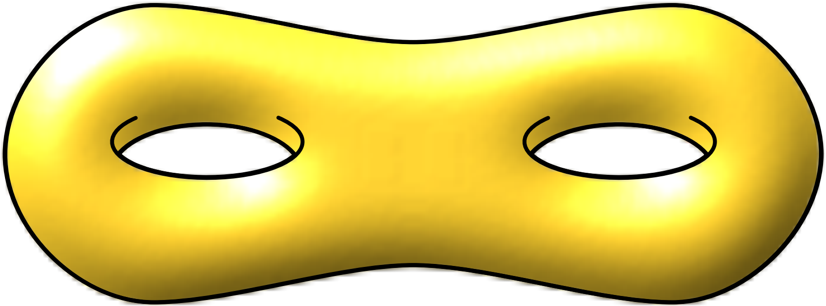
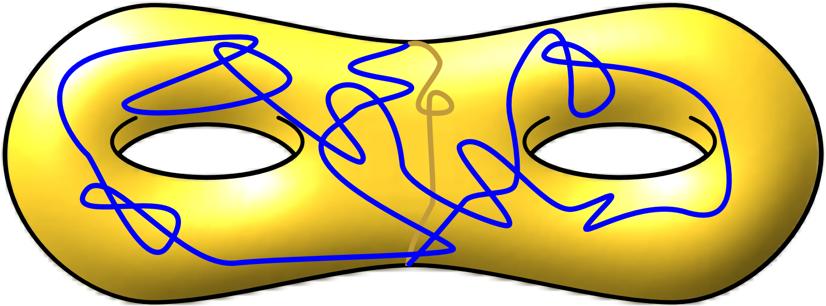
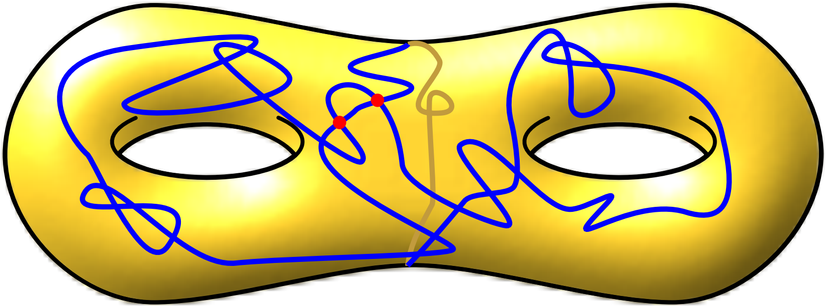
Autour du nombre géométrique d'intersection

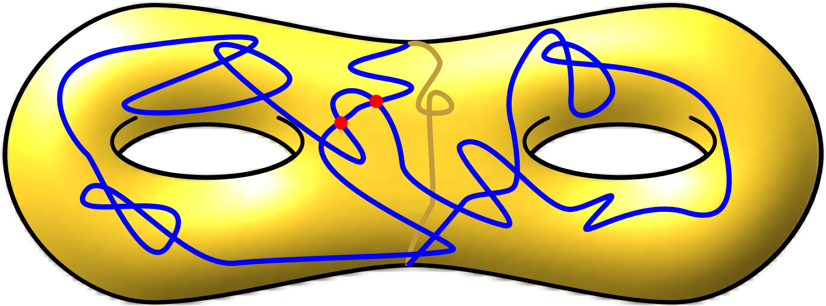
Francis Lazarus
GIPSA-Lab, Grenoble, CNRS

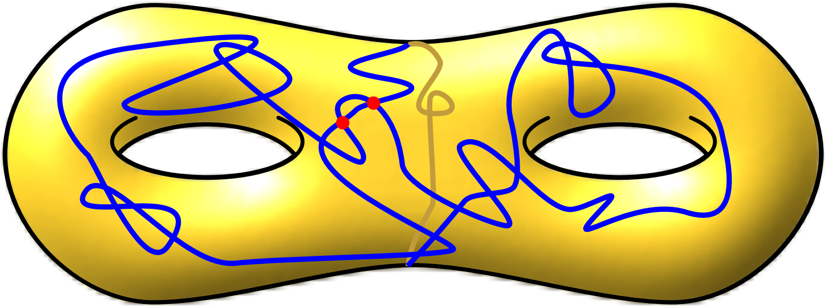
Lionel Walden, Les Docks de Cardiff, 1894

