1.1 Planar Graphs

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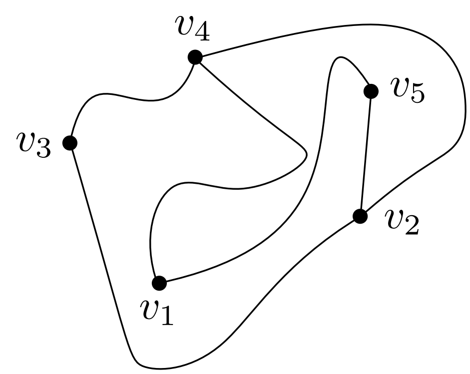
> *I want to express a radical alternative that I learned from Sir Michael Atiyah. His view was that the most significant aspects of a new idea are often not contained in the deepest or most general theorem which they lead to. Instead, they are often embodied in the simplest examples, the simplest definition and their first consequences.*
>
> David Mumford

One major way that graph theory interacts with geometry is through the study of graphs that can be drawn, or *embedded*, in Euclidean spaces with certain constraints. For example, given a graph $G = (V, E)$, one can ask if it can be drawn in the plane in such a way that no two edges of G cross each other.



We define the notion of a drawing more precisely. An *embedding* of a graph $G = (V, E)$ in the plane consists of functions that map its vertices and edges to subsets of the plane. First, the function $\phi_V : V \to \mathbb{R}^2$ maps each vertex $v \in V$ to a point $\phi_V(v) \in \mathbb{R}^2$. Then for each edge $e = (u, v) \in E$, the continuous function $\phi_e : [0, 1] \to \mathbb{R}^2$ maps e to a continuous curve in $\mathbb{R}^2$ between the mappings of u and v, i.e., $\phi_e(0) = \phi_V(u)$ and $\phi_e(1) = \phi_V(v)$. We will assume that ϕ_V is injective, and that no curve for any edge e passes through any of the points of the vertices, unless the vertex is an endpoint of e.

An embedding of G is called planar if the curves of every two edges of E are disjoint, except possibly at a common vertex. A graph $G = (V, E)$ is called *planar* iff there exists a planar embedding of G. Any such embedding of G partitions $\mathbb{R}^2$ into connected components, called the *faces* of the embedding. Each face is bounded by elements of V and E. The unbounded face is called the *outer face* of the embedding, and all other faces are the *inner faces*. The *size* of a face is its number of bounding vertices.

Basic Properties of Planar Graphs

One of the most basic facts about planar graphs is the following:

Lemma 1.1 (Euler's formula). *Let $G = (V, E)$ be a connected planar graph, and consider a planar embedding with the set of faces F. Then*

$$|V| - |E| + |F| = 2.$$

Proof. The proof is by induction on the number of cycles in G. The base case is when G is a tree, in which case $|F| = 1$ (the outer face), $|E| = |V| - 1$ and the relation holds. Otherwise, let $G' = (V', E')$ be the graph obtained by removing any edge $e \in E$ belonging to a cycle in G, and let F' be the resulting set of faces. Then $|V'| = |V|$, $|E'| = |E| - 1$ and $|F'| = |F| - 1$. By induction, $|V'| - |E'| + |F'| = 2$, and we get the desired relation for G. □

Euler's formula implies that the number of faces is the same for any embedding of a planar graph. It also implies an upper-bound on the number of edges in a planar graph:

Lemma 1.2. *Let $G = (V, E)$ be a planar graph on n vertices and m edges, and where the size of each face is at least $k \geq 3$ for an integer k. Then $m \leq \frac{k}{k-2}(n-2)$. In particular, any planar graph has at most $3n - 6$ edges.*

Proof. Consider any planar embedding of G, and let F be the set of faces of this embedding. The proof is by double-counting the sum of the sizes of the faces of G:

$$k \cdot |F| \leq \sum_{f \in F} \big(\text{size of } f\big) \leq 2m.$$

From Euler's relation, $|F| = 2 - n + m$,

$$\begin{aligned} k \cdot (2 - n + m) &\leq 2m \\ m &\leq \frac{k}{k-2}(n-2). \end{aligned}$$

