GEOMGRAPHS: Algorithms and Combinatorics of Geometric Graphs
MPRI 2024-2025

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These are the lecture notes for my half of the course GEOMGRAPHS. The other half is taught by Luca Castelli Aleardi and the slides and exercise sheets for his half are available on the course's webpage.

Some practicalities:

- The course is on Thursdays, from 8:45 to 11:45, in Sophie Germain room 1002.
- There is an exercise session in the middle of each lecture, after the half-time break.
- The class will be graded with a final written exam.
- There will be two optional exercise sheets, with points contributing extra credits for the final grade.

This aim of this course is to provide an overview of the combiantorial, geometric and algorithmic properties of embedded graphs: we start with ***planar graphs*** and then move on to ***surface-embedded graphs***. These are graphs that can be drawn without crossings in the plane or on more complicated surfaces, see Figure 1. Therefore they form a ***combinatorial*** object with a ***topological*** constraint. The objective of the course is to ***explore how topology interacts with combinatorics and algorithms*** on this very natural class of objects. Some questions we will explore are:

1. What are the combinatorial consequences of being planar? Are there combinatorial characterizations?
2. How to test algorithmically whether a graph is planar? If yes, how to draw the graph?
3. Can one exploit planarity to design better algorithms for planar graphs than for general graphs?
4. How do the previous answers generalize to graphs embedded in these more complicated surfaces?
5. How to solve certain topological questions algorithmically on surfaces?

While the focus of the course stays very theoretical (e.g., mostly with a theorem, lemma, proof structure), embedded graphs are of great interest for the practically-oriented mind, as they appear everywhere, for example in road networks (where underpasses and bridges can be modeled using additional topological features), chip design or the meshes that are ubiquitous in computer graphics or computer aided design. In all these applications, there is a strong need

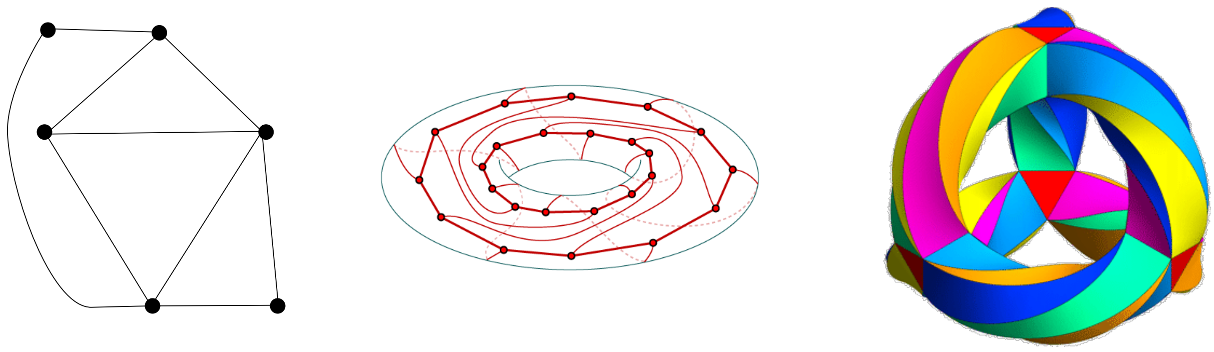

Figure 1: Graphs embedded in the plane, the torus, and the three-holed torus.

for a theoretical understanding of embedded graphs, as well as algorithmic primitives related to their topological features. Additionally, embedded graphs are an important lens to study graphs in general, since any graph can be embedded on some surface. This is especially the case in graph minor theory, where embedded graphs play an absolutely central role (but we barely touch on this topic).

The class is taught alternatively by Luca and myself: his class focuses on combinatorial aspects and graph-drawing questions, while my half covers algorithmic questions and topological aspects of surface-embedded graphs.

These lecture notes follow quite closely the material taught in my half of the class. This is actually their point, as in my experience lecture notes with too much content can easily get overwhelming (especially when one misses a class). Thus we refrain from (too many) digressions and heavy references. The tone aims to be conversational. I will do my best however to add each week the missing details for the proofs which may have been a bit handwavy during the lectures. The content has a strong overlap with the lecture notes of a previous iteration of the class (by Éric Colin de Verdière), and with those for a course on Computational Topology I co-taught with Francis Lazarus in 2016-2018, and we refer to, and strongly recommend those for the missing digressions and references.

1 Planar Graphs

1.1 A partial recap of the first lecture

A ***planar graph*** is a graph that can be embedded, i.e., drawn without crossings, in the plane, or equivalently the sphere. Throughout this course, it is quite important that we allow graphs to have multiple edges and loops, which is handled seamlessly by the previous definition. A graph without multiple edges nor loops is called a ***simple graph***. A ***plane graph*** is an embedding of a planar graph, and two plane graphs are considered equivalent if there is a homeomorphism of the plane sending one to the other one (or intuitively, if one can continuously deform one into the other without adding crossings). Note that a planar graph can correspond to multiple different plane graphs: think for example of a graph with a single vertex and multiple loops.