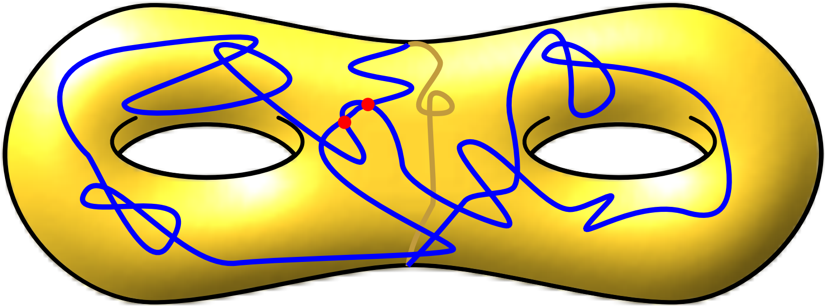


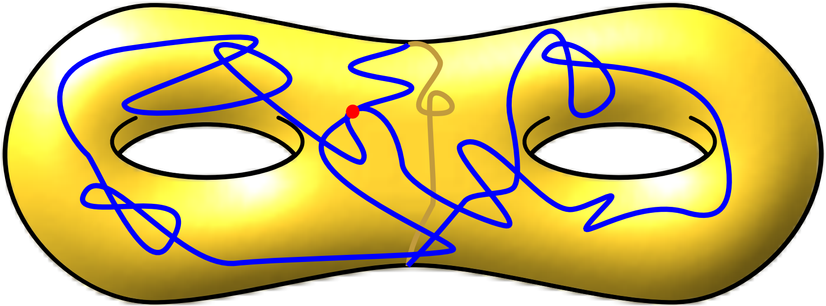


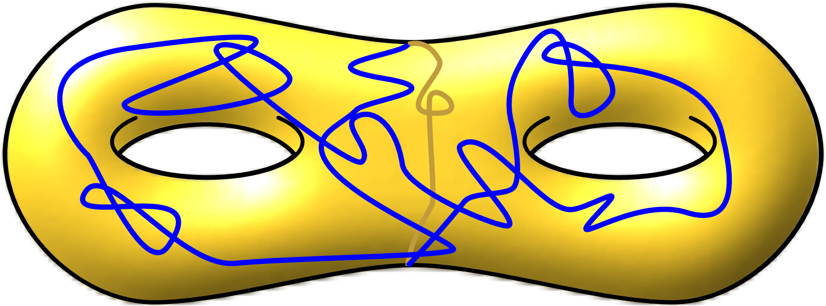


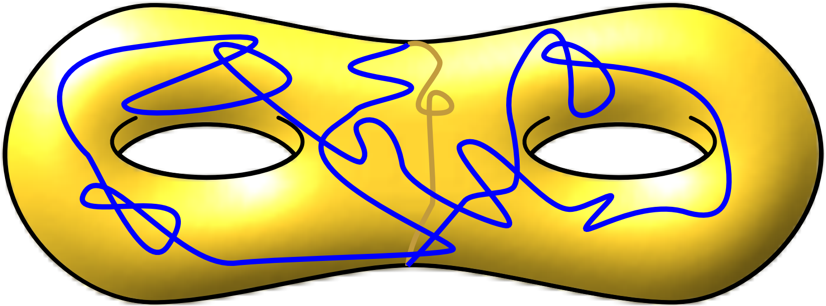


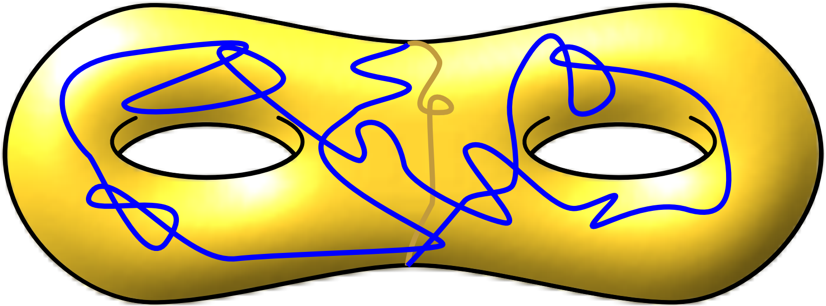


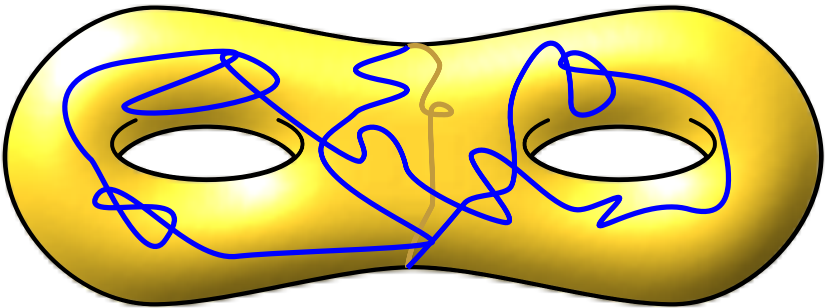


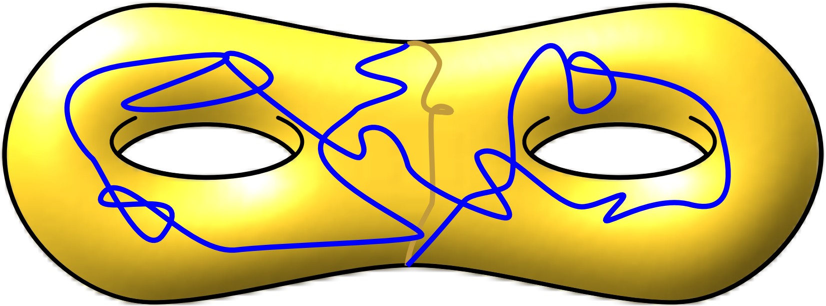


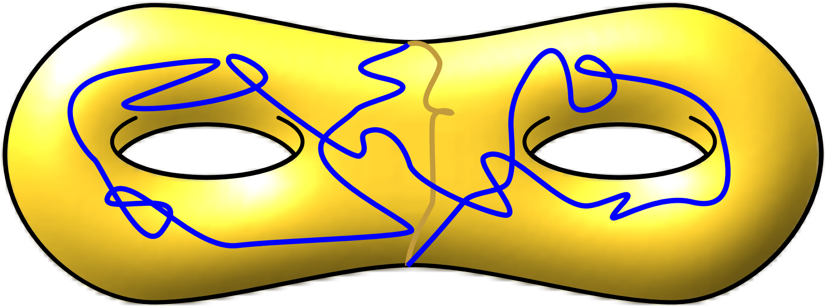


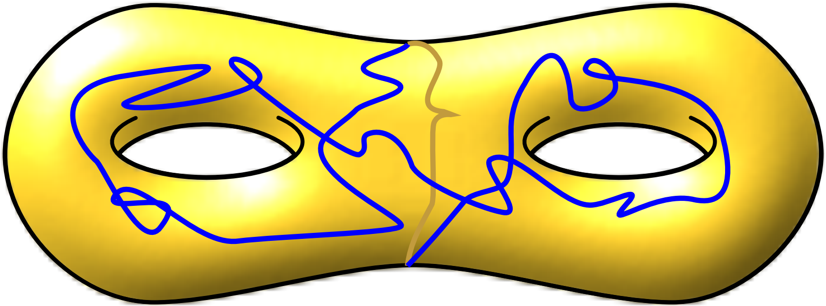


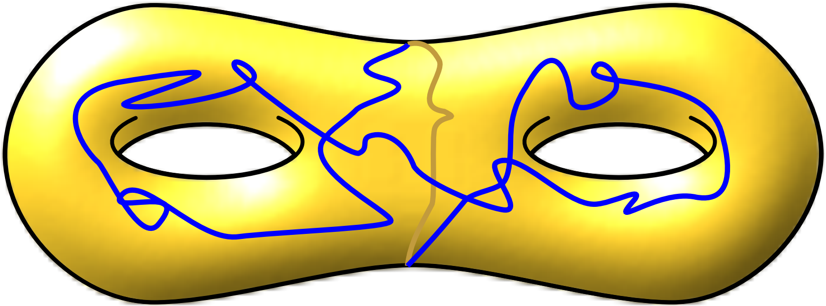


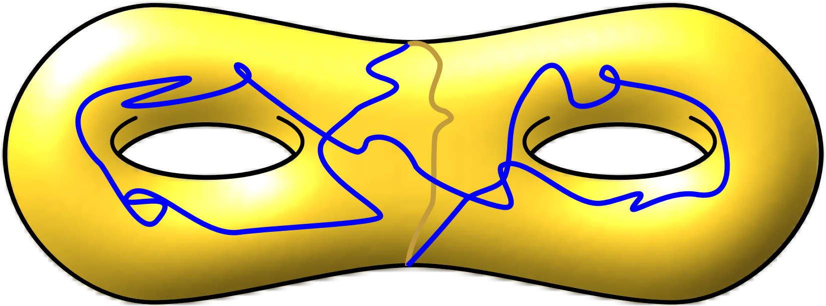


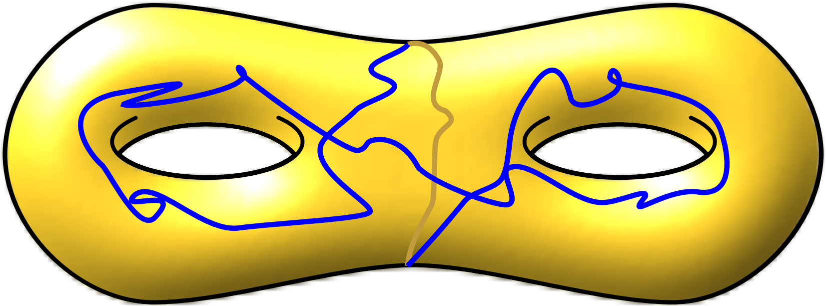


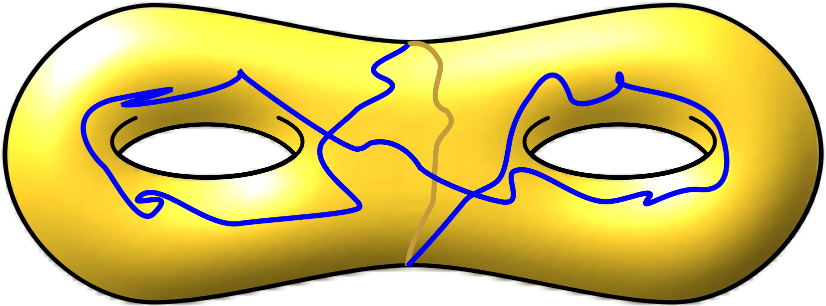


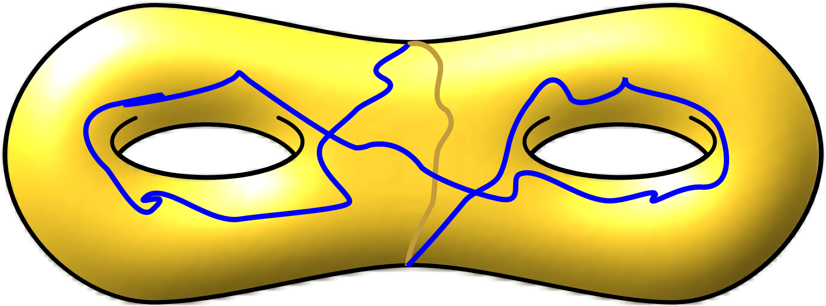


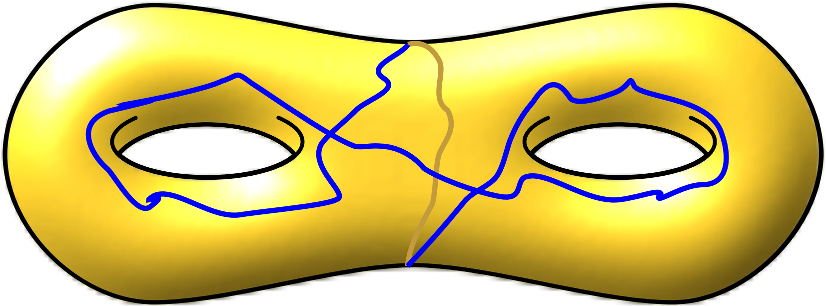


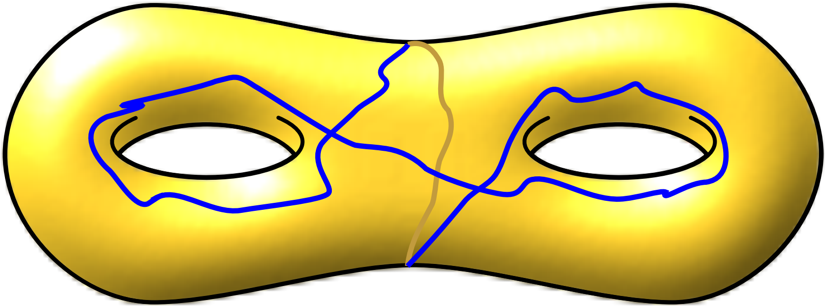


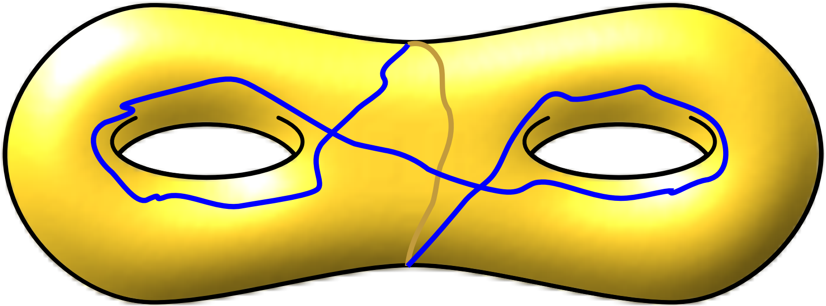


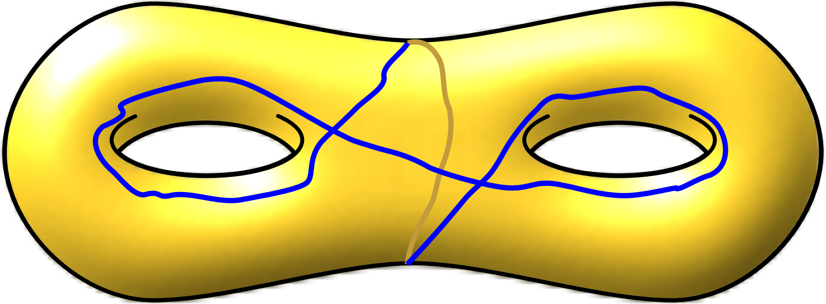


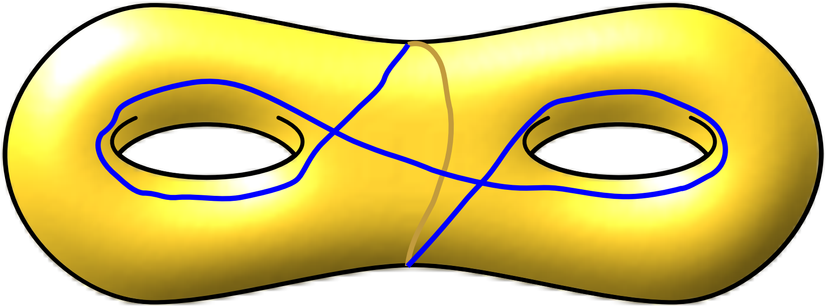


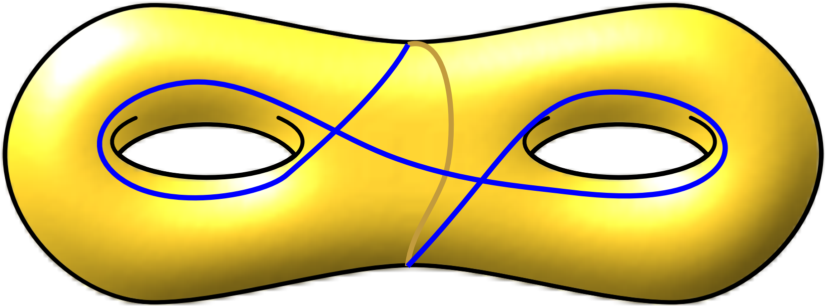


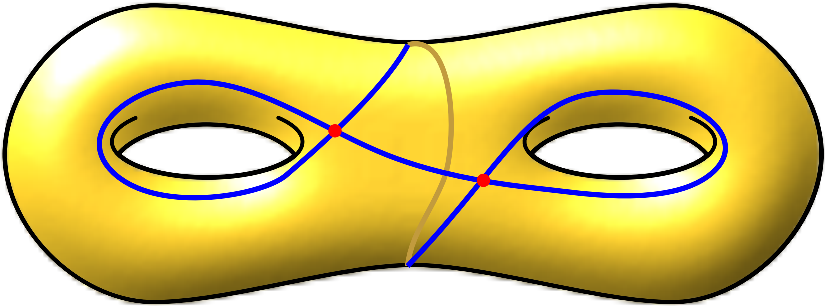


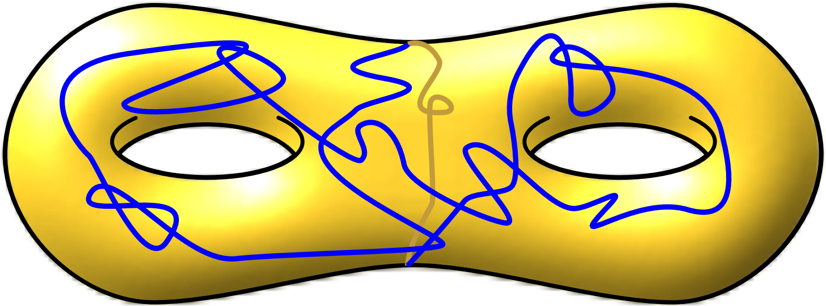


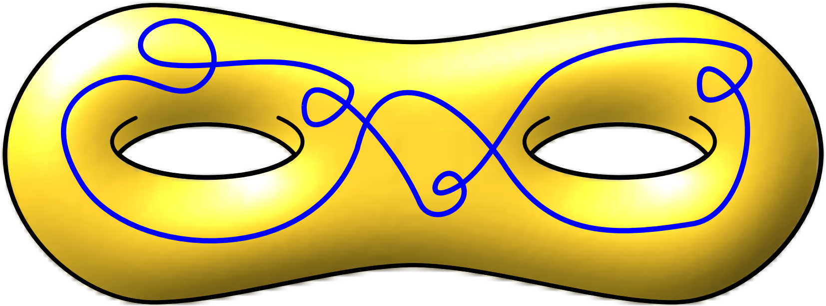


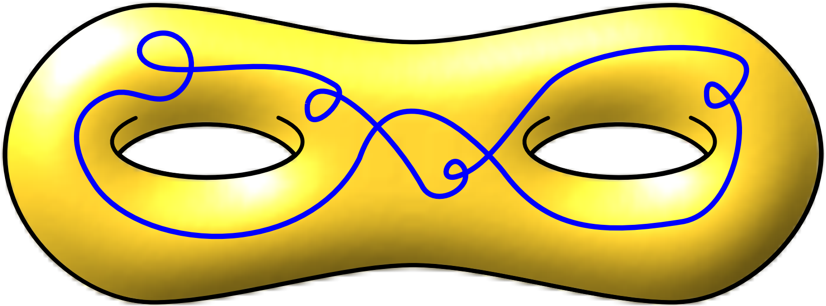


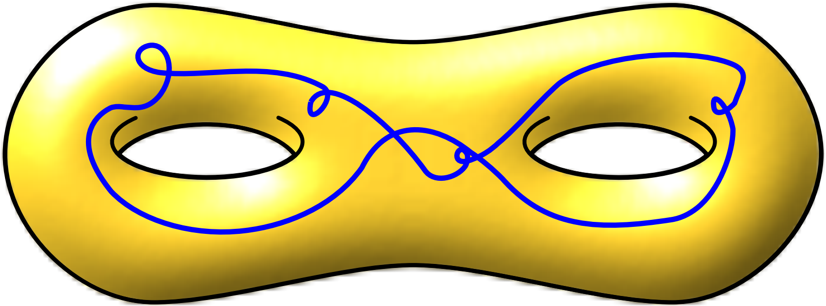


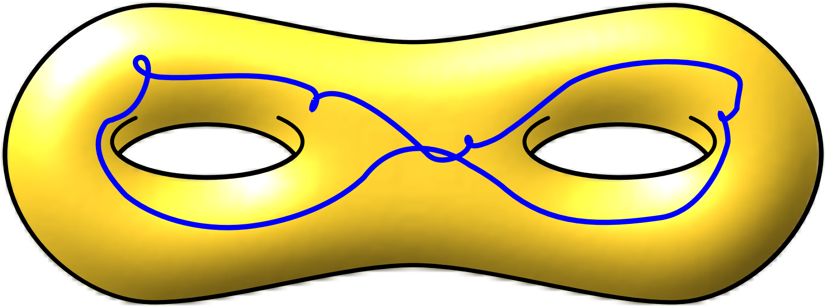


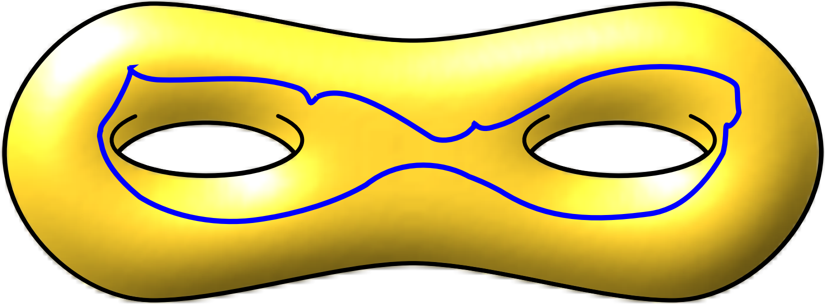


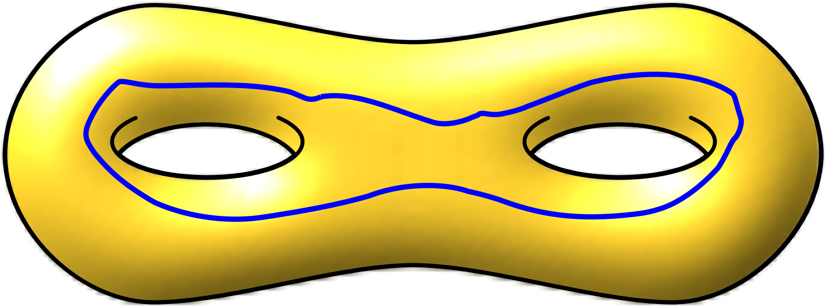


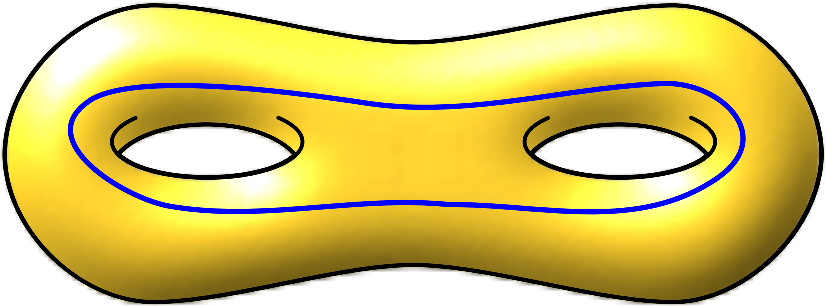


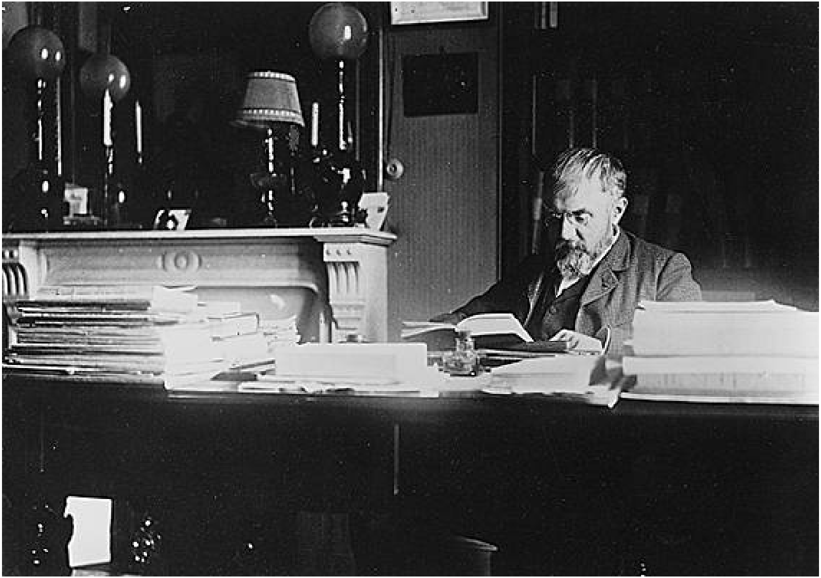












1854 – 1912

ANALYSIS SITUS

Journal de l'École Polytechnique, t. 1, p. 1-121 (1895).

Introduction.

La Géométrie à n dimensions a un objet réel; personne n'en doute aujourd'hui. Les êtres de l'hyperespace sont susceptibles de définitions précises comme ceux de l'espace ordinaire, et si nous ne pouvons nous les représenter, nous pouvons les concevoir et les étudier. Si donc, par exemple, la Mécanique à plus de trois dimensions doit être condamnée comme dépourvue de son objet, il n'en est pas de même de l'Hypergéométrie.

La Géométrie, en effet, n'a pas pour unique raison d'être la description immédiate des corps qui tombent sous nos sens : elle est avant tout l'étude analytique d'un groupe; rien n'empêche, par conséquent, d'aborder d'autres groupes analogues et plus généraux.

Mais pourquoi, dira-t-on, ne pas conserver le langage analytique et le remplacer par un langage géométrique, qui perd tous ses avantages dès que les sens ne peuvent plus intervenir. C'est que ce langage nouveau est plus concis ; c'est ensuite que l'analogie avec la Géométrie ordinaire peut créer des associations d'idées fécondes et suggérer des généralisations utiles.

Peut-être ces raisons ne sont-elles pas suffisantes? Ce n'est pas assez, en effet, qu'une science soit légitime : il faut que l'utilité ne puisse en être contestée. Tant d'objets divers sollicitent notre attention, que les plus importants ont seuls droit de l'obtenir.

Aussi y a-t-il des parties de l'Hypergéométrie auxquelles il n'y a pas lieu de beaucoup s'intéresser : telles sont, par exemple, les recherches sur la courbure des surfaces dans l'espace à n dimensions. On est sûr d'avance d'obtenir les

COMPLÉMENT A L'ANALYSIS SITUS

Rendiconti del Circolo Matematico di Palermo, t. 13, p. 285-343 (1899).

