** (moving at speed $\mathbf{s'}$) can win whatever **R** (moving at speed $\mathbf{s}$) does

Remark

If $\mathbf{s} < \mathbf{s'}$, every graph is $(\mathbf{s},\mathbf{s'})$-cop-win

Theorem (C., Chepoi, Nisse, Vaxès '11)

A graph G is $(\mathbf{s},\mathbf{s'})$-cop-win if and only if there exists a $(\mathbf{s},\mathbf{s'})$-dismantling order $v_1, v_2, \ldots, v_n$ such that

$$\forall i > 1, \exists j < i, B_s(v_i, G \setminus v_j) \cap X_i \subseteq B_{s'}(v_j)$$

$X_i = \{v_1, v_2, \ldots, v_i\}$

Two kinds of (s, s')-dismantlability

An ordering $v_1, v_2, \ldots, v_n$ of the vertices of $V(G)$ is

- (s, s')-dismantling if

$$\forall i > 1, \exists j < i, B_s(v_i, G \setminus v_j) \cap X_i \subseteq B_{s'}(v_j)$$

- $(s, s')^*$-dismantling if

$$\forall i > 1, \exists j < i, B_s(v_i, G) \cap X_i \subseteq B_{s'}(v_j)$$

Remarks

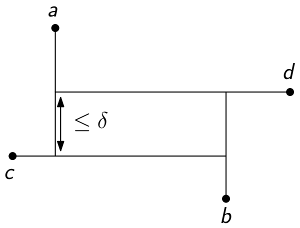
- (s, s')-dismantling $\implies$ $(s, s-1)$-dismantling if $s' < s$
- $(s, s')^*$-dismantling $\implies$ (s, s')-dismantling
- $(s, s-1)$-dismantling $\implies$ $(s, s-1)^*$-dismantling
- G is $(s, s)^*$-dismantlable iff G^s is dismantlable

δ-hyperbolic graphs

A graph (or a metric space) is δ-hyperbolic if for every four points a, b, c, d,

$$d(a,b)+d(c,d) \leq \max\{d(a,c)+d(b,d), d(a,d)+d(b,c)\}+2\delta$$

The hyperbolicity δ^* of a graph G is the minimal value of δ such that G is δ-hyperbolic



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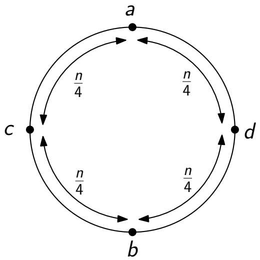
Examples:

- Trees and cliques are 0-hyperbolic
- Cycles are $\frac{n}{4}$-hyperbolic
- Square grids are $\sqrt{n} - 1$-hyperbolic
- Chordal graphs are 1-hyperbolic [Brinkmann, Koolen, Moulton '01]



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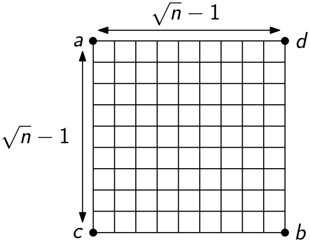

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Remark

- The hyperbolicity of G measures how G is metrically close from a tree
- There exist many definitions of δ-hyperbolicity; they are equivalent up to a multiplicative factor

Why is hyperbolicity an interesting parameter ?

A notion from Geometric Group Theory [Gromov '87]

- for δ-hyperbolic group, the word problem is solvable in linear time (it is undecidable for general groups)

Some large scale graphs are known to be of small hyperbolicity

- the Internet topology can be embedded into a hyperbolic space [Boguna et al. '10]
- the map of the AS of the Internet has small hyperbolicity [Cohen et al. ' 13]

Efficient algorithms exist for graphs of small hyperbolicity

- Greedy routing algorithms can be expected to perform very well [Papadopoulos et al. '09]
- Routing labeling schemes with small labels and small additive error [Chepoi et al '12]