□

Planarity Testing Criterion

Clearly planar graphs are a small subset of all possible graphs. Two important examples of non-planar graphs are K_5 and $K_{3,3}$. From Lemma 1.2, it follows that K_5 is not planar, as it has 5 vertices yet with greater than $3 \cdot 5 - 6 = 9$ edges; similarly $K_{3,3}$ is not planar, as it has 6 vertices, no triangular face (so $k = 4$), and yet greater than $2 \cdot 6 - 4 = 8$ edges. Surprisingly, any graph not containing a subdivision[†] of these two subgraphs is planar:

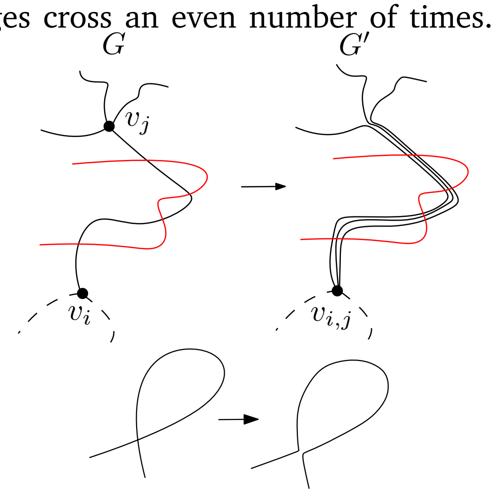
Theorem 1.3 (Kuratowski's theorem)**.** *A graph is planar iff it does not contain a subdivision of K_5 and $K_{3,3}$.*

Kuratowski's theorem can be used to prove another important theorem, the Hanani-Tutte theorem, which states that if a graph G can be embedded in the plane such that every edge has an even number of crossings, then G is planar. Here we prove a slightly weaker variant directly by induction.

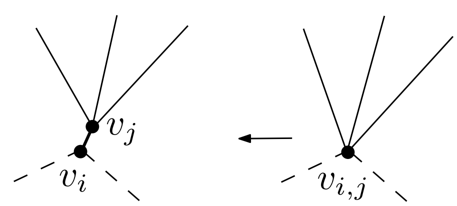
[†]Recall that given $G = (V, E)$ and an edge $e = \{u, v\} \in E$, the *subdivision of* e yields a graph $G' = (V', E')$, where $V' = V \cup \{w\}$, and $E' = E \setminus \{\{u, v\}\} \cup \{\{u, w\}, \{w, v\}\}$. A graph H is a subdivision of G if it can be derived from G by a sequence of subdivisions.

Theorem 1.4 (Weak Hanani-Tutte theorem)**.** *Let $G = (V, E)$, $|V| = n, |E| = m$, be a graph for which there exists an embedding such that every pair of edges cross an even number of times. Then G is a planar graph.*

Proof. Fix an embedding such that every pair of edges cross an even number of times. Then we will inductively modify this embedding to construct a planar embedding of G. For the inductive argument to work, we will prove the statement in a slightly stronger form: (i) we show it more generally for multigraphs, and (ii) that the planar embedding has the same sequence of edges around each vertex as the given embedding.



Let $e = \{v_i, v_j\}$ be any edge in G. Contract e by moving v_j to the position of v_i to get a new graph G' with the merged vertex $v_{i,j}$ at the position of v_i. The embedding for each edge previously incident to v_j is updated by extending it along the edge e to reach v_i. There might exist self-intersections of an edge in this modified embedding, but they can be removed (see figure). Note also that even if G was a graph, G' could be a multigraph (as a vertex with an edge to both v_i and v_j in G now has two edges to v_i in G'). Every pair of edges still cross an even number of times in this embedding of G', and so by induction, G' has a planar embedding. This embedding can now be extended to an embedding of G by splitting $v_{i,j}$ in G' to two vertices v_i, v_j of G connected by a small-enough line segment. Here we need the stronger inductive hypothesis that the embedding of G' preserves the order of edges around each vertex, as then all the edges incident to v_j are contiguous around $v_{i,j}$ (same for the edges incident to v_i) in the planar embedding of G' and so can be assigned to v_j after the split without causing any additional intersections.



The base case is a graph G' consisting of a single vertex v with multiple self edges. Then, as every pair of edges cross an even number of times, there must exist an edge e with an embedding that leaves and enters v consecutively in the clockwise ordering of the edges leaving/entering v[†]. Then $G' \setminus \{e\}$ has, by induction, a planar embedding with the same sequence of edges around v. Adding e back, by a small-enough curve that does not intersect any other edge, gives the required embedding for G'. □

[†]Take the edge e with the smallest number of edges in the clockwise ordering of its two endpoints around v. If there is an edge that leaves between the two endpoints of e, it must enter also between the two endpoints, but this contradicts the choice of e.

