Shortcuts for the Circle

Sang Won Bae, Mark de Berg, Otfried Cheong,
Joachim Gudmundsson, Christos Levcopoulos

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≫ When adding k edges to the n-vertex cycle (n even), the diameter of the resulting graph is at least $\frac{n}{k+2} - 3$, and there is a way to add k edges to achieve diameter $\frac{n}{k+2} - 1$ [Chung, Garey 1984].

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- Also algorithmic questions: Find the "best" edge(s) to be added to a given network fast.

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- We are allowed to add k shortcuts, for a given (small) number k.
- A shortcut is a chord of the circle that can be used to connect points on the circle.
- The goal is to minimize the diameter of the resulting "graph". The diameter is the maximum of $d_S(p, q)$ over all pairs of points p, q on the circle.

Our results

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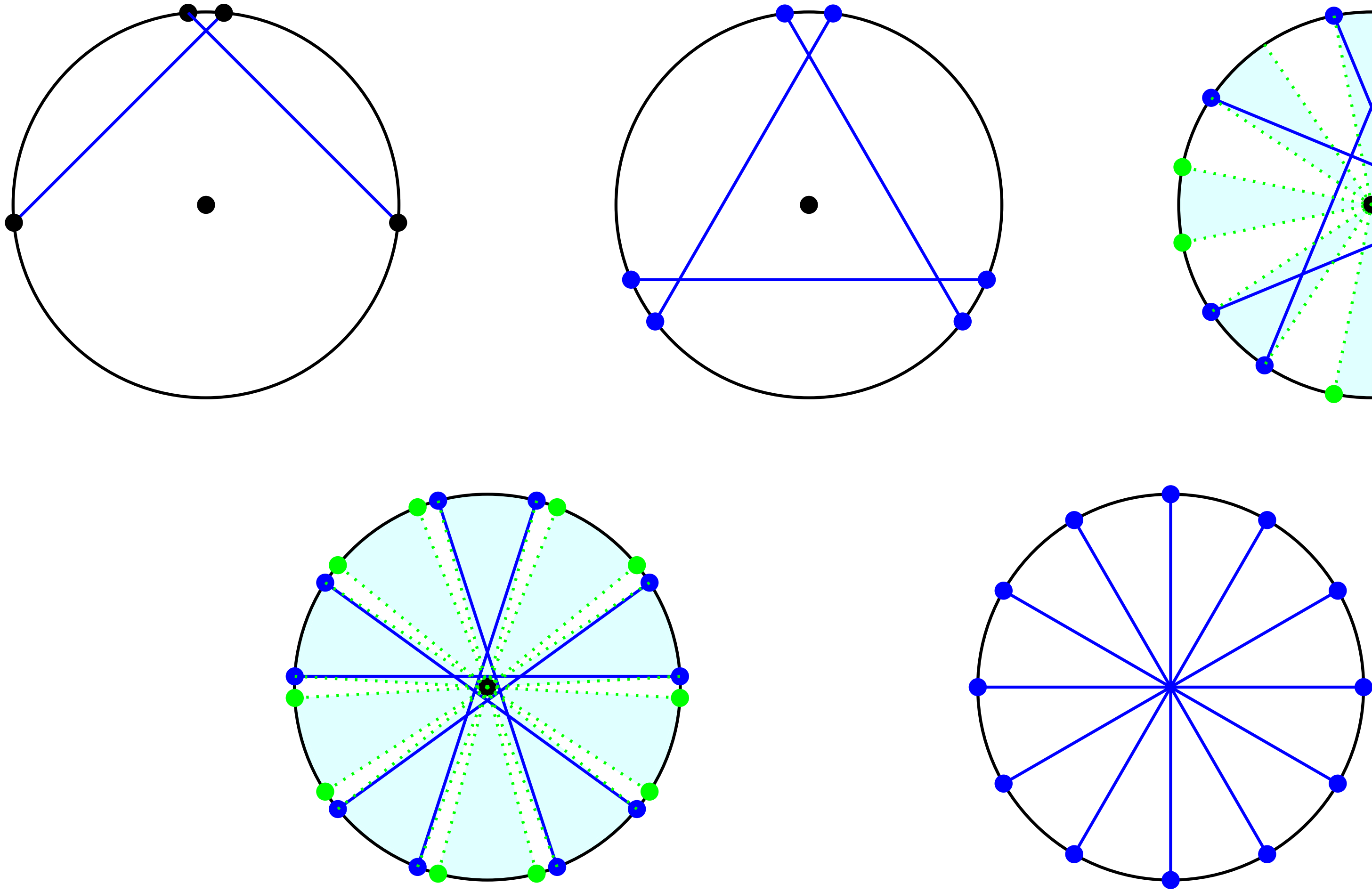