§ I. — Introduction.

Dans le *Journal de l'École Polytechnique* (volume du centenaire de la fondation de l'École, 1894) j'ai publié un Mémoire intitulé *Analysis situs*, où j'étudie les variétés de l'espace à plus de trois dimensions et les propriétés des nombres de Betti. C'est à ce Mémoire que se rapporteront les renvois que je serai amené à faire fréquemment dans la suite, en mentionnant seulement le titre *Analysis situs*.

Dans ce Mémoire se trouve énoncé le théorème suivant : *Pour toute variété fermée, les nombres de Betti également distants des extrêmes sont égaux.*

Le même théorème a été énoncé par M. Picard dans sa *Théorie des fonctions algébriques de deux variables*.

M. Heegaard vient de revenir sur ce même problème dans un travail très remarquable, publié en langue danoise, sous le titre « *Forstudier til en topologisk teori för de algebraiske Sammenhäng* » (Copenhague, det Nordiske Forlag Ernst Bojesen, 1898). D'après lui le théorème en question est inexact et les démonstrations sont sans valeur.

Avant d'examiner les objections de M. Heegaard, il convient de faire une distinction. Il y a deux manières de définir les nombres de Betti.

Considérons une variété V que je supposerai, par exemple, fermée; soient $v_1, v_2, \ldots, v_n$, n variétés à p dimensions, faisant partie de V. Je suppose qu'on ne puisse pas trouver de variété à $p+1$ dimensions faisant partie de V et dont $v_1, v_2 \ldots, v_n$ constituent la frontière complète; mais que, si on leur adjoint une $(n+1)^{\text{ième}}$ variété à p dimensions que j'appellerai v_{n+1} et qui fera

SECOND COMPLÉMENT

A

L'ANALYSIS SITUS

Proceedings of the London Mathematical Society, t. 32, p. 277-308 (14 juin 1900).

Introduction.

J'ai publié dans le *Journal de l'École Polytechnique* un travail intitulé « *Analysis situs* »; je me suis occupé une seconde fois du même problème dans un Mémoire portant pour titre « *Complément à l'Analysis situs* », et qui a été imprimé dans les *Rendiconti del Circolo Matematico di Palermo*.

Cependant la question est loin d'être épuisée, et je serai sans doute forcé d'y revenir à plusieurs reprises. Pour cette fois, je me bornerai à certaines considérations qui sont de nature à simplifier, à éclaircir et à compléter les résultats précédemment acquis.

Les renvois portant simplement une indication de paragraphe ou de page se rapporteront au premier Mémoire, celui du *Journal de l'École Polytechnique*; les renvois où ces indications seront précédées de la lettre c se rapporteront au Mémoire des *Rendiconti*.

Quant aux renvois aux paragraphes du présent Mémoire, je les ferai précéder des lettres 2c.

SUR

CERTAINES SURFACES ALGÉBRIQUES; TROISIÈME COMPLÉMENT A L'ANALYSIS SITUS

Bulletin de la Société Mathématique de France, t. 30, p. 49-70 (1902).

Proposons-nous d'étudier, au point de vue de l'*Analysis situs*, la surface

$$z = \sqrt{F(x, y)}, \tag{1}$$

où $F(x, y)$ est un polynome. Nous supposerons que la courbe

$$F(x, y) = 0$$

ne présente que des points ordinaires où $\frac{dF}{dx}$ n'est pas nul, ou bien où $\frac{dF}{dx} = 0$, mais sans que $\frac{d^2F}{dx^2}$ ni $\frac{dF}{dy}$ s'annulent, ou des points doubles ordinaires où

$$\frac{dF}{dx} = \frac{dF}{dy} = 0,$$

mais sans que $\frac{d^2F}{dx^2}$ ni $\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dx\,dy}\right)^2$ s'annulent.

Considérons d'abord y comme une constante. Alors l'équation (1) représentera une courbe algébrique et les coordonnées x et z d'un point de cette courbe pourront s'exprimer, comme on le sait, comme des fonctions fuchsiennes d'une même variable auxiliaire u. Considérons le groupe fuchsien correspondant et le polygone fuchsien qui l'engendre. En général, le polygone fuchsien qui correspond à une courbe de genre p est un polygone de $4p$ côtés, et l'on

SUR

LES CYCLES DES SURFACES ALGÉBRIQUES; QUATRIÈME COMPLÉMENT A L'ANALYSIS SITUS

Journal de Mathématiques, t. 8, p. 169-214 (1902).

§ 1. Introduction.

Les beaux travaux de M. Picard sur les *Surfaces algébriques* ont mis depuis longtemps en évidence l'importance de la notion des cycles à une, deux ou trois dimensions. J'ai pensé qu'on pourrait appliquer à cette question les principes que j'ai exposés dans l'*Analysis situs* et ses deux premiers compléments (*Journal de l'École Polytechnique*, tome du centenaire; *Rendiconti del Circolo Matematico di Palermo*, t. XIII; *Proceedings of the London Mathematical Society*, vol. XXXII) et j'ai obtenu ainsi certains résultats partiels que j'ai déjà énoncés dans une Note aux *Comptes rendus* et qui viennent compléter, sur certains points, ceux de M. Picard.

Je rappelle qu'étant donné une variété V fermée à p dimensions, je trace sur cette variété d'autres variétés, fermées ou non, d'un moins grand nombre de dimensions; je désigne par W_q une variété à q dimensions tracée de la sorte sur V.

Si ΣW_q est un ensemble de variétés à q dimensions et ΣW_{q-1} un ensemble de variétés à $q-1$ dimensions, la congruence

$$\Sigma W_q \equiv \Sigma W_{q-1}$$

signifie (par définition) que ΣW_{q-1} forme la frontière complète de l'ensemble

CINQUIÈME COMPLÉMENT

A

L'ANALYSIS SITUS

Rendiconti del Circolo matematico di Palermo, t. 18, p. 45-110 (1904).

1. J'ai déjà eu souvent l'occasion de m'occuper d'*Analysis situs*; j'ai d'abord publié un Mémoire sur ce sujet dans le tome du *Centenaire du Journal de l'École Polytechnique*; ce Mémoire a été suivi de quatre compléments qui ont paru dans le tome XIII des *Rendiconti del Circolo Matematico di Palermo*, dans le tome XXXVII des *Proceedings of the London Mathematical Society*, dans le *Bulletin de la Société Mathématique de France* en 1901, et enfin dans le *Journal de Liouville* en 1901.

Je reviens encore aujourd'hui sur cette même question, persuadé qu'on n'en pourra venir à bout que par des efforts répétés et qu'elle est assez importante pour les mériter.

Cette fois je me suis borné à l'étude de certaines variétés à trois dimensions, mais les méthodes que j'ai employées pourront être sans doute d'un usage plus général. En passant je me suis étendu assez longuement sur certaines propriétés des courbes fermées que l'on peut tracer sur les surfaces fermées de l'espace ordinaire.

Le résultat final que j'avais en vue est le suivant. Dans le second complément j'ai montré que pour caractériser une variété il ne suffit pas de connaître ses nombres de Betti, mais que certains coefficients que j'ai appelés coefficients de torsion (second complément, § 5, p. 363) jouaient un rôle important.

On pourrait alors se demander si la considération de ces coefficients suffit;

Nous sommes donc conduit à une contradiction, ce qui veut dire que l'hypothèse faite au début était absurde et que *le cycle* K *doit être bouclé*.

C. Q. F. D.

4. Nous avons vu au paragraphe précédent qu'il est relativement aisé de reconnaître si un cycle donné est homologue à un cycle non bouclé, ou si deux cycles donnés sont respectivement homologues à deux cycles qui ne se coupent pas. Nous allons dans le présent paragraphe examiner une question analogue :

Comment reconnaître si un cycle donné est *équivalent* à un cycle non bouclé, ou si deux cycles donnés sont équivalents à deux cycles qui ne se coupent pas ?

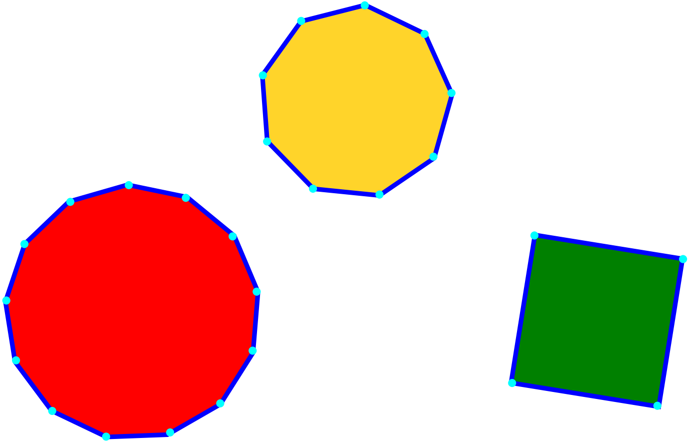
Mais avant d'aborder cette question, revenons sur la définition de l'équivalence.

Jusqu'ici nous avons toujours entendu cette équivalence de la façon suivante :

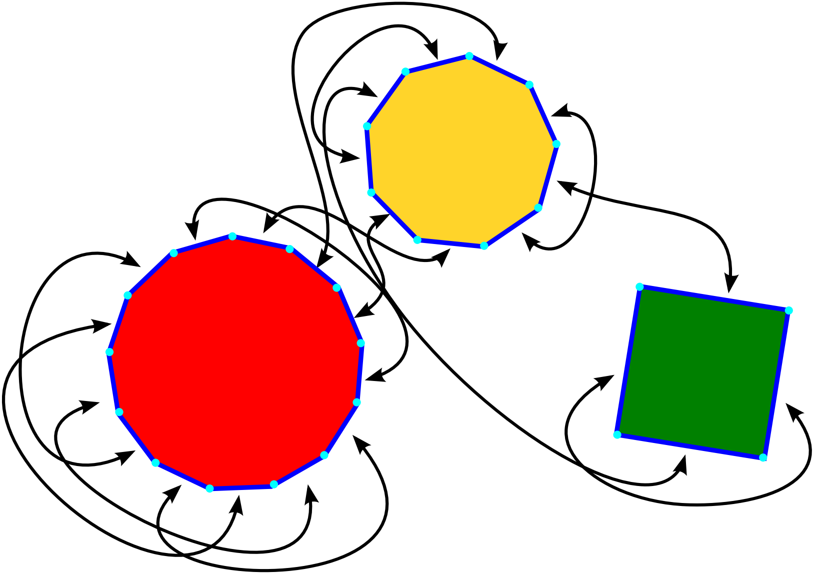
Quand nous écrivons

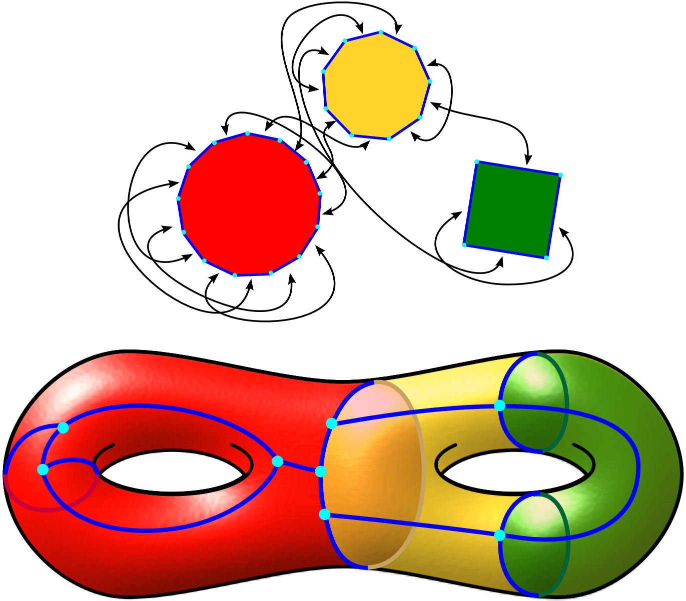
$$C \equiv C',$$

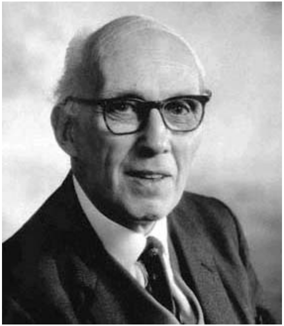
What is a surface?



What is a surface?







Tibor Radó
1895 – 1965

Über den Begriff der Riemannschen Fläche.

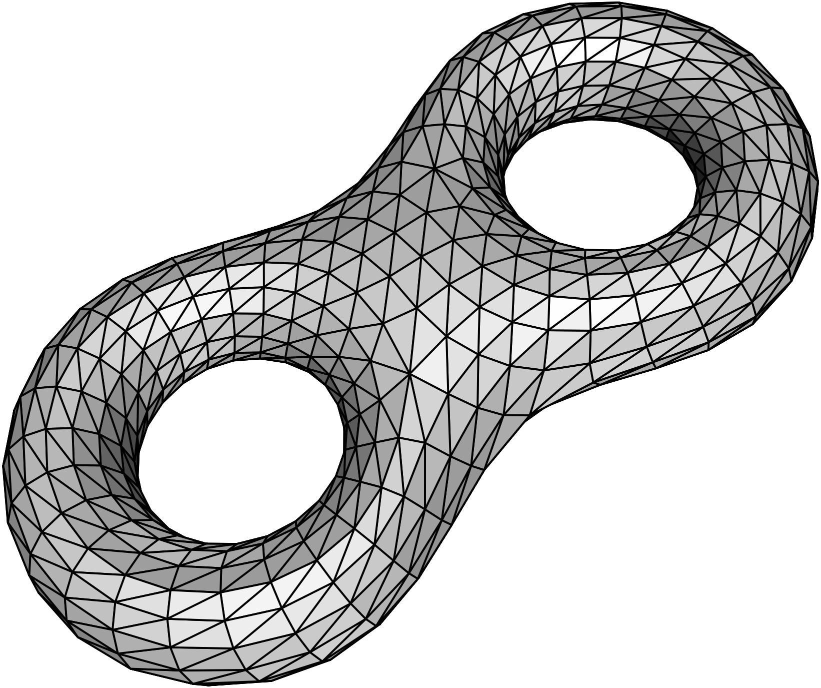
Von Tibor Radó in Szeged.

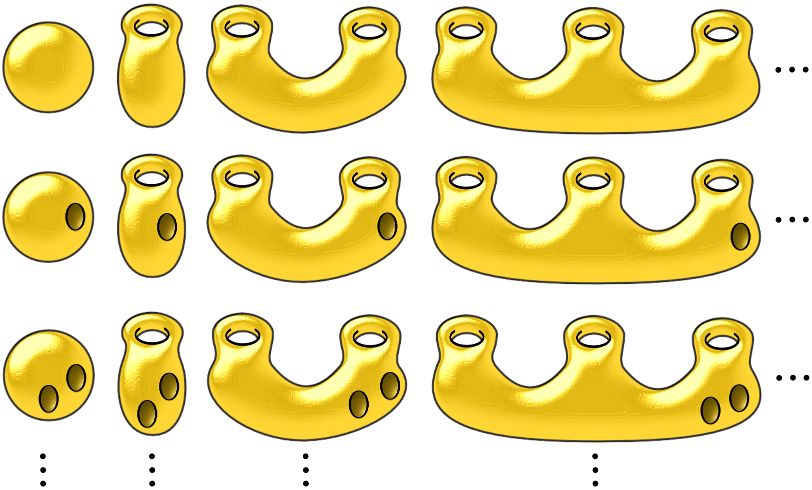
Einleitung.