Some Other Properties of Planar Graphs

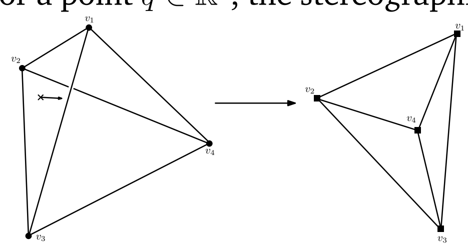
A *proper coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of V such that the two vertices of every edge $e \in E$ have different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors required for a proper coloring of G. It is easy to see that any planar graph $G = (V, E)$ has $\chi(G) \le 6$: G has at most $3|V| - 6$ edges, i.e., $\sum_v \deg(v) \le 6|V| - 12$, and so there exists a vertex with degree at most 5. Inductively color $G \setminus \{v\}$ with 6 colors, and then assign v the color missing among the neighbors of v. This bound can be improved, and the following famous theorem gives a precise answer for the chromatic number of planar graphs:

Theorem 1.5 (Four color theorem)**.** *Let G be a planar graph. Then $\chi(G) \le 4$. Furthermore, there exist planar graphs G for which $\chi(G) = 4$*†.

Since vertices of the same color class are independent, an immediate corollary of this theorem is:

Corollary 1.6. *Any planar graph on n vertices has an independent set of size at least $n/4$.*

Let C be a convex polyhedron in $\mathbb{R}^3$ and let P be its set of vertices. Define the graph $G = (P, E)$ with the following set of edges: $\{p_i, p_j\} \in E$ iff the segment $p_i p_j$ lies on the boundary, denoted ∂C, of C. G is called the 1-*skeleton* of C. For a point $q \in \mathbb{R}^3$, the stereographic projection $\pi_q : C \to \mathbb{R}^2$ w.r.t. q projects ∂C onto $\mathbb{R}^2$, where $\pi_q(p)$ is the intersection of the line qp with the xy-plane. Note that the pre-image of any point $p' \in \mathbb{R}^2$ consists of at most two points in ∂C. In particular, if one picks q to be close enough to a facet of C, then the edges of E map to pairwise non-crossing set of segments in $\mathbb{R}^2$, called the *Schlegel diagram* of C). Thus we can conclude that:



Theorem 1.7. *The 1-skeleton of a convex polytope in $\mathbb{R}^3$ is a planar graph.*

Questions

(solutions)

1. Let L be a set of n lines in the plane, no two of which are parallel, and with no three passing through a common point. Let P be the set of $\binom{n}{2}$ intersection points of L. Let $G = (P, E)$ be the graph on P such that $(p_i, p_j) \in E$ iff p_i and p_j are consecutive intersection points along some line $l \in L$.

   (a) Using the existence of a low-degree vertices in G, prove that $\chi(G) \le 5$.

†For example, K_4 is a planar graph and requires 4 colors.

(b) Prove that $\chi(G) \leq 3$.

2. Let $G = (V, E_1 \cup E_2)$ be a graph, where (V, E_1) and (V, E_2) are planar graphs. Then show that $\chi(G) \leq 12$.

3. Prove that a planar graph is bipartite if and only if in its planar embedding, all faces have even length.

4. Let $G = (V, E)$ be a planar graph with k connected components. Let F be the set of faces in a fixed planar embedding of G. Then prove that $|V| - |E| + |F| = k + 1$.

5. Let P be a set of n points in $\mathbb{R}^3$ in convex position, and let G be the 1-skeleton of P. Then prove that for any halfspace H, the subgraph of G induced by the points in $H \cap P$ is connected.

6. Let C be a convex polyhedron with 12 vertices, and where each facet is a triangle. Show that if each facet of C is labelled with a nonnegative integer such that the sum of these integers over all faces is 39, then there must exist a vertex whose two adjacent faces have the same label.

7. Let L be a set of n lines in the plane. The arrangement of L induces a set F of faces. Let $G = (F, E)$ be the planar graph on F where $\{f_i, f_j\} \in E$ iff the two faces f_i, f_j are adjacent in the arrangement, i.e., their boundaries share a common line in L. Prove that $\chi(G) \leq 2$. Show the same is true for regions induced by circles instead of lines.