In this talk

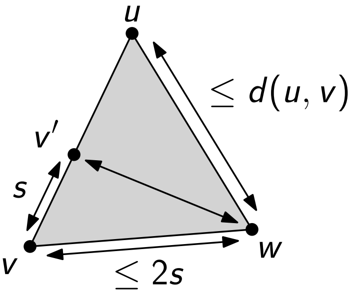
- Characterization of hyperbolicity via cop and robber games
  - δ-hyperbolic graphs are $(2s, s + 2\delta)$-cop-win for any s
  - $(s, s - 1)$-cop-win graphs are $64s^2$-hyperbolic
- An efficient algorithm to approximate the hyperbolicity of a graph

δ-hyperbolic graphs are $(2s, s + 2\delta)$-cop-win

Proposition (from Chepoi, Estellon ’07)

Any δ-hyperbolic graph is $(2s, s + 2\delta)^*$-dismantlable, and thus $(2s, s + 2\delta)$-cop-win

- Consider any BFS ordering of $V(G)$ from a vertex u
- For all v, let v' be a vertex on a shortest path from v to u such that $d(v, v') = s$



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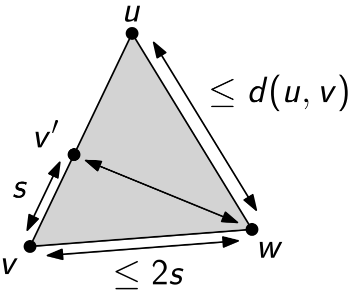
Let $w \in B_{2s}(v) \cap X_v$

$$\begin{aligned} d(u, v') + d(v, w) &\leq d(u, v') + 2s \\ &\leq d(u, v) + s \\ d(v, v') + d(u, w) &\leq s + d(u, v) \end{aligned}$$



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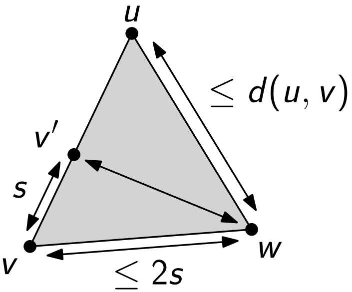

Consequently,

$$\begin{aligned} d(v',w) + d(u,v) &\leq s + d(u,v) + 2\delta \\ d(v',w) &\leq s + 2\delta \end{aligned}$$

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Question

Is any (s, s')-cop-win graph $f(s)$-hyperbolic (when $s' < s$) ?

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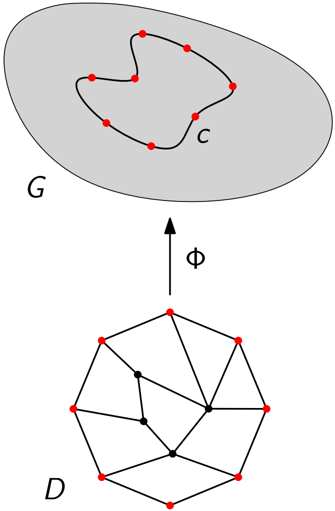
Theorem

G is $(s, s - 1)$-cop-win $\implies$ G is $64s^2$-hyperbolic

Another characterization of hyperbolicity

For a cycle c, (D, Φ) is an N-filling of c if

- D is a 2-connected planar graph
- every internal face of D has at most $2N$ edges
- $\Phi : D \to G$ is a simplicial map
- $\Phi(\partial D) = c$



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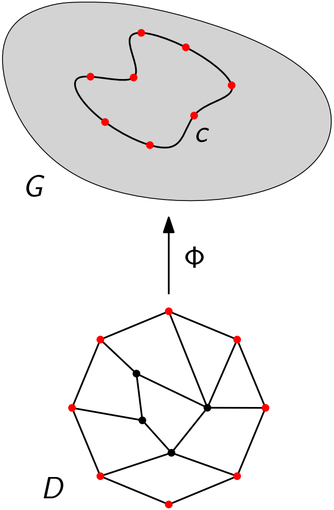
- The area of (D, Φ) is the number of faces of D
- $\text{Area}_N(c)$ is the minimum area of an N-filling of c
- $\ell(c)$ is the length of c