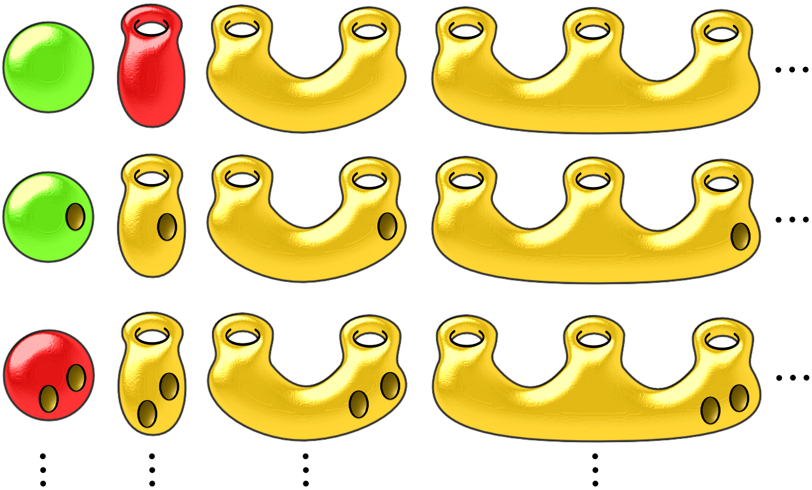
Die vorliegende Arbeit enthält eine Untersuchung, zu welcher ich beim Studium des grundlegenden Werkes des Herrn Weyl über *Die Idee der Riemannschen Fläche* geführt wurde. Bekanntlich wird in diesem Buche der Begriff der Riemannschen Fläche zum ersten Male in vollkommen strenger Weise erklärt, und zwar wie folgt. Eine Riemannsche Fläche ist eine zweidimensionale Mannigfaltigkeit, welche trianguliert werden kann, und für welche eine konforme Abbildung im Kleinen mitgegeben ist. Wir werden diese Begriffe eingehend besprechen (§§ 1 und 4), müssen aber gleich an dieser Stelle die Forderung der *Triangulierbarkeit* doch etwas genauer betrachten, um unser Problem formulieren zu können.

Der Ausdruck *zweidimensionale Mannigfaltigkeit* wird nicht von allen Autoren in demselben Sinne gebraucht. Sie ist jedenfalls ein zusammenhängender topologischer Raum im Hausdorffschen Sinne, welcher im Kleinen der *xy*-Ebene homöomorph ist; aber es wird manchmal noch die Forderung an sie gestellt, sie soll dem zweiten Hausdorffschen Abzählbarkeitsaxiom genügen. Wollte man den Ausdruck zweidimensionale Mannigfaltigkeit bei der Erklärung der Riemannschen Fläche in diesem Sinne verstehen, so wäre die Forderung der Triangulierbarkeit überflüssig. Unter Voraussetzung dieses Abzählbarkeitsaxioms bietet nämlich die Triangulierung einer zweidimensionalen Mannigfaltigkeit keine prinzipielle, sondern nur technische Schwierigkeiten, und die explicite Forderung der Triangulierbarkeit würde einfach bedeuten dass man mit möglicherweise umständlichen, aber im Grunde ganz einfachen Betrachtungen keine Zeit verlieren will.

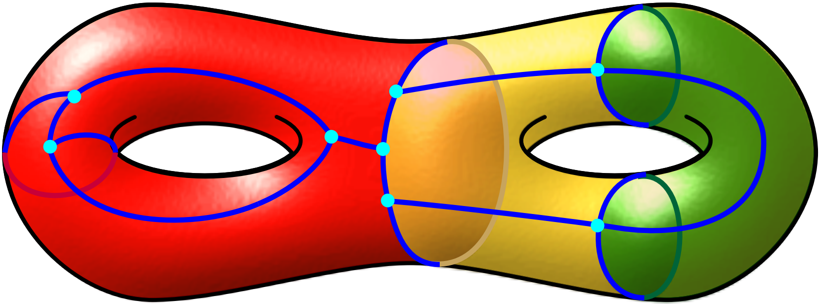
1925

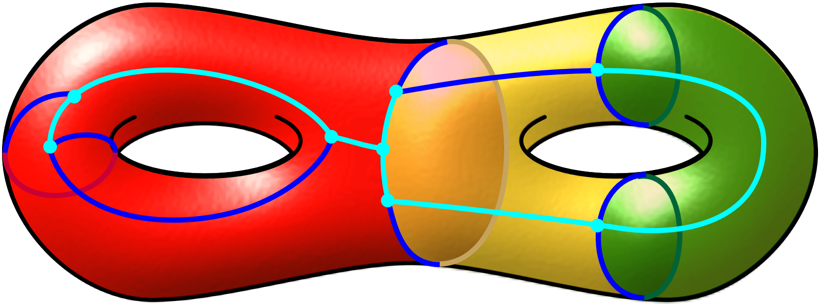


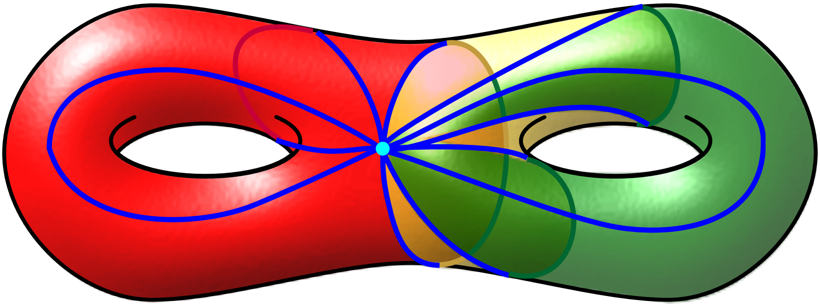


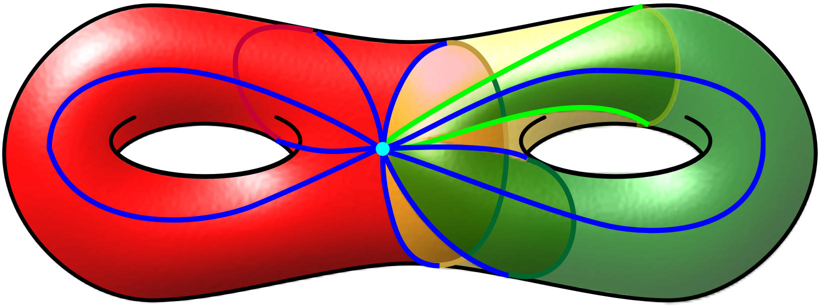


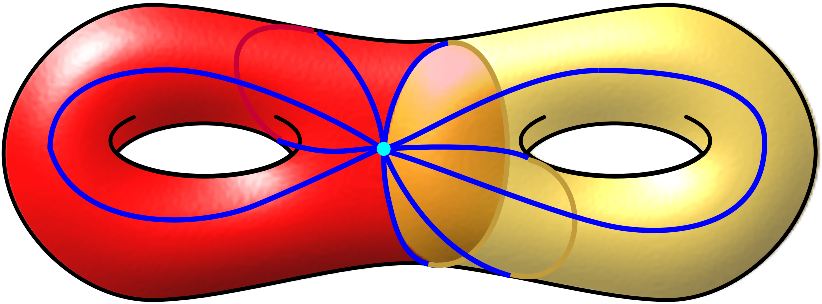
$$S - A + F = \chi = 2 - 2g - b$$

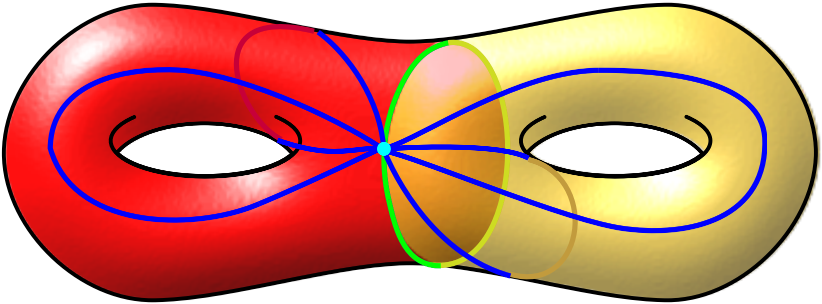


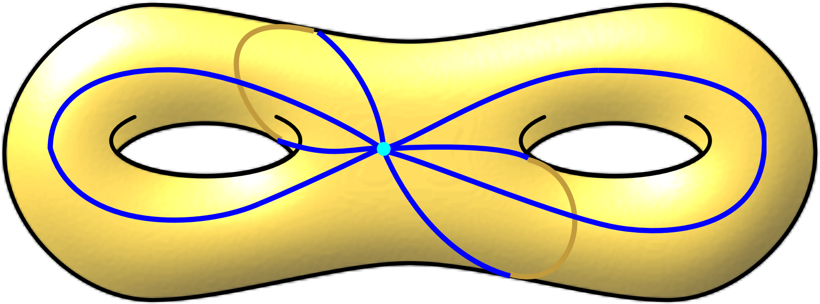


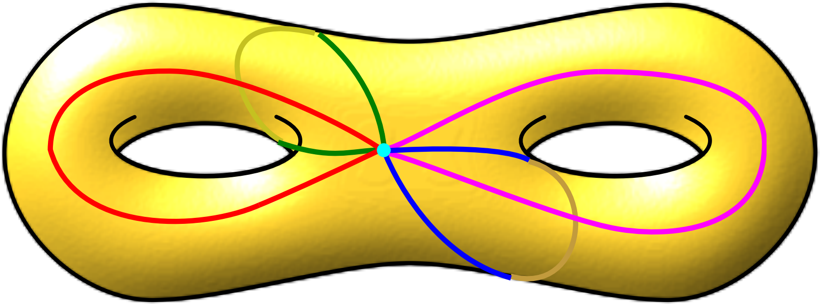


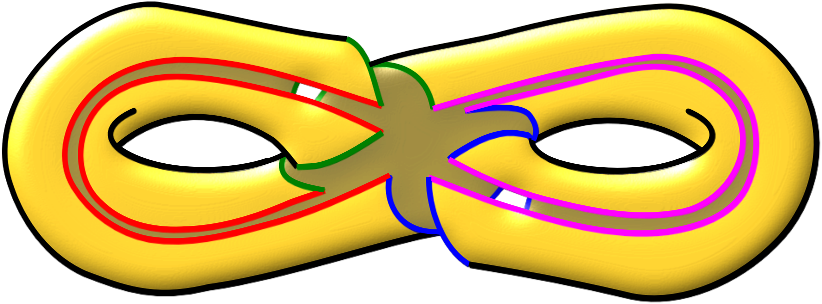


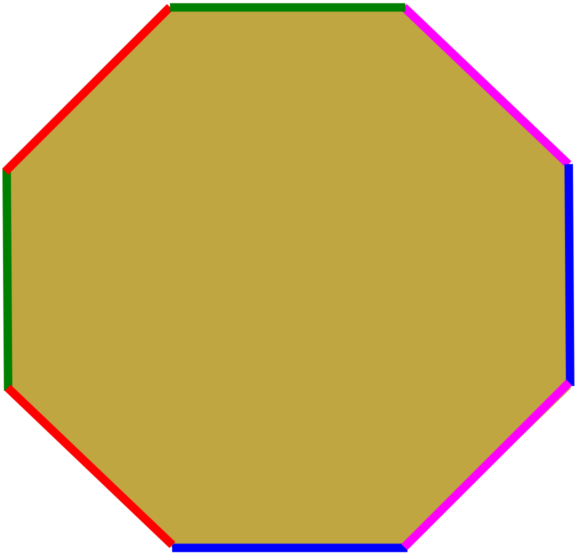




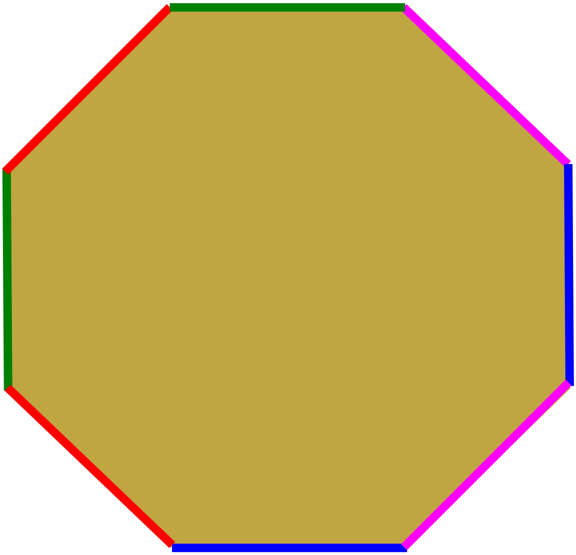


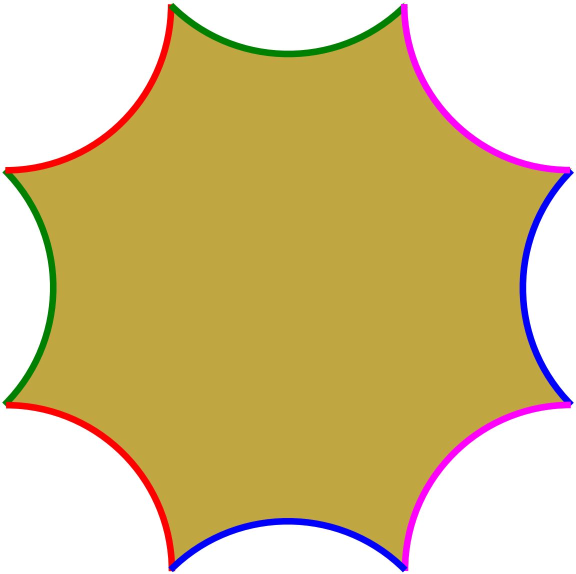




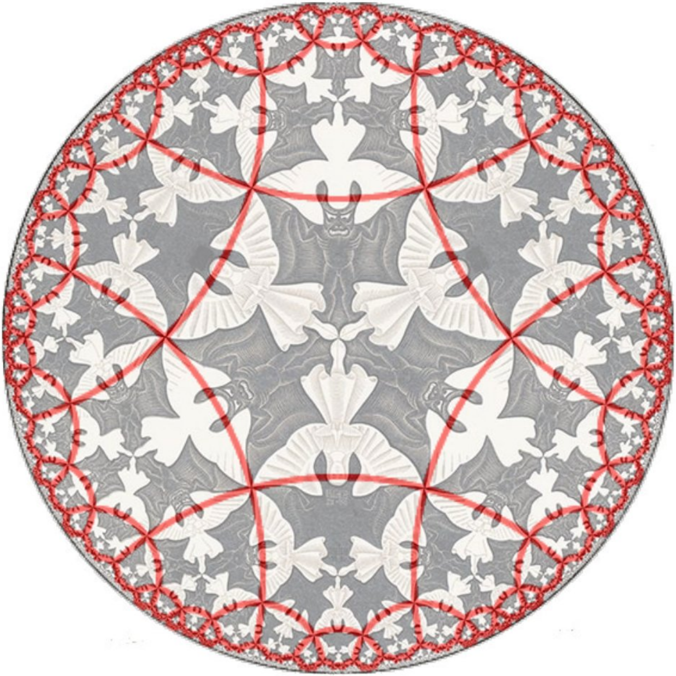


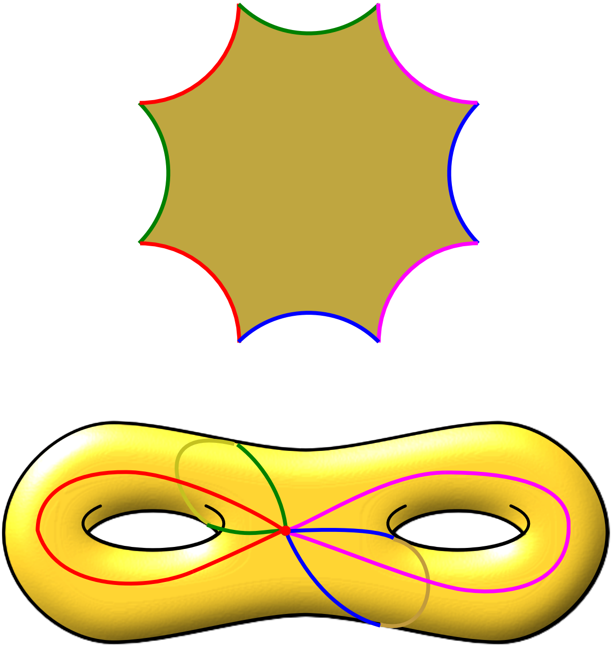
On peut se placer, dans l'étude de la question qui nous occupe, à plusieurs points de vue différents. Représentons d'abord notre surface par un polygone fuchsien R_0 de la première famille, construisons les différents transformés de ce polygone par les transformations du groupe fuchsien correspondant G; ces transformés rempliront le cercle fondamental. Un cycle quelconque C sera alors représenté par un arc de courbe MM', allant d'un point M à un de ses transformés M'. Deux cycles proprement équivalents seront représentés par deux arcs de courbe MPM' et MQM' ayant mêmes extrémités et réciproque-