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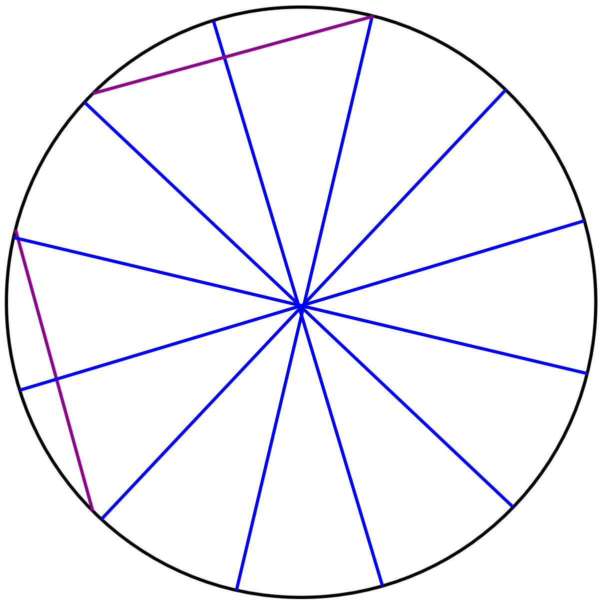

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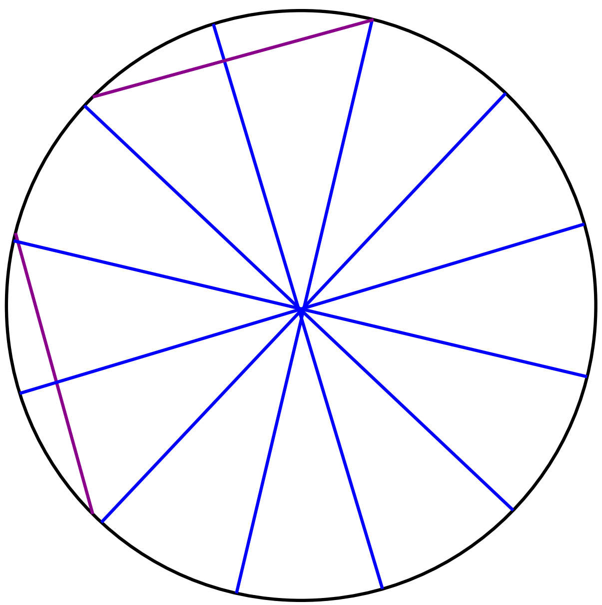

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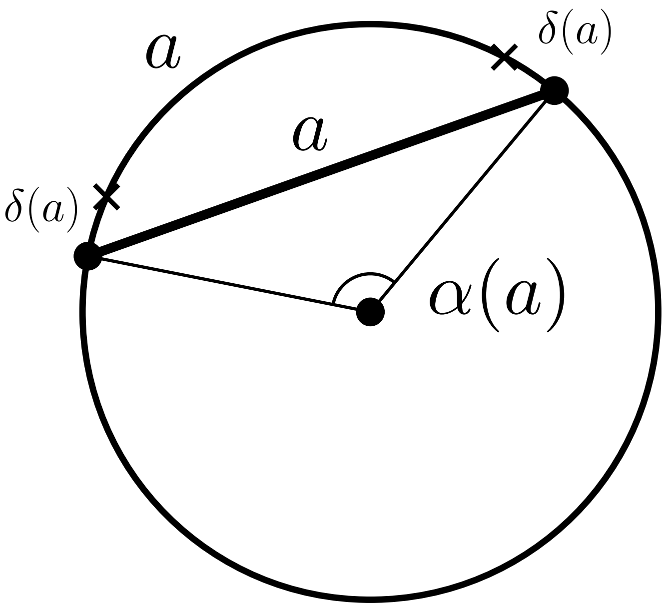
My first paper to come with a Python script to perform numerical calculations:
http://github.com/otfried/circle-shortcuts

What one shortcut can do for you



Shortcut of length $a \in [0, 2]$ spans angle

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Set $\delta(a) = (\alpha(a) - a)/2$, so that $\alpha(a) = a + 2\delta(a)$.

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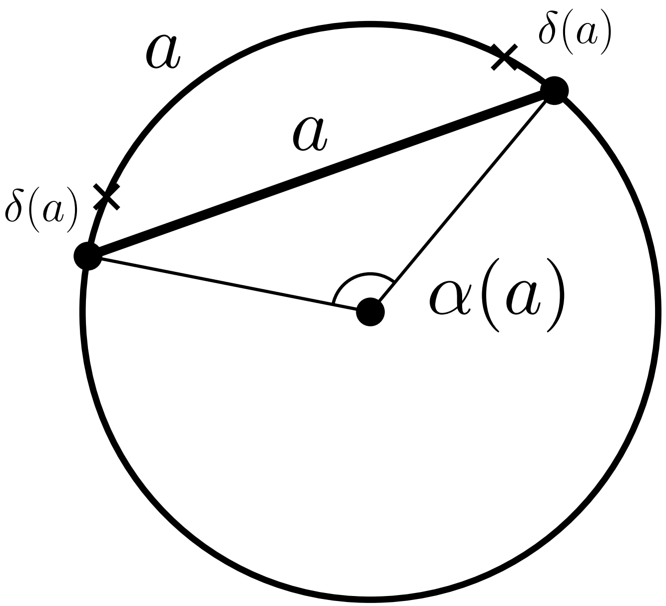