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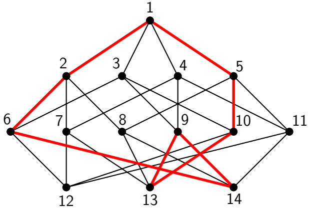
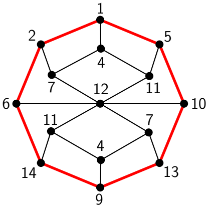


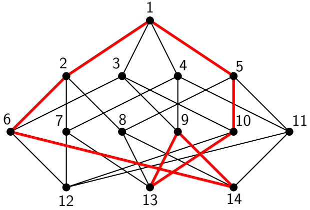
- The area of (D, Φ) is the number of faces of D
- $\text{Area}_N(c)$ is the minimum area of an N-filling of c
- $\ell(c)$ is the length of c



Another characterization of hyperbolicity

For a cycle c, (D, Φ) is an *N*-filling of c if

- D is a 2-connected planar graph
- every internal face of D has at most $2N$ edges
- $\Phi : D \to G$ is a simplicial map
- $\Phi(\partial D) = c$



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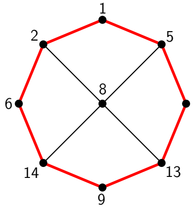

Linear Isoperimetric Inequality

A graph G satisfies the linear isoperimetric inequality, if there exists $K \in \mathbb{N}$ and N such that

$$\forall c,\ \text{Area}_N(c) \leq K\ell(c)$$

Theorem (Gromov)

- *G is δ-hyperbolic* $\implies \forall c, Area_{16\delta}(c) \leq \ell(c)$
- $\forall c, Area_N(c) \leq K\ell(c) \implies$ *G is $O(K^2N^3)$-hyperbolic*

For a proof, see [Bridson and Haefliger]

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For a proof, see [Bridson and Haefliger]

Proposition

When $K \in \mathbb{Q}$,
$\forall c, \text{Area}_N(c) \leq \lceil K\ell(c) \rceil \implies G$ is $(32KN^2 + \frac{1}{2})$-hyperbolic

$(s, s')^*$-dismantl. $\implies$ lin. isoperimetric inequality

Theorem

If G is $(s, s')^*$-dismantlable with $s' < s$,

$$\forall c, Area_{s+s'}(c) \leq \left\lceil \frac{\ell(c)}{2(s - s')} \right\rceil$$

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- v: the last vertex of c in the dismantling order
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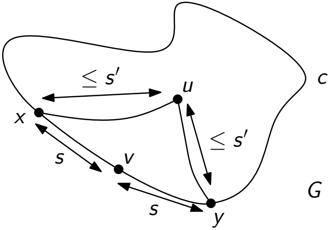

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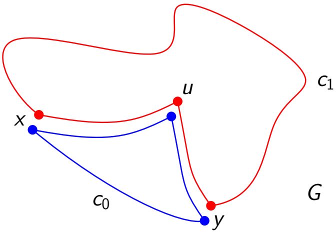

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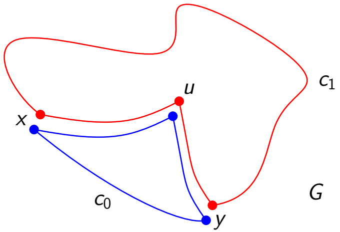

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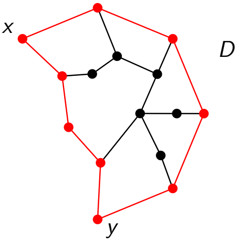

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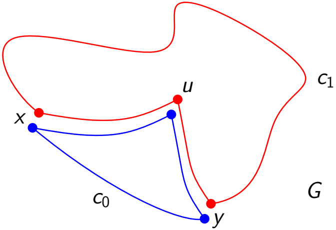

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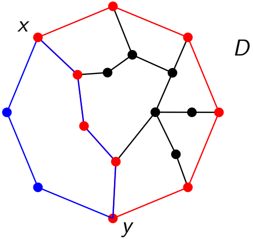
- $\ell(c_1) \leq \ell(c) - 2(s-s')$
- $\text{Area}_{s+s'}(c) \leq 1 + \left\lceil \frac{\ell(c_1)}{2(s-s')} \right\rceil \leq \left\lceil \frac{\ell(c)}{2(s-s')} \right\rceil$