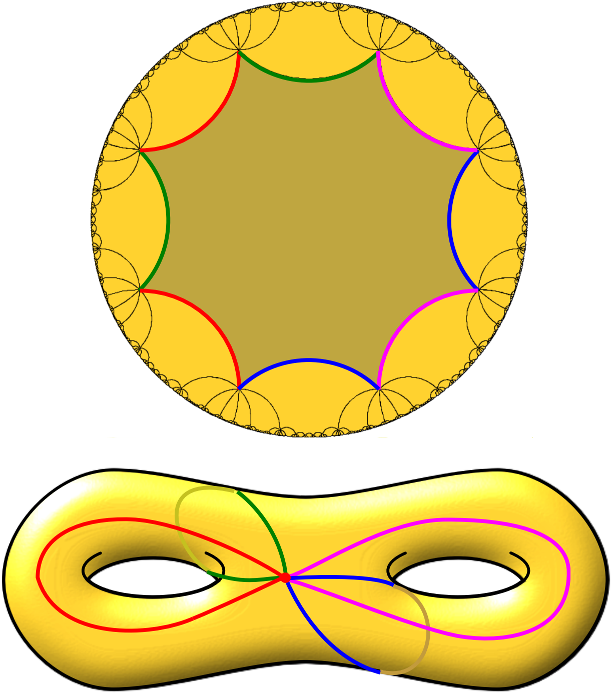


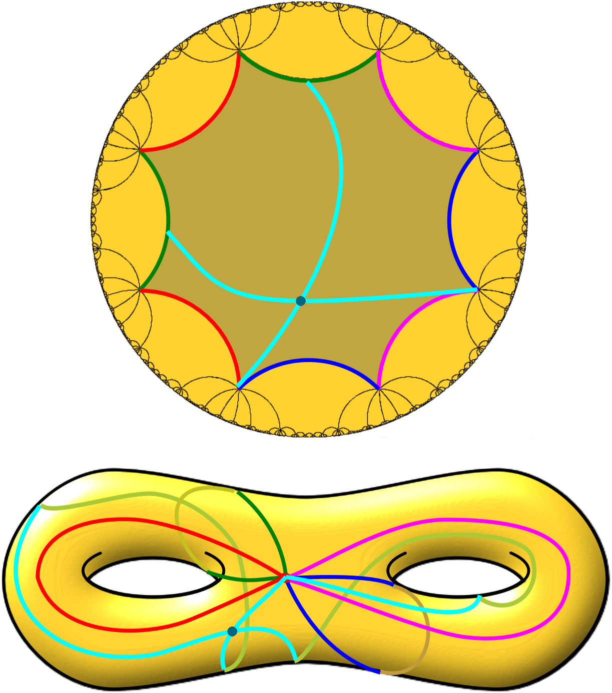


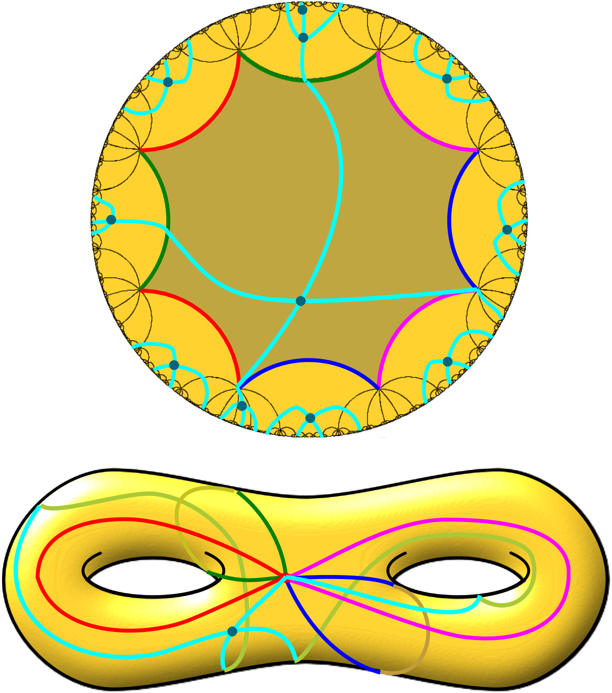


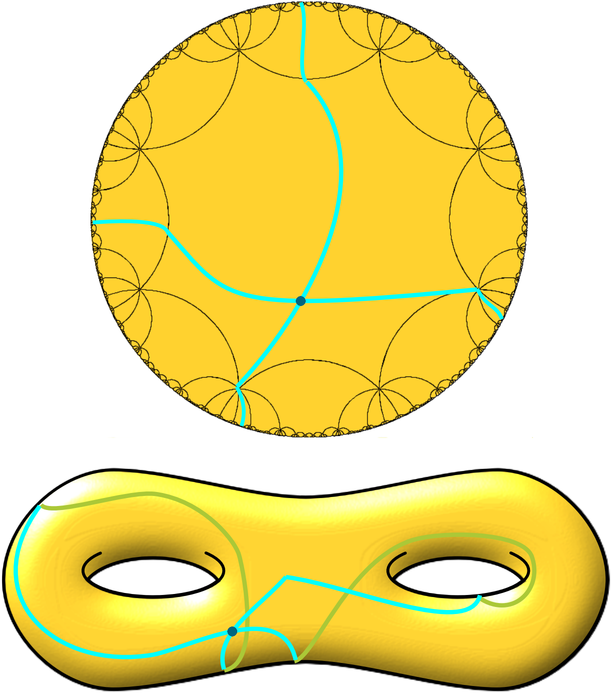


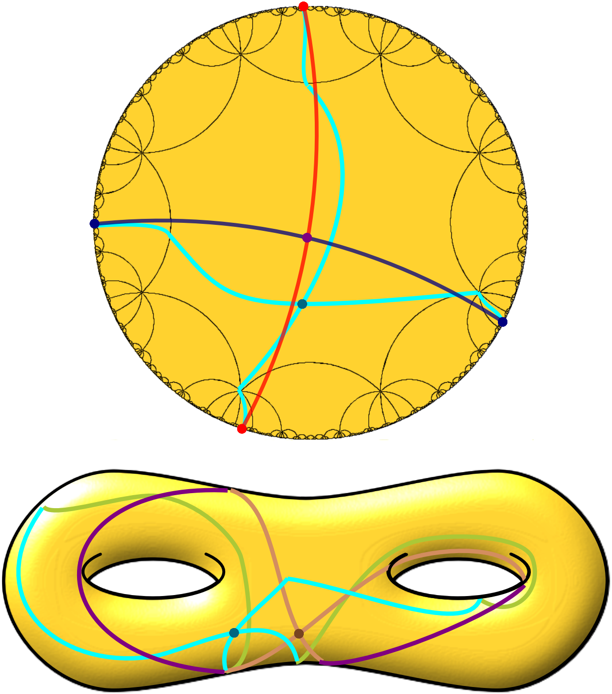


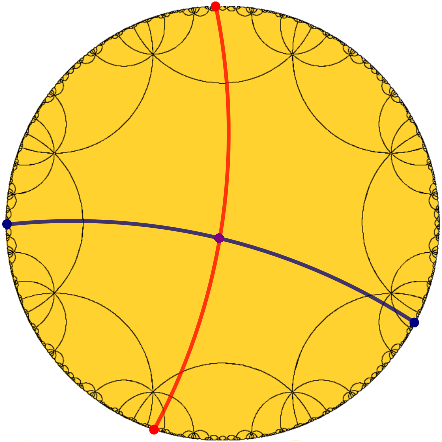
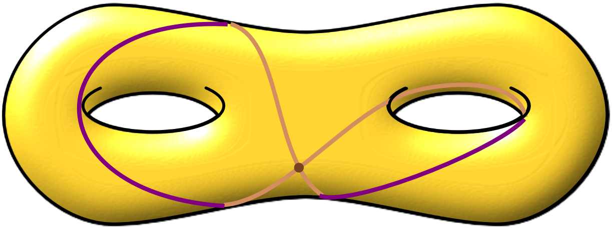










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ALGORITHMS FOR JORDAN CURVES ON COMPACT SURFACES

BY BRUCE L. REINHART*

(Received October 20, 1960)

As is well known, free homotopy classes of mappings of the circle into a space correspond to conjugacy classes in the fundamental group. If the space is two-dimensional, such a free homotopy class may or may not contain a simple closed (that is, Jordan) curve. Our purpose here is to give an algorithm for determining which free homotopy classes admit such a curve in the case of compact surfaces of negative Euler number. Our algorithm is applicable to orientable and non-orientable surfaces, with or without boundary. This problem was first discussed by Dehn [4] and later by Baer [1] and Goeritz [5]. All of these studies are based on cutting the surface into spheres with 3 holes and deriving a (hopefully) canonical form for each free homotopy class by pasting together curves on each sphere with holes. Except for the surface of genus 2 without boundary, no complete answer has been achieved in this way. We shall treat the problem globally by imbedding the fundamental group into the group of motions of the hyperbolic plane, in the spirit of Poincaré [14] and numerous works of Nielsen. As a preliminary, we give in §1 a simple method for assigning to each word in the usual generators of the fundamental group a curve on the surface which has double points only in the neighborhood of the base point, and no other multiple points. We call such a curve an indicating curve for the word. In § 2, we apply the fact that each motion of the hyperbolic plane which arises in our problem leaves fixed a unique geodesic, its axis. The projection of an axis onto the surface is the unique geodesic lying in the free homotopy class containing the motion of which it is axis. For orientable surfaces, a free homotopy class admits a simple closed curve if and only if its geodesic is simple [14, p. 467], while

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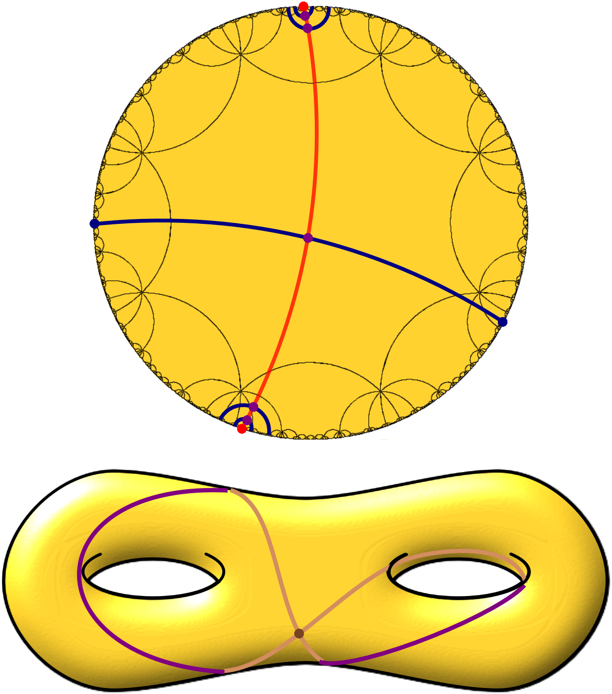
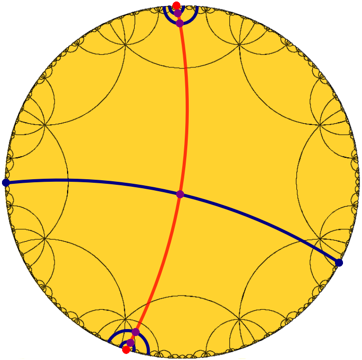
	closed surface	**counting**	**free homotopy**	**special feature**
Chillingworth '69				winding number
Birman & Series '84			✓	retraction onto a graph
Cohen & Lustig '87		✓	✓	retraction onto a graph
Lustig '87	✓	✓	✓	canonical representative
de Graaf & Schrijver '97	✓	✓	✓	Reidemeister moves
Paterson '02	✓	✓	✓	Reidemeister moves
Gonçalves et al. '05	✓	✓	✓	algebraic approach





- If a curve c is primitive its lifts are uniquely defined by their limit points.
- If τ is the hyperbolic translation corresponding to a lift $\tilde{c}_0$ of a primitive c then

$$\imath(c) = |\{\text{set of pairs of limit points crossing } \tilde{c}_0\}/\langle\tau\rangle|$$

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- If τ is the hyperbolic translation corresponding to a lift $\tilde{c}_0$ of a primitive c then

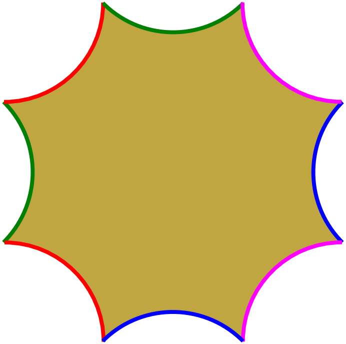
$$\imath(c) = |\{\text{set of pairs of limit points crossing } \tilde{c}_0\}/\langle\tau\rangle|$$

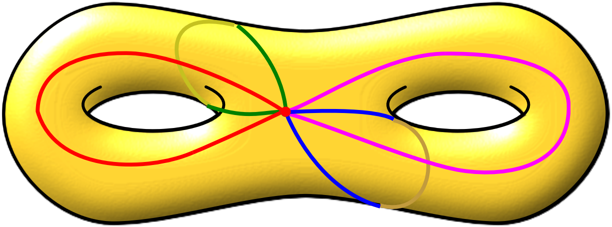
The plan

For a given curve c

1. Determine the primitive root of c.
2. Count the number of classes of crossing pairs of limit points (for the root of c).
3. Use adequate formula if c is not primitive.