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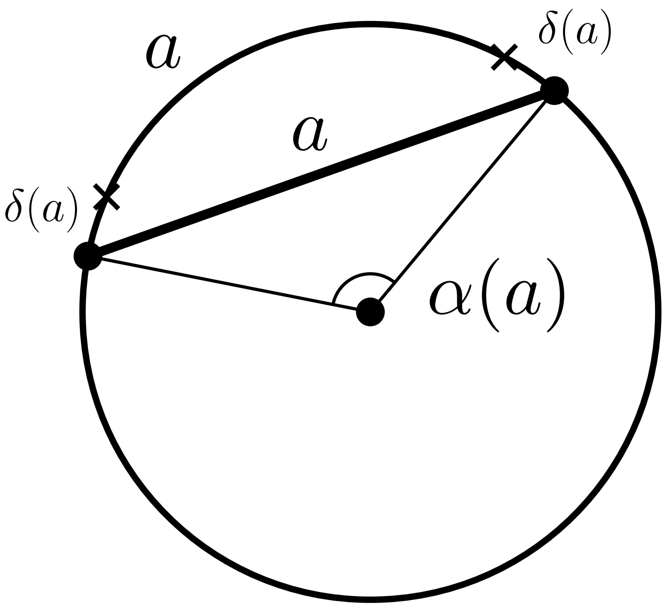

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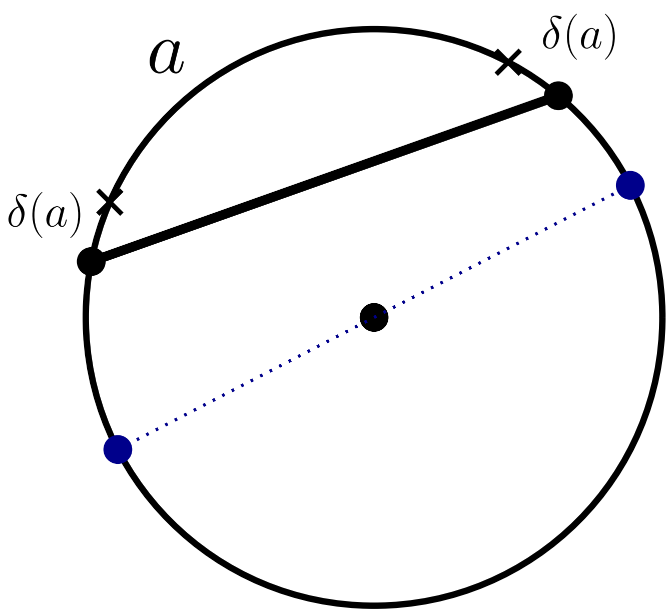

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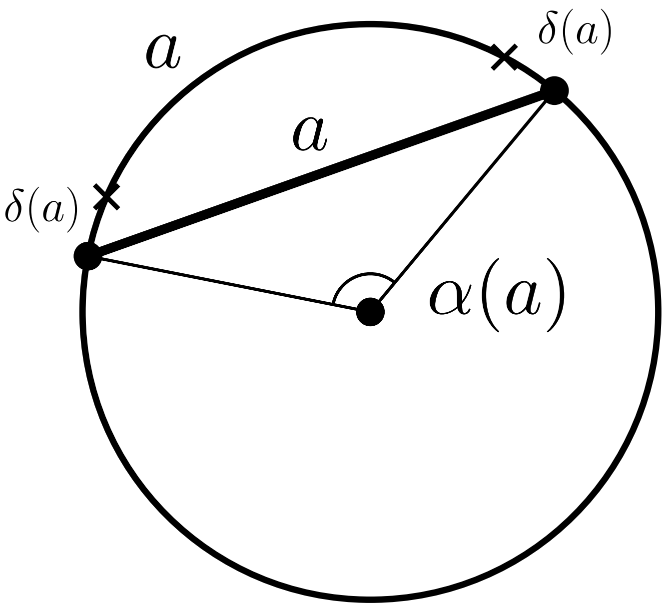

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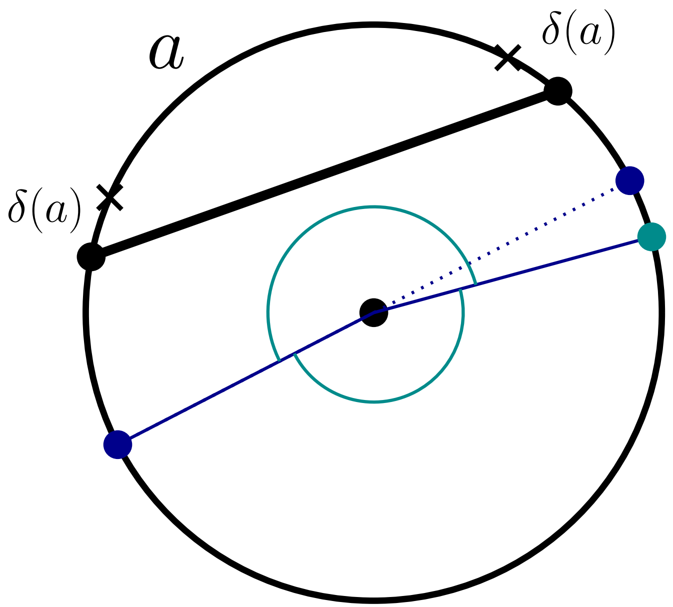