Compared to just a general graph, a plane graph has an additional combinatorial structure: there are now ***faces***, which are the connected components of the complement $\mathbb{R}^2 \setminus G$. This additional structure interacts with the initial graph in multiple ways:

- The ***Euler characteristic*** stipulates that for any connected plane graph, we have $v - e + f = 2$, where v, e and f are respectively the number of vertices, edges and faces.
- This implies that planar graphs are ***sparse***: for $v \geq 3$, $e \leq 3v - 6$ (and in general $e \leq 3v$. So planar graphs are very particular compared to general graphs.
- To any plane graph G we can associate a ***dual graph*** G^*, whose vertices are the faces of G and whose edges connect adjacent faces of G (with multiplicity and loops if needed). Note that the dual of a simple graph is in general not simple, and that the dual of graph depends on the embedding.
- The combinatorial data of the faces is actually all that is needed to encode a plane graph. There are various data structures one can use to do that. In these notes, we will simply consider that we have such a data structure that allows us to do "intuitive" operations in the natural time: for example moving from an edge incident to a vertex to the next edge in the circular order in constant time, or listing the k edges adjacent to a face in time $O(k)$, etc.

All of the properties of planar and plane graphs boil down to "intuitive" (but hard to prove) topological properties of the plane: the ***Jordan curve theorem*** shows that any simple (i.e., non self-intersecting) closed curve in the plane separates it into two components, one bounded and one unbounded. The ***Jordan Schoenflies theorem*** shows that the bounded component is homeomorphic to a disk. These theorems are hard to prove because simple closed curves in the plane can be very complicated, as pictured in Figure 2. If one restricts our attention, say, to polygonal curves, then the proofs become much easier. This leaves us with two alternatives for the class: either we define all our graphs to have edges as polygonal segments, and then all the intuitive facts are easy to prove (there is such a proof for the Jordan curve theorem in my older lecture notes), or we take the general definition and then trust these hard theorems that everything works. I leave you to choose your preferred option.

In particular, all the bounded faces of a connected plane graph are (homeomorphic to) a disk. We say that such a graph is ***cellularly embedded*** (this definition is quite useless for plane graphs but will become useful on other surfaces). We will see later on that the failure of such nice topological properties for more complicated surfaces leads to interesting topological questions.

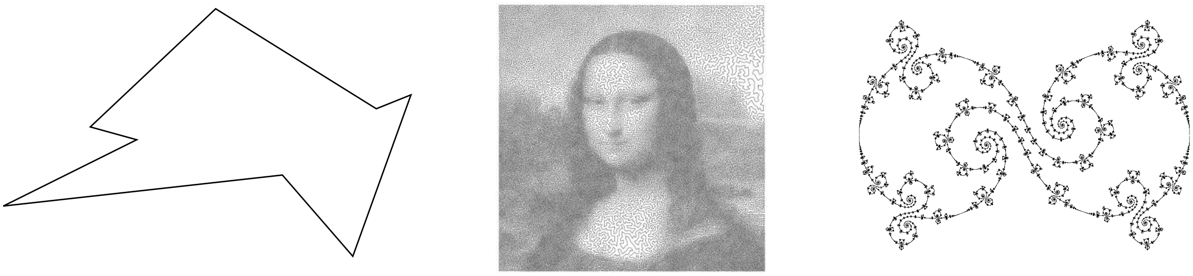

Figure 2: Three simple closed curves in the plane

We also recall the famous theorem of Kuratowski, which will not be proved in this class (nor in Luca's) (you can find a proof in my older lecture notes).

Theorem 1.1 (Kuratowski, 1929)**.** *A graph is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$ *as a subgraph.*

Finally, the Fàry theorem shows that every plane graph can be realized with edges drawn as straight lines. This is proved in Luca's half as a corollary of the Tutte embedding theorem (there are also easier direct proofs).

1.2 Coloring

The sparsity has the following nice implication. A k-(vertex) ***coloring*** of a simple graph is an assignment of k colors to the vertices so that no two adjacent vertices share a common color.

Proposition 1.2. *Simple planar graphs are* 6*-colorable.*

Proof. We prove the result by induction on the number of vertices. For low values, this is immediate. For the induction step, pick a vertex z of degree less than 6 which exists because of the sparsity, and color inductively $G \setminus z$. Then the five neighbors have at most 5 different colors, and we can color z with one of the remaining colors. □

The following improvement is on the exercise sheet, and relies on more than just the Euler characteristic and the sparsity.

Exercise 1.3. Prove that simple planar graphs are 5-colorable. Hint: look at paths connecting non-adjacent neighbors of a degree 5 vertex.

As is well-known, planar graphs are actually 4-colorable, and we will not prove this in the course.

1.3 The crossing lemma

A ***drawing*** of a graph if just a continuous map $f : G \to \mathbb{R}^2$, that is, a drawing of the graph on the plane where crossings *are* allowed. We will only consider drawings ***in general position***: that means that vertices must be mapped to distinct points, edges do not intersect vertices except at their endpoints, edges only intersect transversely and at most two edges intersect at each point. Note that such a drawing in general position induces a plane graph, where every crossing has been replaced by a vertex of degree 4. As we said earlier, via Fàry's theorem, such a plane graph can be assumed to have its edges realized as straight lines. So without loss of generality, we can and we will assume that this is the case in all our drawings: this implies that all the edges are drawn as polygonal segments (which may bend at crossings).