A combinatorial framework





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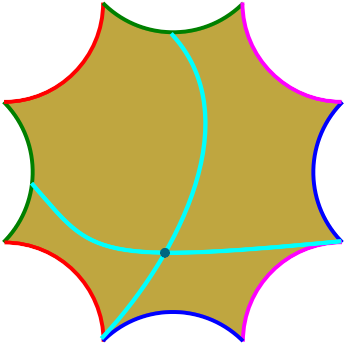

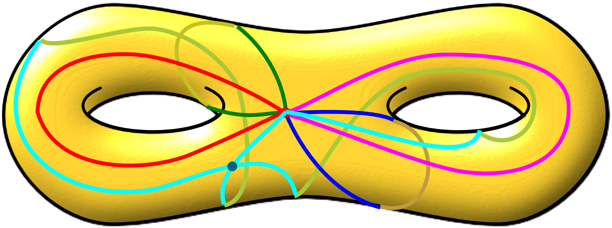

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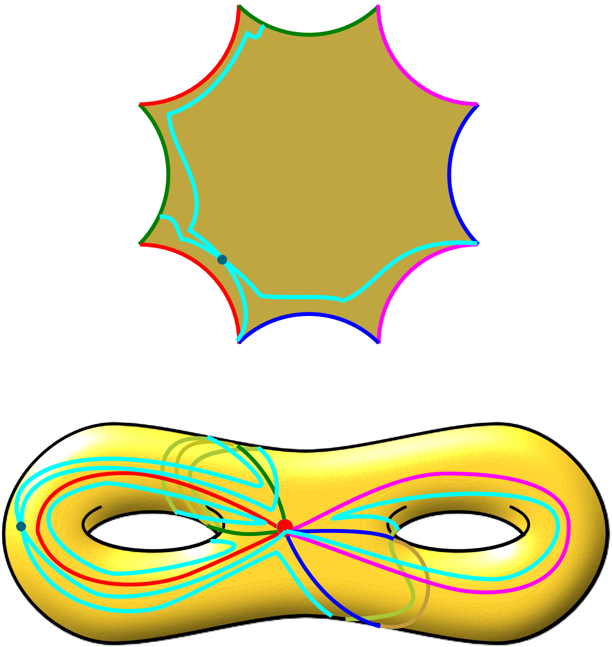

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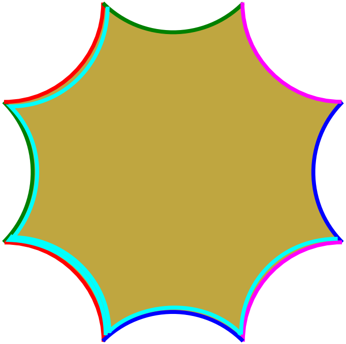
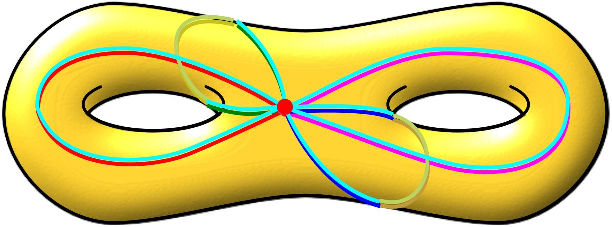
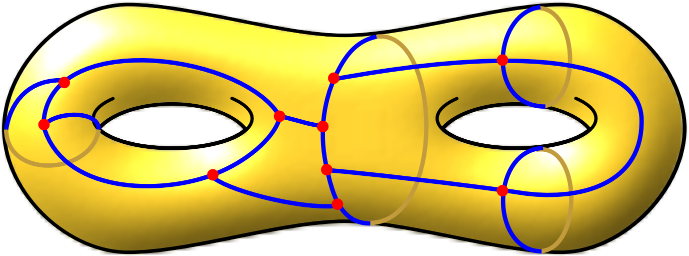




A combinatorial framework: elementary homotopies



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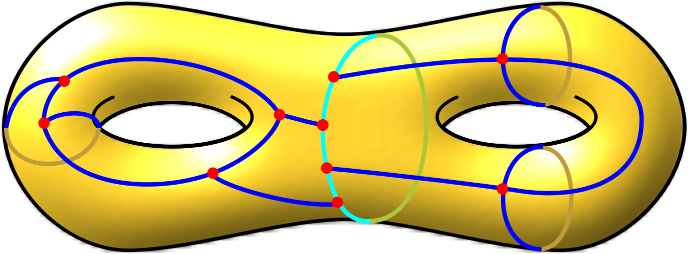

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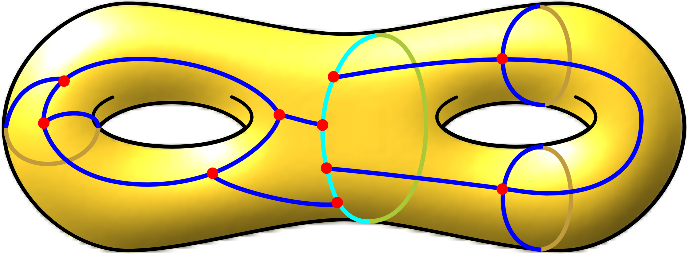
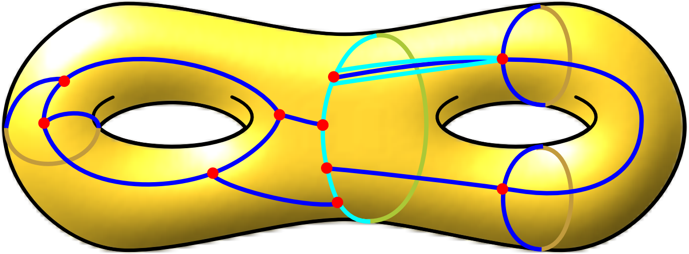


Two rules:

A combinatorial framework: elementary homotopies



Two rules:

- add/delete a spur along an edge

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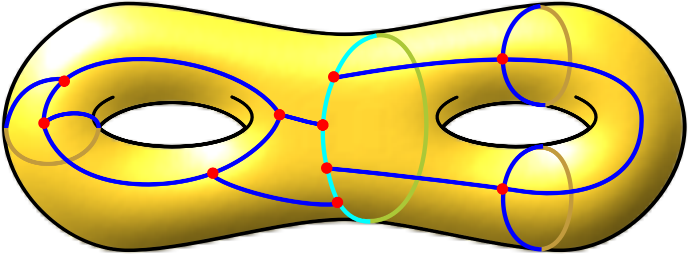

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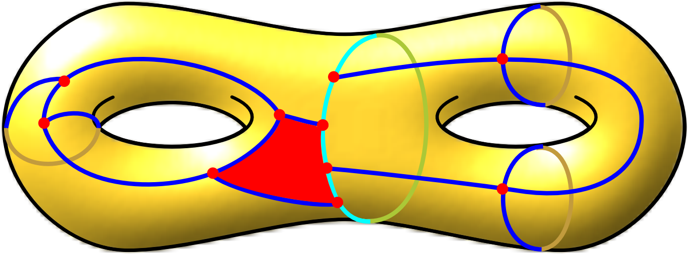
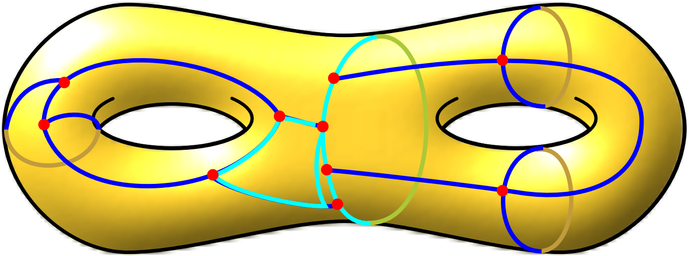


Two rules:

- add/delete a spur along an edge
- add/delete a facial walk

A combinatorial framework: elementary homotopies



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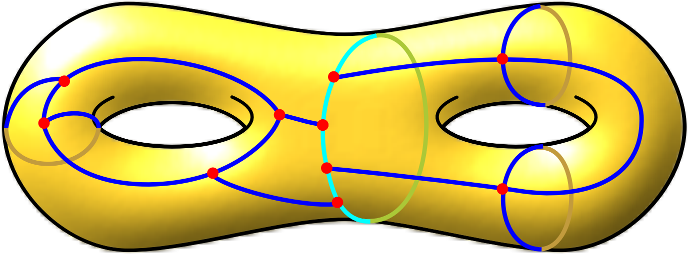

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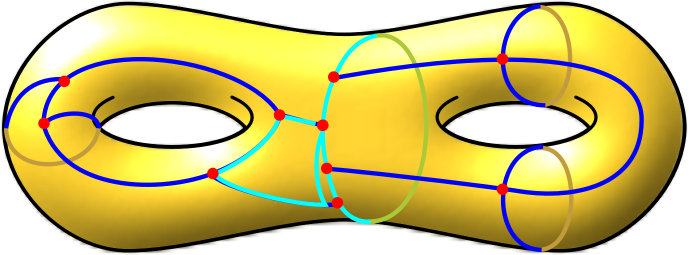

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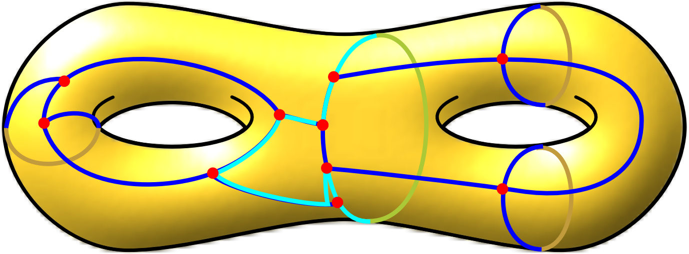

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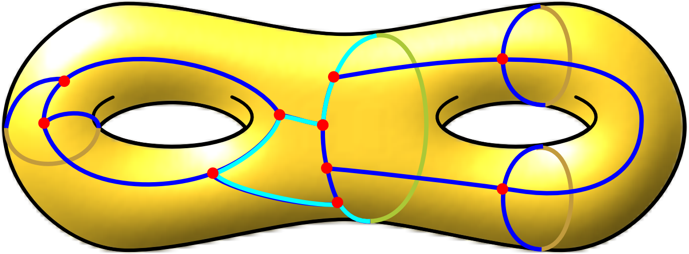
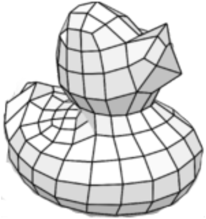


Two rules:

- add/delete a spur along an edge
- add/delete a facial walk

Hyp : All faces are quadrilaterals.



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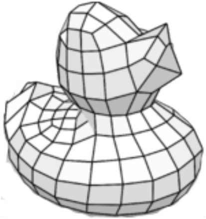

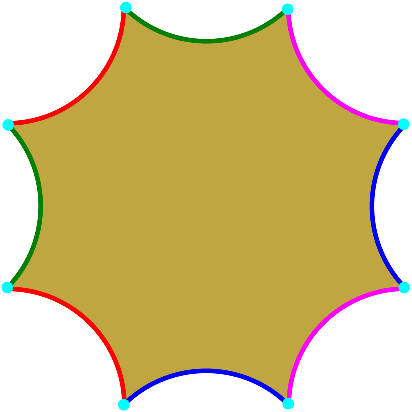

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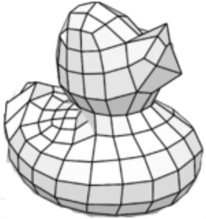

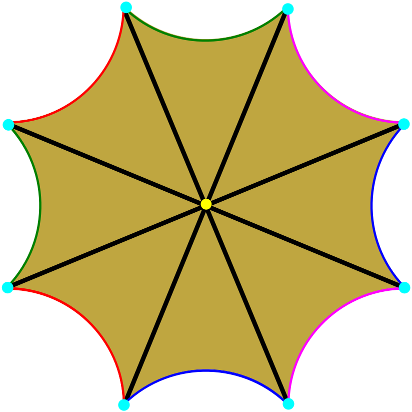

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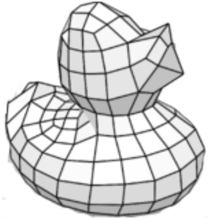

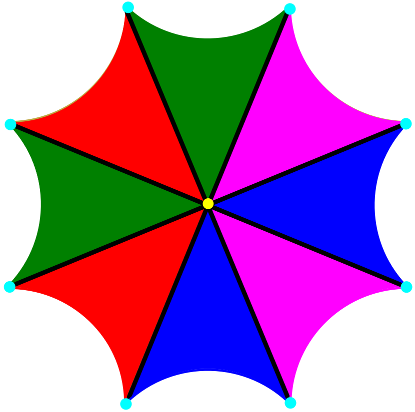

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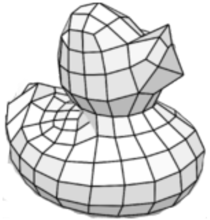
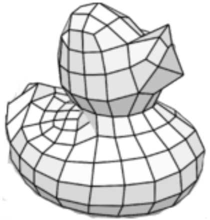


Discrete curvature

$$\kappa_s = 1 - \frac{d_s}{2} + \frac{c_s}{4}$$

$d_s :=$ degree of s and $c_s :=$ number of incident corners.

Hyp : All faces are quadrilaterals.



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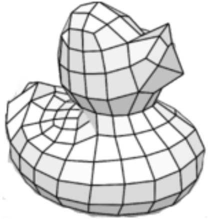
Combinatorial Gauss-Bonnet Theorem

$$\sum_{s \in S} \kappa_s = \chi$$

PROOF. $\sum_{s \in S} \kappa_s =$

Hyp : All faces are quadrilaterals.



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PROOF. $\sum_{s \in S} \kappa_s = S -$

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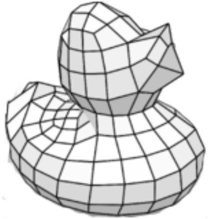

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PROOF. $\sum_{s \in S} \kappa_s = S - A +$

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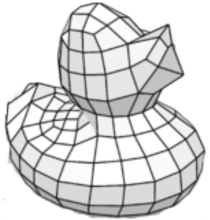

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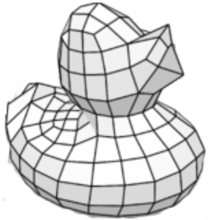
Combinatorial Gauss-Bonnet Theorem

$$\sum_{s \in S} \kappa_s = \chi$$

PROOF. $\sum_{s \in S} \kappa_s = S - A + 4F/4$

Hyp : All faces are quadrilaterals.



Discrete curvature

$$\kappa_s = 1 - \frac{d_s}{2} + \frac{c_s}{4}$$

$d_s :=$ degree of s and $c_s :=$ number of incident corners.

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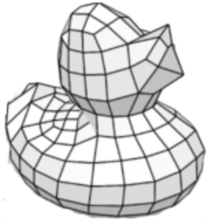

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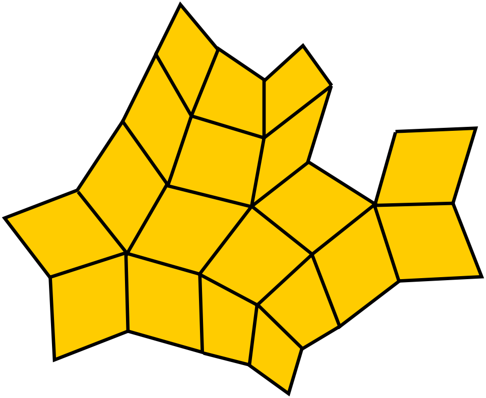
Hyp : All faces are quadrilaterals and all internal vertices have degree ≥ 4.



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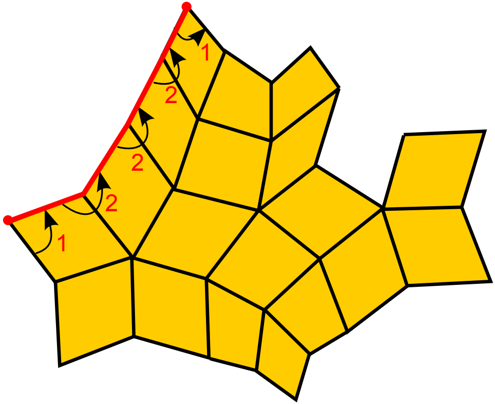

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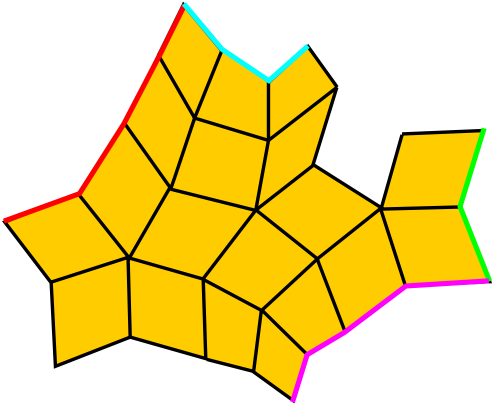

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The boundary of a non-singular disk has at least 4 brackets.