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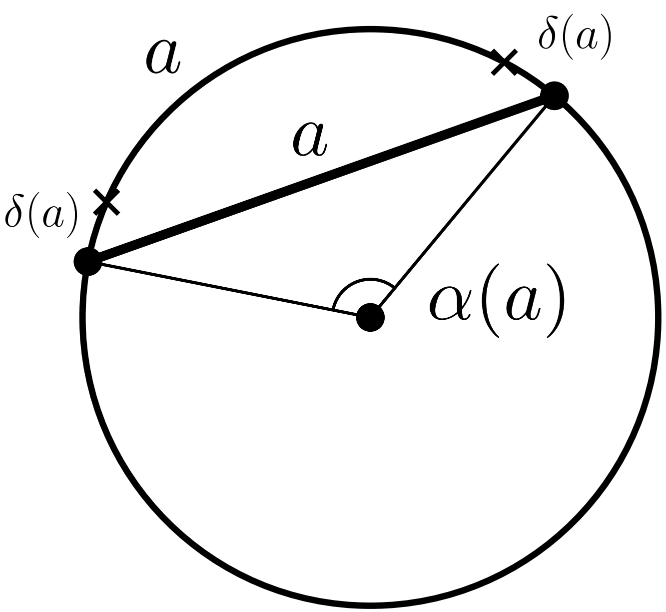

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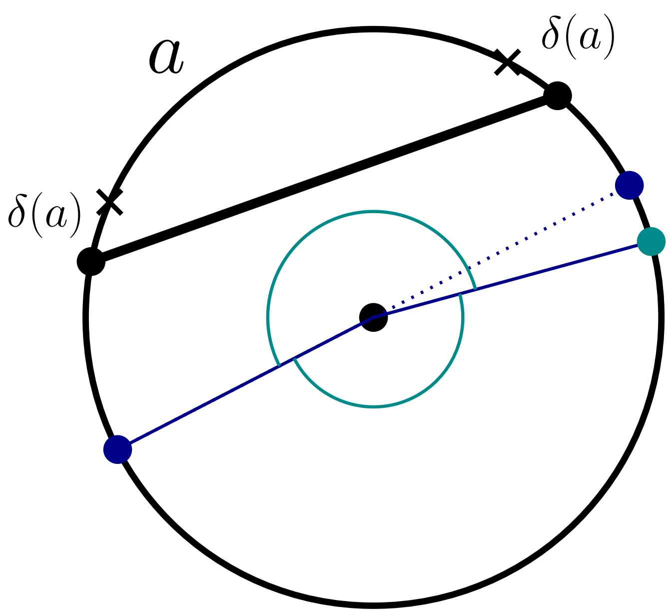

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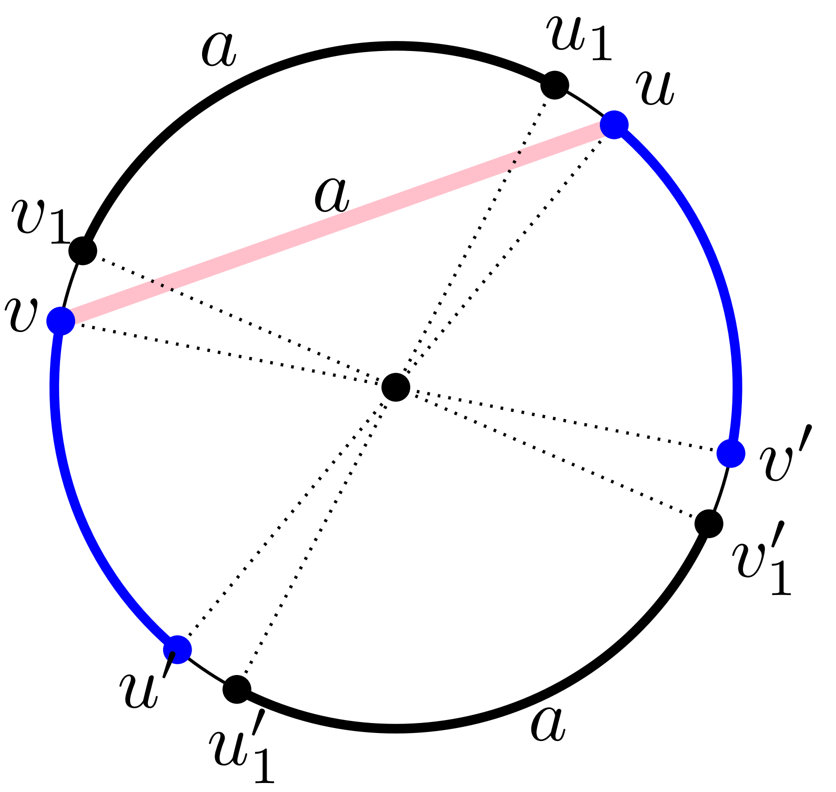
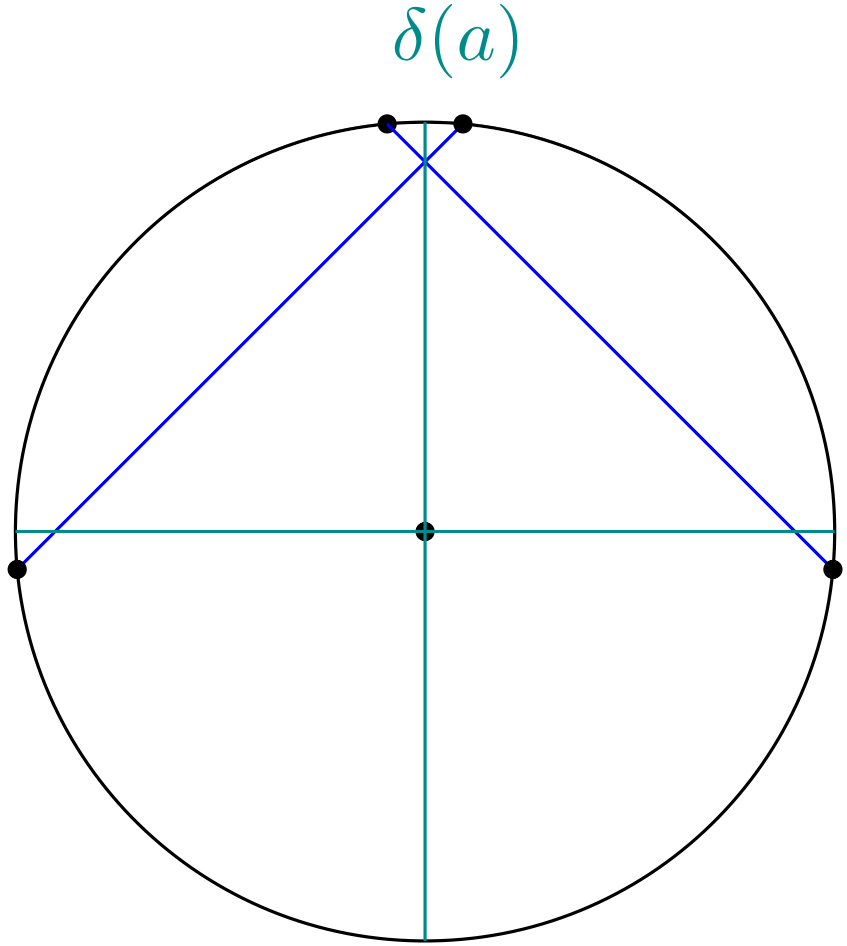
Indeed, the optimal solution for up to six shortcuts uses shortcuts of equal length a and the resulting diameter is $\pi - \delta(a)$.