(s, s')-cop-win graphs are hyperbolic

Theorem

G is $(s, s')^*$*-dismantlable with* $s' < s \implies \delta^*(G) \leq 16\frac{(s+s')^2}{s-s'} + \frac{1}{2}$

Corollary

G is $(s, s-1)$*-cop-win* $\implies$ *G is* $64s^2$*-hyperbolic*

Computing the hyperbolicity

Assume the distance-matrix of G has been computed

Computing the hyperbolicity $\delta^*(G)$

- 4 points condition: $O(n^4)$

Computing an approximation of $\delta^*(G)$

- fixing one point: a 2-approx. in $O(n^3)$

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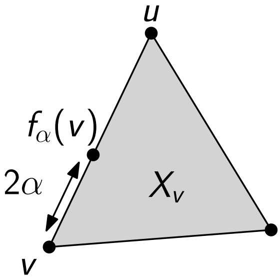
Theorem

From the distance-matrix of G, one can compute a constant approximation of $\delta^*(G)$ *in* $O(n^2)$

Approximation Algorithm for δ^*

Approx-$\delta^*(G,\alpha)$

Consider a BFS ordering $\prec$ of $V(G)$ from any vertex u ;
For all v, let $f_\alpha(v)$ be on a shortest path from v to u such that $d(v, f_\alpha(v)) = 2\alpha$;
for *all* $v \in V$ **do**
  if $B_{4\alpha}(v,G) \cap X_v \not\subseteq B_{3\alpha}(f_\alpha(v),G)$ **then**
    return NO
return YES;



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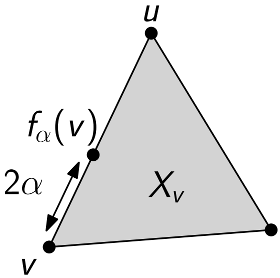

NO $\prec$ is not $(2(2\alpha), 2\alpha + \alpha)^*$-dismantling
$\implies \delta^* > \frac{\alpha}{2}$

YES G is $(4\alpha, 3\alpha)^*$-dismantlable
$\implies \delta^* \leq 16\frac{(7\alpha)^2}{\alpha} + \frac{1}{2} = 784\alpha + \frac{1}{2}$

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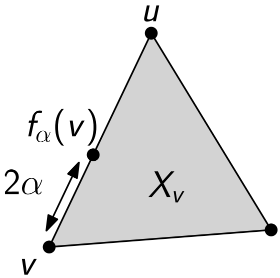

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We can find α^*

$$\alpha^*/2 \leq \delta^* \leq 784\alpha^* + \frac{1}{2}$$

1569-approx. of $\delta^*(G)$

Approximation Algorithm for δ^*

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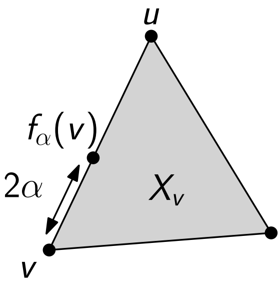

Complexity: **Approx-**$\delta^*(G,\alpha)$ runs in time $O(n^2)$

Proposition

One can compute a 1569-approximation of δ^* in time $O(n^2 \log \delta^*)$

Approximation Algorithm for δ^*

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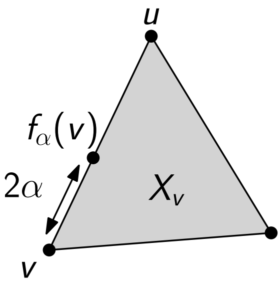

We can avoid to recompute everything when we increase α

Theorem

One can compute a 1569-approximation of δ^* in time $O(n^2)$

Conclusion

- Characterization of hyperbolicity via a cop and robber game
  Different notions that are qualitatively equivalent
  - (s, s')-cop-win graphs
  - (s, s')-dismantlability
  - $(s, s')^*$-dismantlability
  - bounded hyperbolicity
- Links between $(s, s')^*$-dismantlability and hyperbolicity hold for infinite graphs
- A constant-factor approximation of the hyperbolicity in $O(n^2)$ (starting from the distance-matrix)