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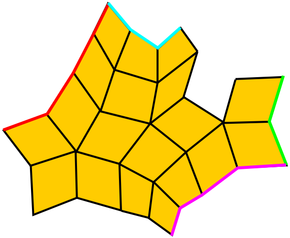

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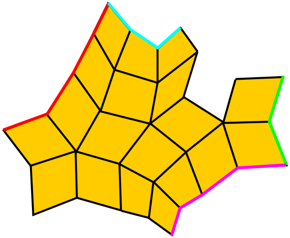

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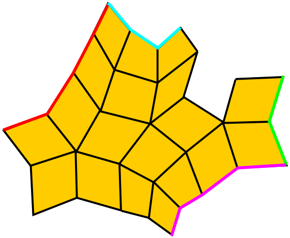

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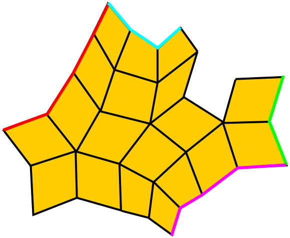

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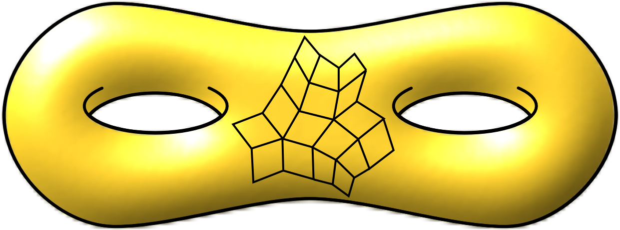

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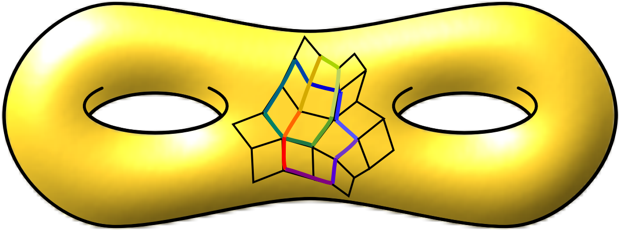

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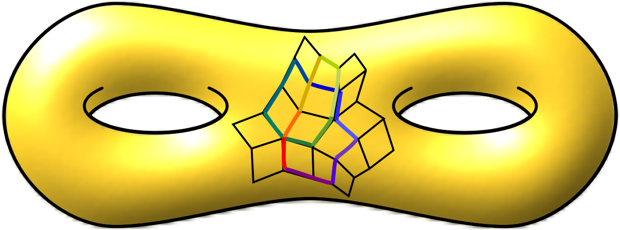

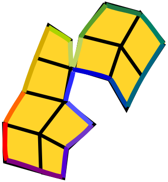

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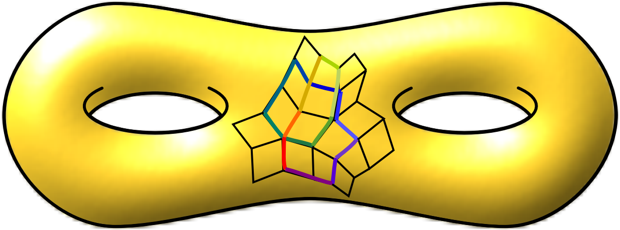

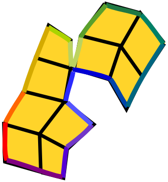

Apply the 4 brackets Theorem to this disk. □

Hyp : All faces are quadrilaterals and all internal vertices have degree > 4.

The 5 brackets Theorem

The boundary of a non-singular disk with at least one interior vertex has at least 5 brackets.

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Canonical representative

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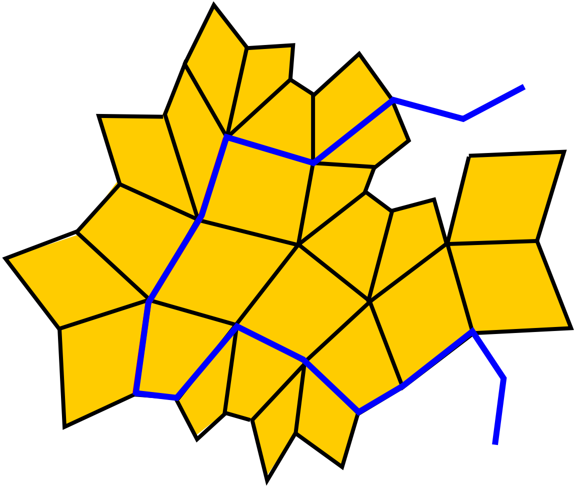

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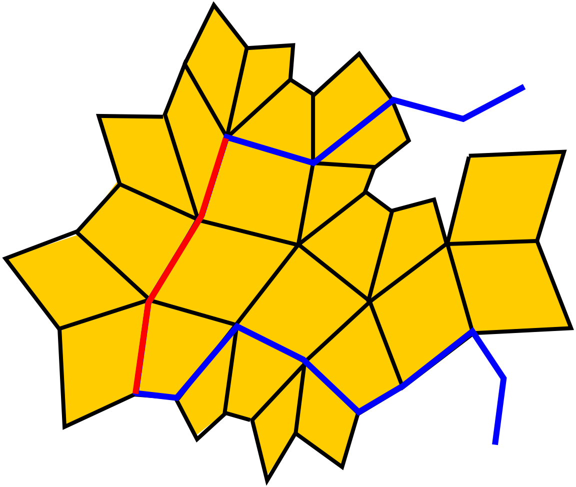

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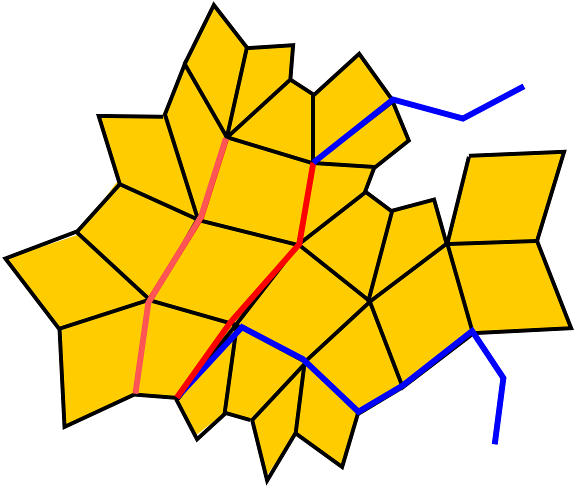

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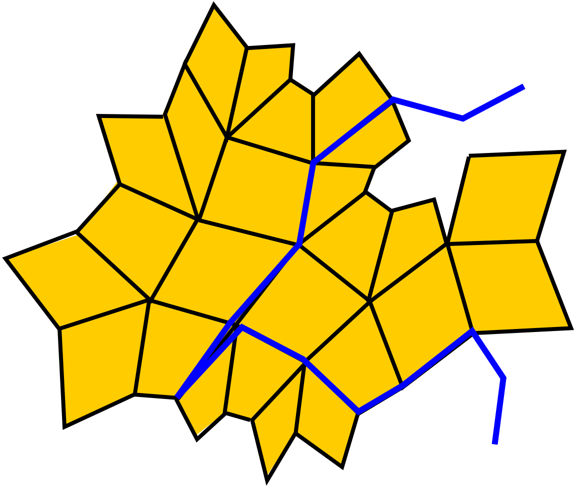

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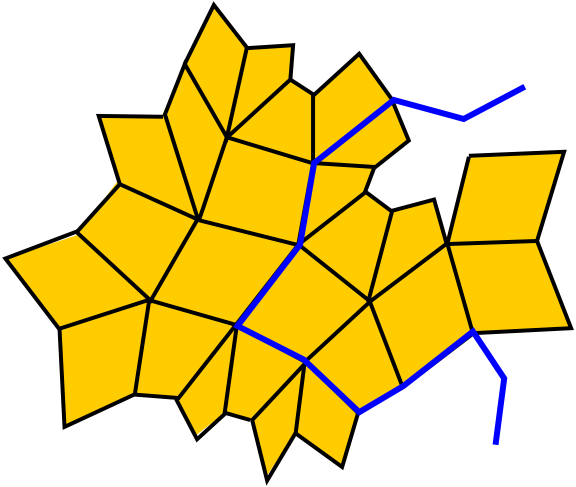

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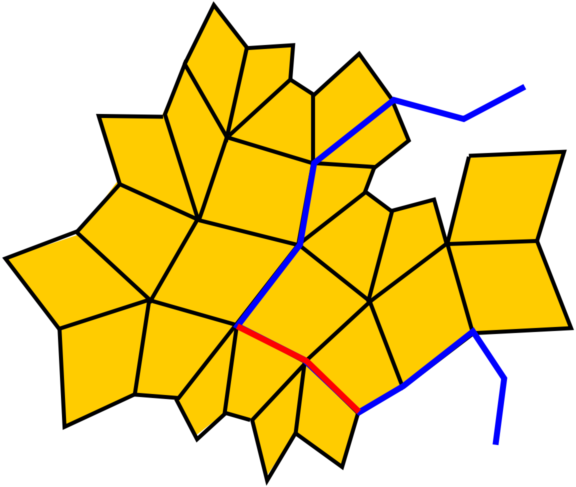

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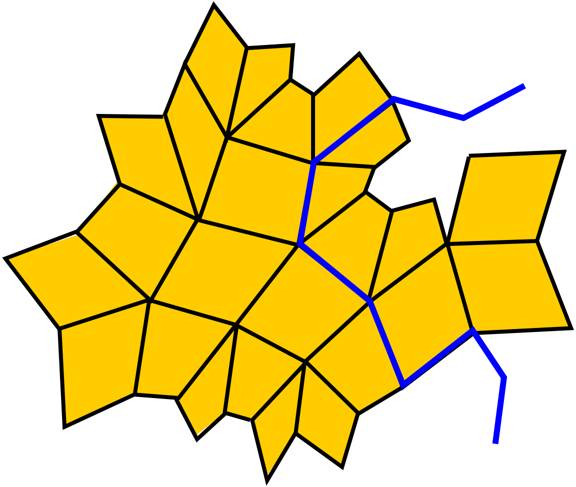

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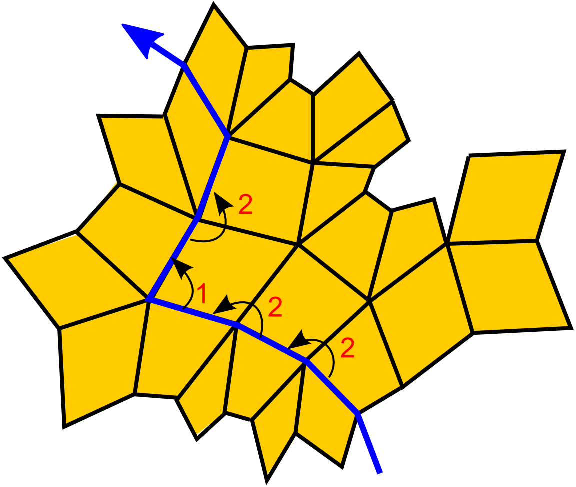

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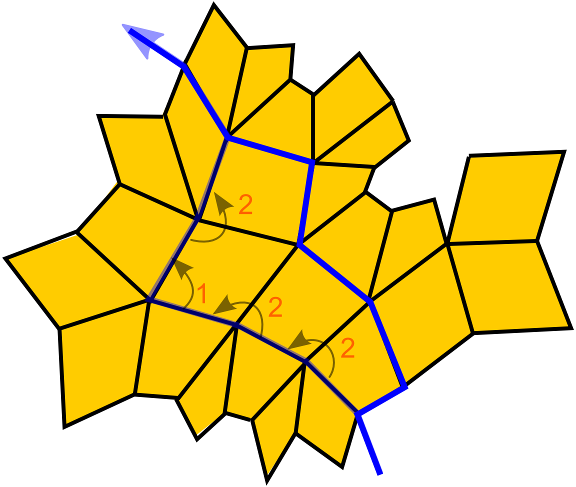

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Canonical representative

L. and Rivaud '12, Erickson and Whittlesey '13

After removing all spurs and brackets and pushing to the right as much as possible, we obtain a canonical representative. It can be computed in linear time.

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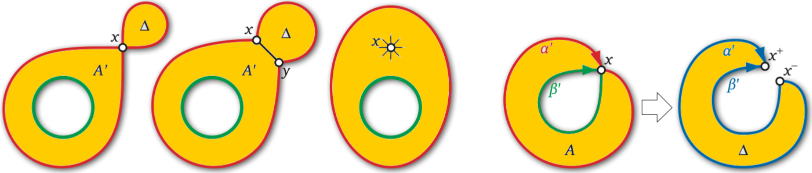

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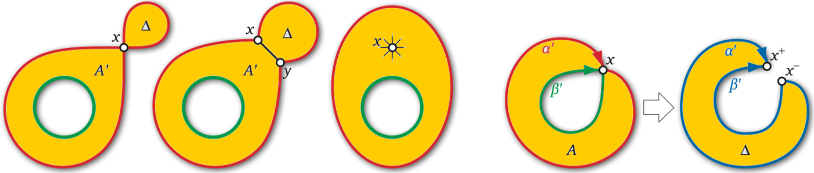

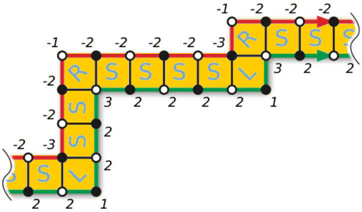

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Corollary I

One can decide if two curves are homotopic in linear time.

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Corollary I

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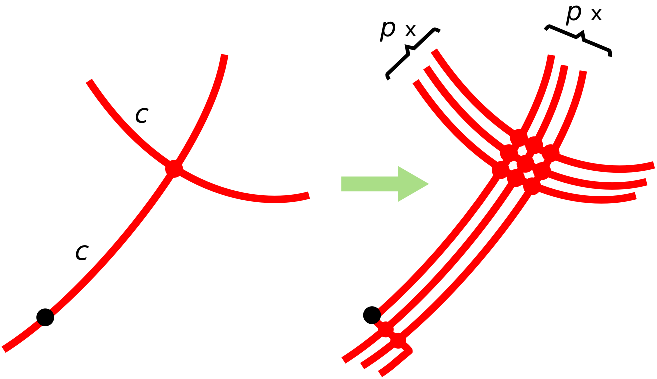
Corollary II

One can compute the primitive root of a curve in linear time.