Umbra and radiance



A shortcut of length a has two umbra arcs of length a.
Points in the umbra can make no use of the shortcut.
Points in the radiance can make full use of the shortcut and save $2\delta(a)$.

Example: Two shortcuts



Parameterizing the interesting pairs

Fix a target diameter of the form $\pi - \delta^*$. We only need to consider pairs (p, q) making an angle in $[\pi - \delta^*, \pi + \delta^*]$.

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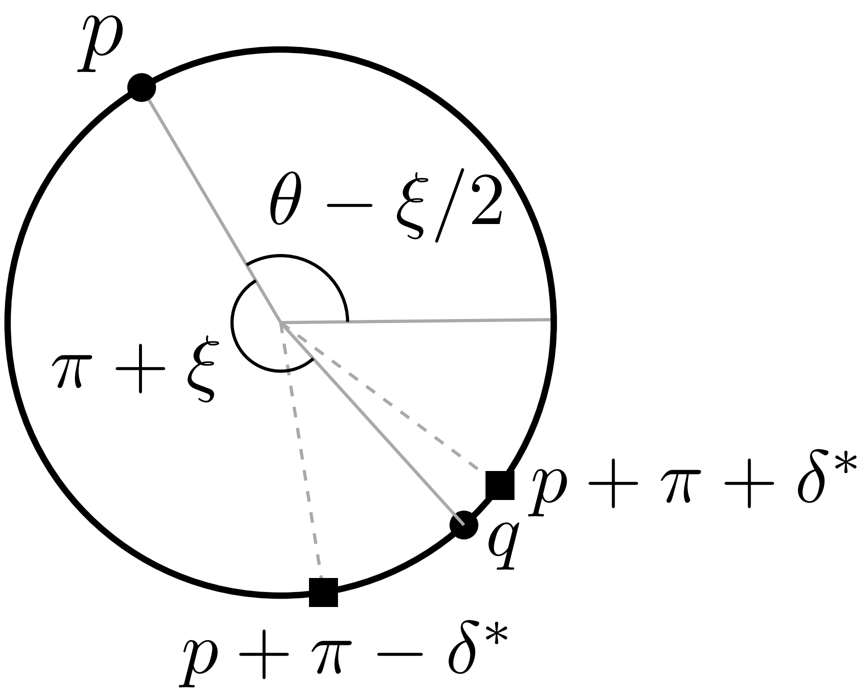
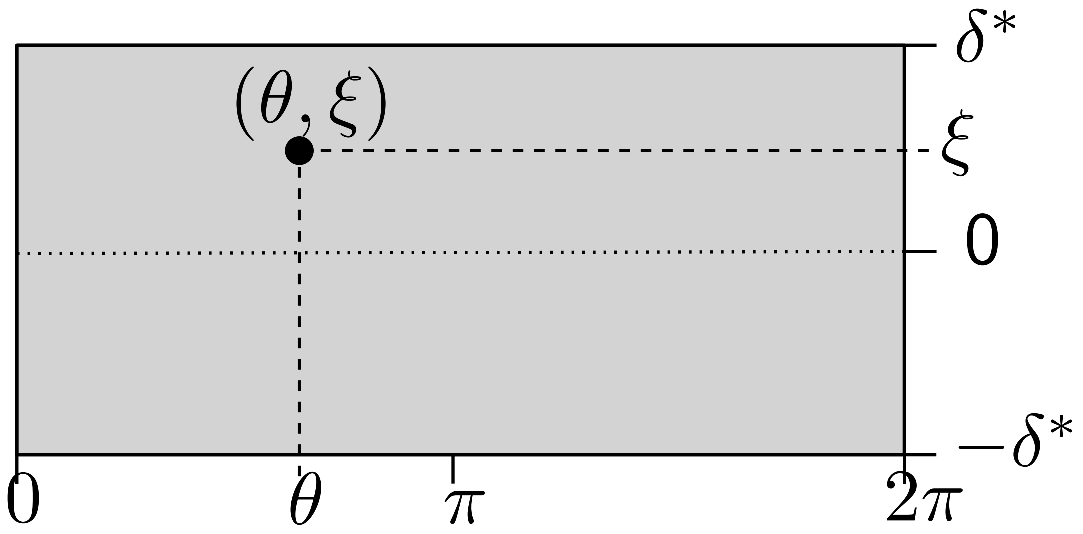
Represent these pairs as the cylinder (θ, ξ) for $\theta \in [0, 2\pi]$, $\xi \in [-\delta^*, \delta^*]$, where $p = \theta - \xi/2$ and $q = \theta + \pi + \xi/2$.