Non-primitive curves

Lemma

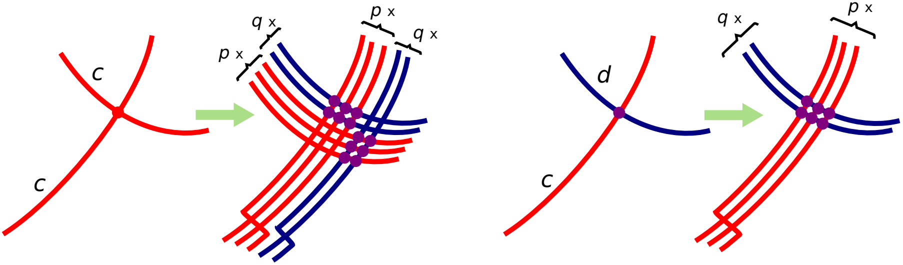
$$\imath(c^p) = p^2 \times \imath(c) + p - 1$$



Non-primitive curves

Lemma

$$\imath(c^p, d^q) = \begin{cases} 2pq \times \imath(c) & \text{if } c \sim d \text{ or } c \sim d^{-1}, \\ pq \times \imath(c, d) & \text{otherwise.} \end{cases}$$



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For a given curve c

1. Determine the primitive root of c.
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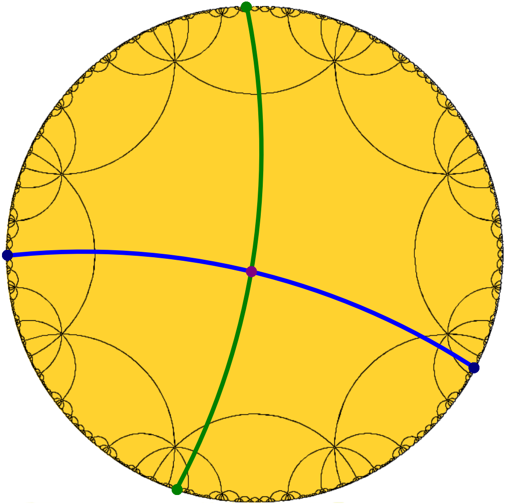
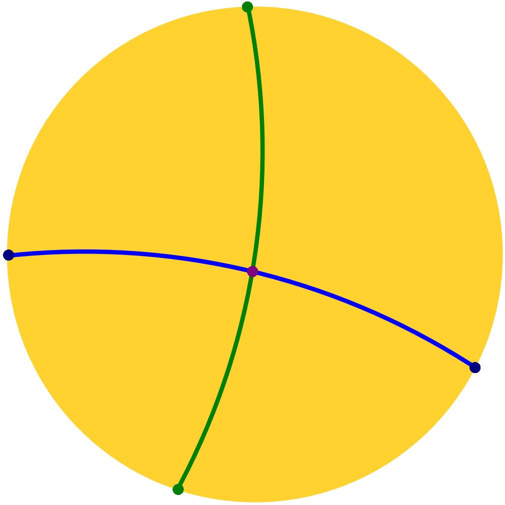
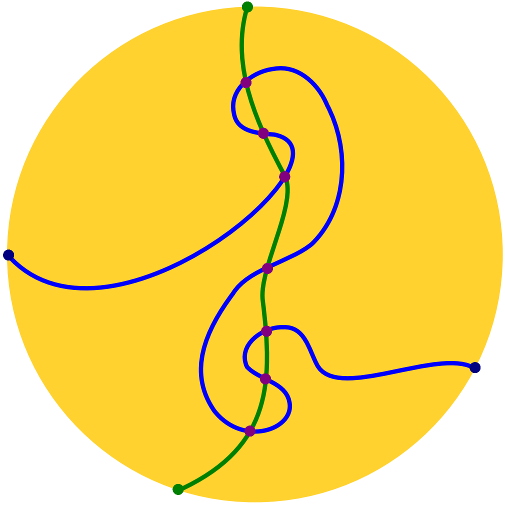
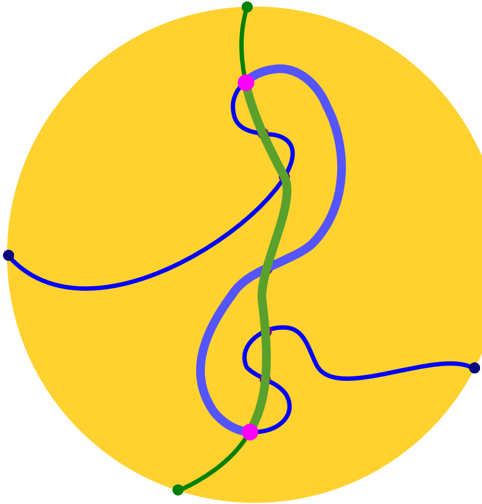






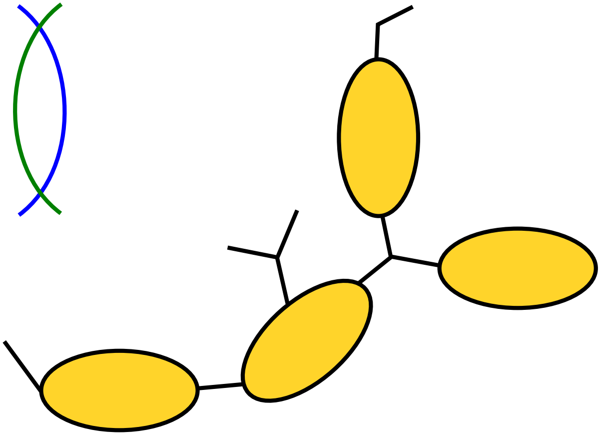


Double paths

A double path is a pair of homotopic paths.

Lemma

Double paths in a canonical representative (resp. a geodesic) are quasi-flat.



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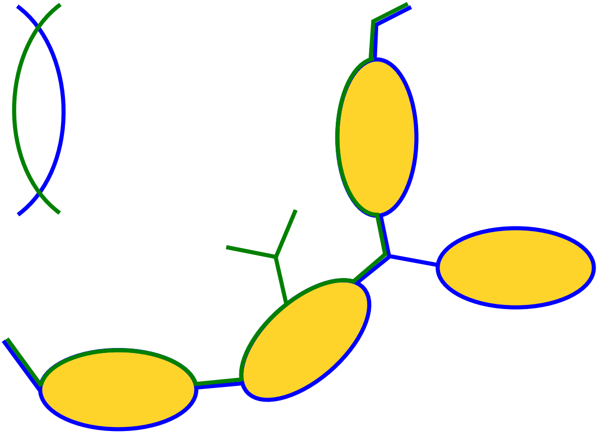

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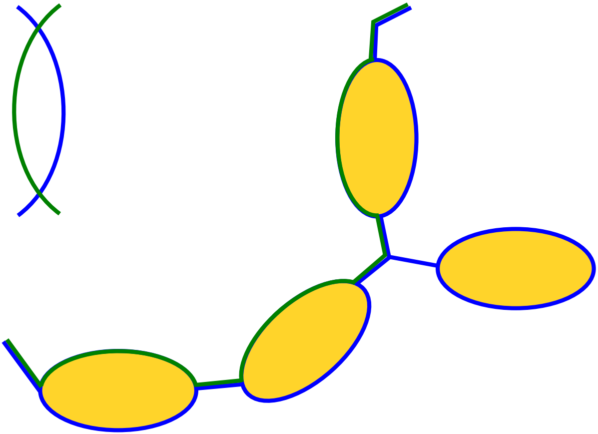

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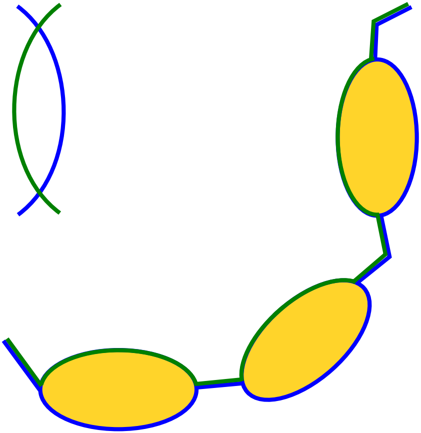

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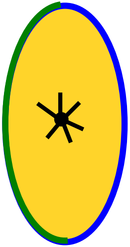

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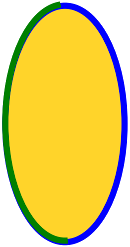

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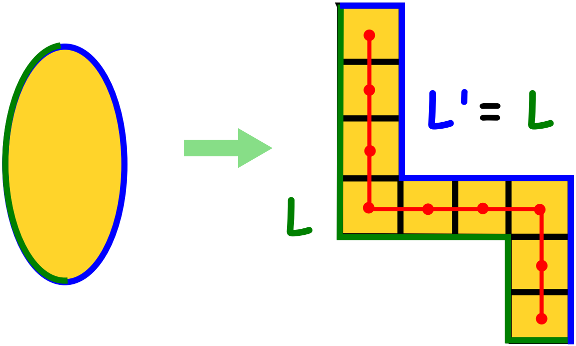

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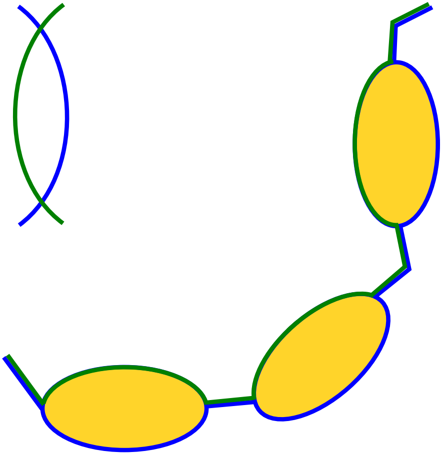

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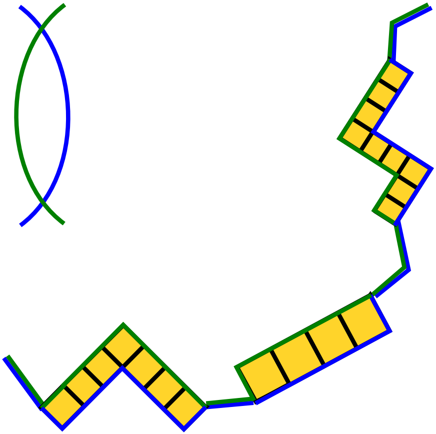

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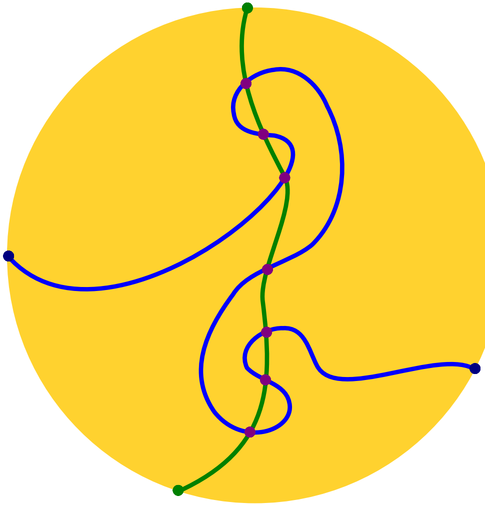

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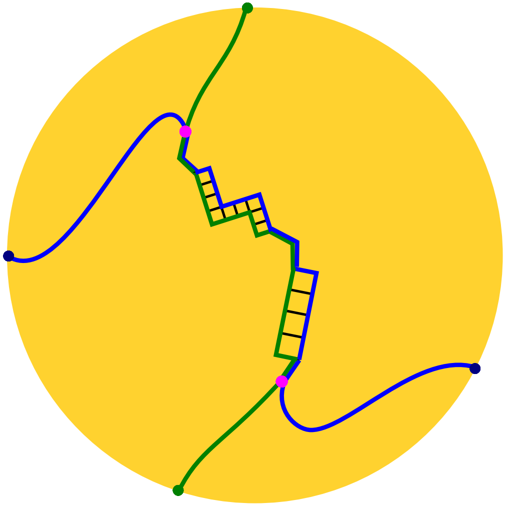

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Lemma

The set of maximal double paths can be computed in quadratic time.

PROOF. A pair of indices (i, j) may occur at most twice in the set of maximal double paths. $\square$

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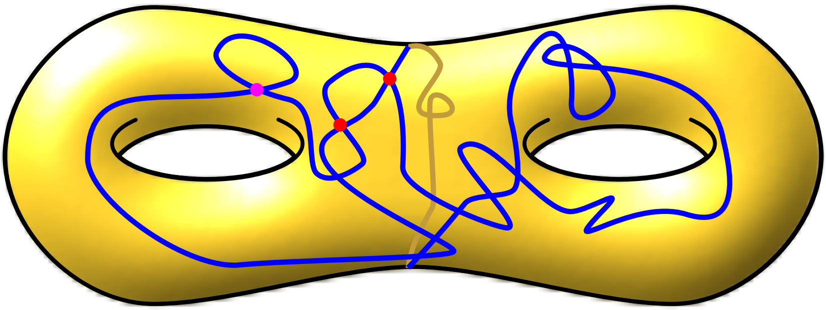
Despré and L. ’16

The geometric intersection number of one (two) curve(s) can be computed in quadratic time.

Computing an actual minimal configuration



Hass and Scott '85

A curve with excess intersection has either a monogon or a singular bigon.

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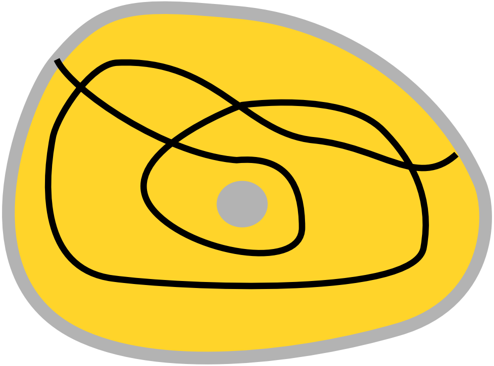

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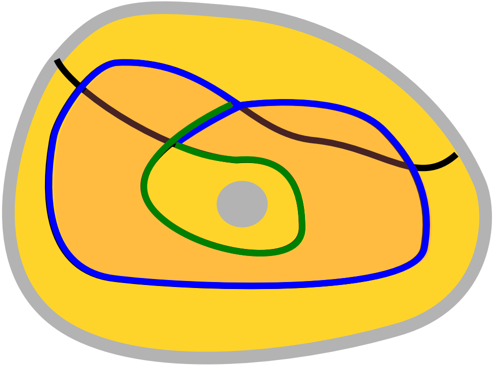

Bigon swapping

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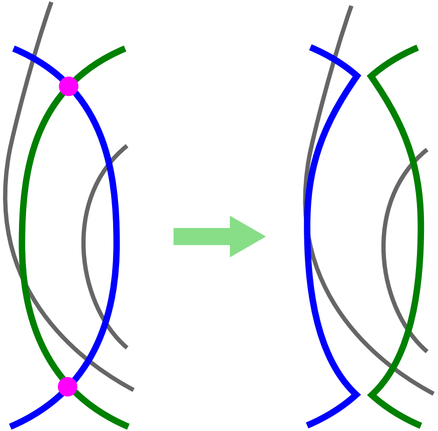

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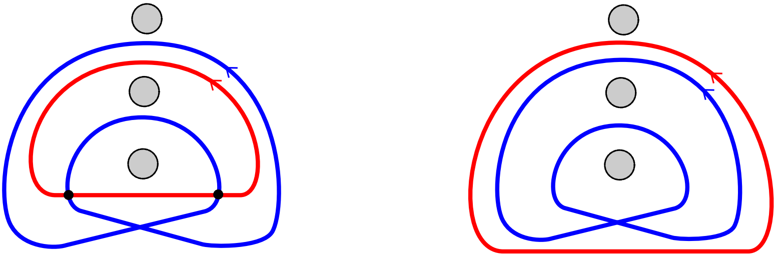
Despré and L. '16

Given c, an homotopic immersion with a minimal number of intersections can be computed in $O(|c|^4)$ time.

Case of two curves



Open problems

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- Design a better algorithm to just decide if the geometric intersection number is null.
- Propose an algorithm to compute a minimal immersion in hyperbolic configuration (cf. Hass and Scott ’99).

Thank you for your attention