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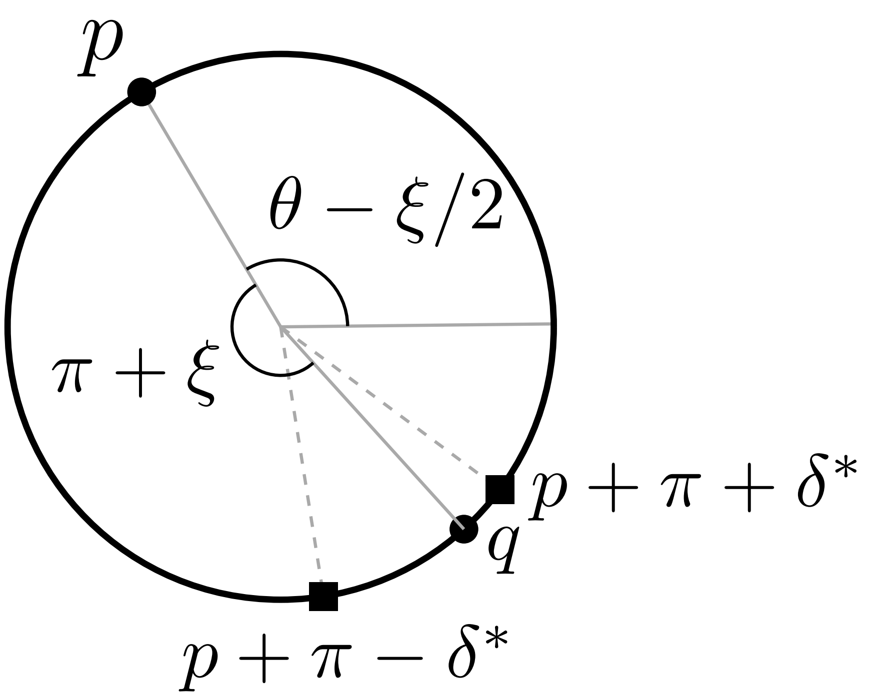

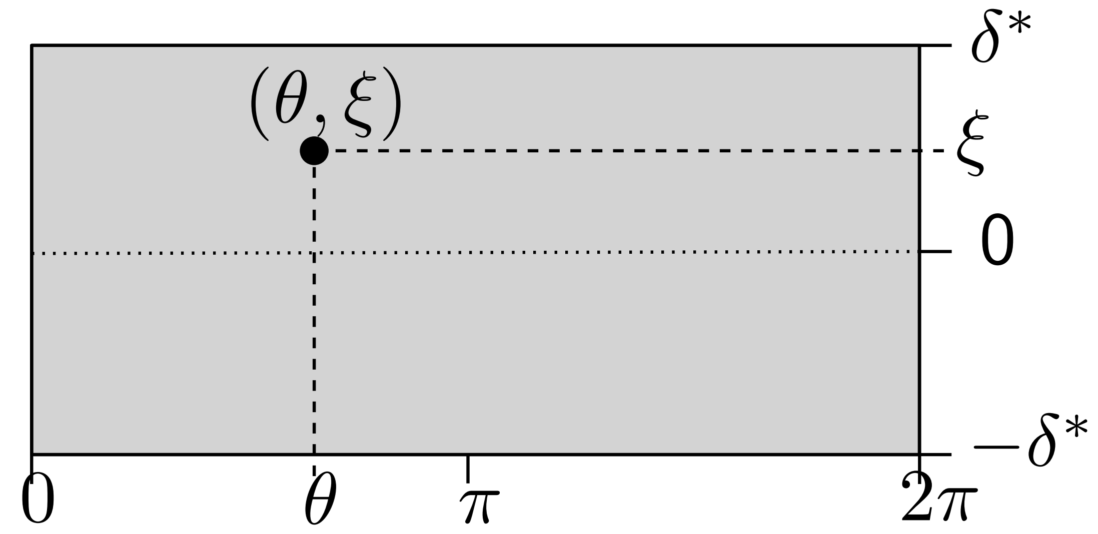

Actually, the "cylinder" is a Möbius-strip.

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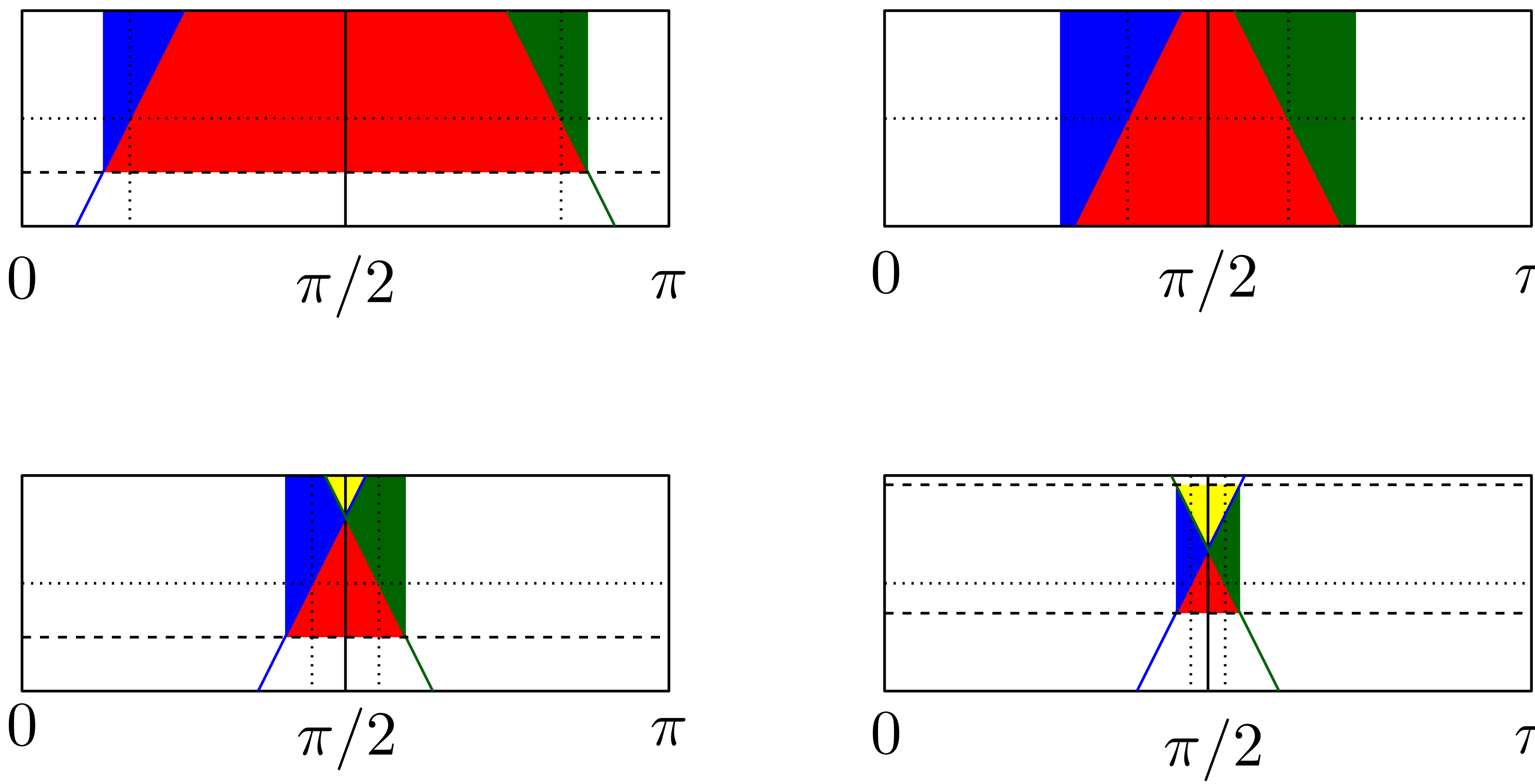
Then $R(s)$ consists of two rectangles of width $\pi - a - \delta^*$ and height $2\min(\delta(a), \delta^*)$.



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Calculus: For a fixed $\delta^* \leq 0.7$, the function $a \mapsto A(a, \delta^*)$ is increasing for $a \leq a^*$ and decreasing for $a \geq a^*$. Its maximum value is $A(a^*, \delta^*) = 4\delta^*(\pi - a^* - \delta^*)$.

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Let a_k^* be the unique solution to

$$a_k^* + \delta(a_k^*) = \frac{k-1}{k}\pi.$$

The magic values

k	a_k^*	δ_k^*	$\mathrm{diam}(S) = \pi - \delta_k^*$
2	1.4782	0.0926	3.0490
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Theorem: For $k \in \{2, 3, 4, 5\}$ and assuming that no pair of points can use more than one shortcut, $\pi - \delta_k^*$ is the optimal diameter and our solution is unique up to rotation.

Using more than one shortcut?

Lemma: Let $S' \subset S$ be the set of shortcuts used by the shortest path for an antipodal pair (p, q). If $d_{S'}(p, q) \leq \pi - \delta_k^*$, where $k \in \{4, 5, 6\}$, then the longest shortcut in S' has length at least λ_k and all the others have length at most σ_k, where σ_k and λ_k with $\sigma_k < \lambda_k$ are the two solutions to the equation $\delta(x) + \delta(\pi - \delta_k^* - x) = \delta_k^*/2$ for $x \in [\pi - \delta_k^* - 2,\ 2]$.

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Lemma: Let S be a set of k shortcuts for $k \in \{3, 4, 5, 6\}$ such that $\mathrm{diam}(S) \leq \pi - \delta_k^*$. Then there is no antipodal pair of points $p, q \in C$ such that the path of length $d_S(p, q)$ uses more than one shortcut.

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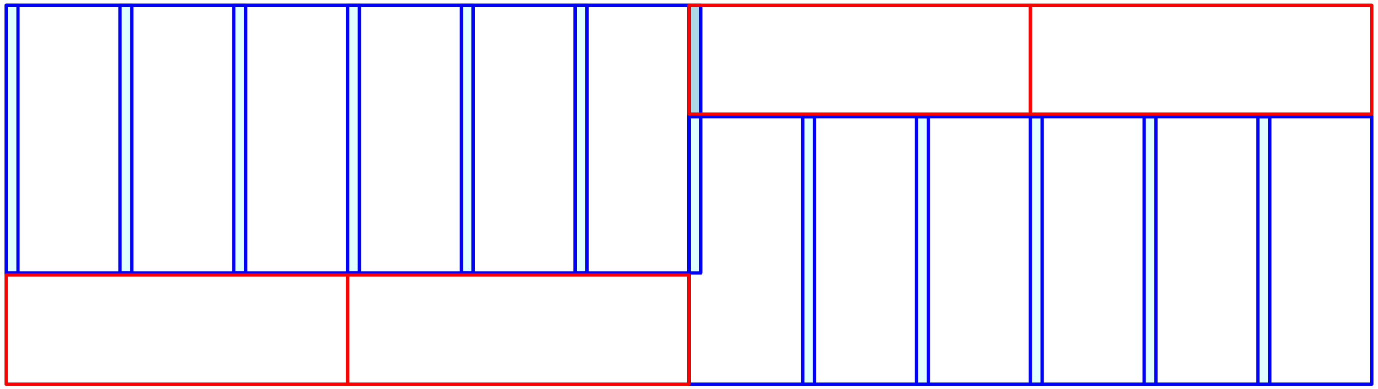
- Therefore coverage of upper boundary is at most

$$5(\pi - \mu - \delta^*) + (\pi - \delta^*) < 2\pi,$$

a contradiction.

Eight shortcuts



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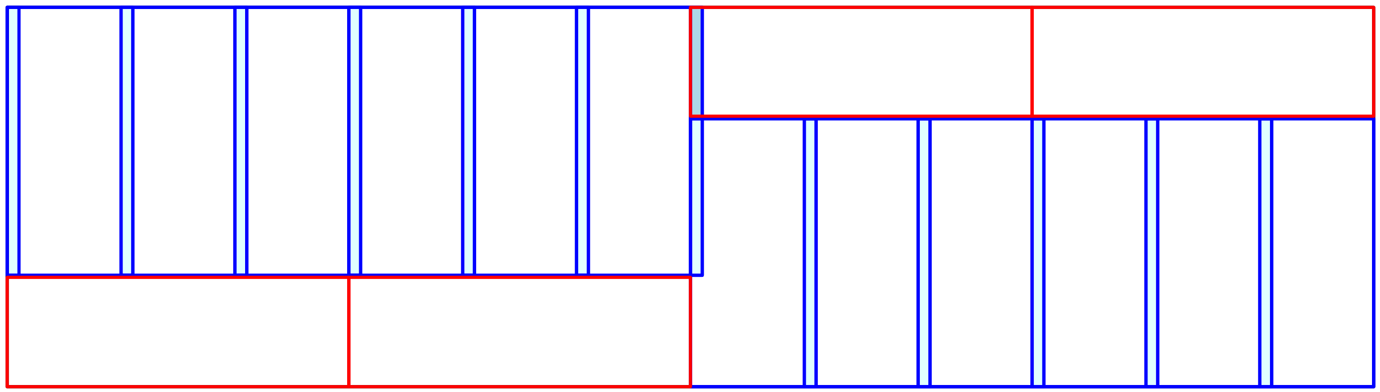
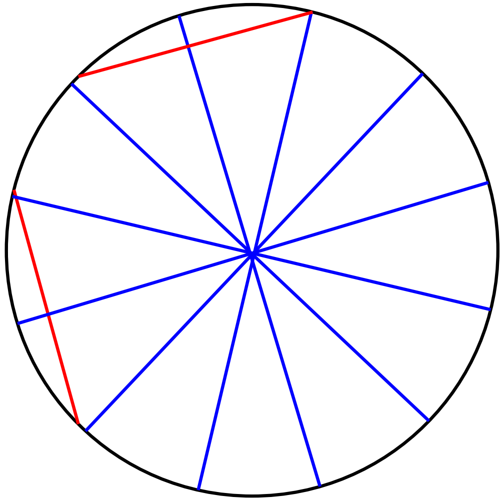


Maximize δ^* with constraints $\pi - a_1 - \delta^* \geq \pi/6$, $\pi - a_2 - \delta^* \geq \pi/2$, and $\delta(a_1) + \delta(a_2) \geq \delta^*$.



We have $a_1 \approx 1.999870869$, $a_2 \approx 0.988571799$ and achieve diameter $\mathrm{diam}(S) \approx \pi - 0.5822245291 = 2.559368125 < \mathrm{diam}(6)$.

More, and open problems

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≫ Are there any other values of k for which $\mathrm{diam}(k) = \mathrm{diam}(k+1)$?