The ***crossing number*** $cr(G)$ of a graph is the minimal number of crossings over all the possible drawings in general position of G. For instance, $cr(G) = 0$ if and only if G is planar. The ***crossing number inequality*** provides the following lower bound on the crossing number.

Theorem 1.4. $cr(G) \geq \frac{|E|^3}{64|V|^2}$ *if* $|E| \geq 4|V|$.

The proof is a surprising application of (basic) probabilistic tools. There is a nice and much more indepth discussion of this proof available on Terry Tao's blog.

Proof. Starting with a drawing of G with the minimal number of crossings, define a new graph G' obtained by removing one edge for each crossing. This graph is planar since we removed all the crossings, and it has at least $|E| - cr(G)$ edges, so we obtain that $|E| - cr(G) \leq 3|V|$ (Note that we removed the -6 to obtain an inequality valid for any number of vertices). This gives in turn

$$cr(G) \geq |E| - 3|V|.$$

This can be amplified in the following way. Starting from G, define another graph by removing vertices (and the edges adjacent to them) at random with some probability $1 - p < 1$, and denote by G'' the resulting graph. Taking the previous inequality with expectations, we obtain $\mathbb{E}(cr(G'')) \geq \mathbb{E}(|E''|) - 3\mathbb{E}(|V''|)$. Since vertices are removed with probability $1 - p$, we have $\mathbb{E}(|V''|) = p|V|$. An edge survives if and only if both its endpoints survives, so $\mathbb{E}(|E''|) = p^2|E|$. Finally a crossing survives if an only if the four adjacent vertices survive[1]. The resulting drawing might not be crossing-minimal but it does imply that $\mathbb{E}(cr(G'') \leq p^4 cr(G)$, which is the inequality we will need. So we obtain

$$cr(G) \geq p^{-2}|E| - 3p^{-3}|V|,$$

and taking $p = 4|V|/|E|$ (which is less than 1 if $|E| \geq 4|V|$) gives the result. ☐

In particular, applying this inequality to dense graphs, and in particular to complete graphs K_n shows that any drawing of the complete graph K_n has $\Omega(n^4)$ crossings. Finding the correct constants is notoriously difficult though, even for that very specific-looking case of complete graphs: the Hill conjecture posits that

$$cr(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor \sim \frac{1}{64}n^4,$$

but the best known lower bound is quite far off[2], at about $cr(K_n) \geq 0.8594\frac{1}{64}n^4$ for sufficiently large n. Likewise, the correct constant is unknown for complete bipartite graphs, and in general there are tons of simple-looking open problems on crossing numbers (see this exhaustive survey by Marcus Schaefer).

[1]There may be less than four adjacent vertices in general, but not in the drawing minimizing the crossing number, since drawings where there is a crossing forming an α shape can be simplified by removing one crossing. So in a crossing-minimal drawing, they do not happen

[2]Note that the fact that 1/64 also appears in Theorem 1.4 is a red herring: the number of edges in a complete graph is $\binom{n}{2}$, so a blind application of the crossing lemma only gives $cr(K_n) \geq \frac{1}{64 \cdot 2^3}n^4$.

1.4 The Hanani-Tutte theorem

Another interesting result of planar drawings is the following surprising (to me at least) theorem. Two edges of a graph are ***independent*** if they are not incident to a common vertex.

Theorem 1.5 (Strong Hanani-Tutte theorem, 1934)**.** *Any drawing in general position of a non-planar graph contains two independent edges that cross an odd number of times.*

Conversely, if one can draw a graph so that all independent edges cross an even number of times, then the graph is planar (!)

Proof. We first claim that it suffices to prove the theorem for K_5 and $K_{3,3}$. Indeed, let G be a non-planar graph. By Kuratowski's theorem, it contains a subdivision of K_5 or $K_{3,3}$. If the theorem is proved for K_5 and $K_{3,3}$, then we can find a pair of disjoint paths in our graph G that cross an odd number of times. This means that there exists a pair of independent edges that cross an odd number of times. Indeed, otherwise, summing all the even number of crossings among all pairs of edges in the pair of paths would give an even number of crossings for the pair of paths.

In order to prove the theorem for K_5 and $K_{3,3}$, we prove the stronger result that in any drawing of these two graphs, the sum of the number of crossings over all independent pairs of edges is odd. The reason is that this quantity mod 2 does not actually depend on the drawing:

Claim 1.6. *For $G = K_5$ or $K_{3,3}$ and any two drawings G_1 and G_2 in general position of the graph G, the quantity*

$$\sum_{\substack{e,e' \\ \textit{independent edges}}} cr(e,e') \mod 2$$

is the same.

Proof. We first describe a general way to deform any G_1 into any G_2, and then prove that the target quantity is invariant throughout this deformation.

As we said earlier, we can assume that in the drawings G_1 and G_2, the edges are polygonal segments. We first move the vertices of G_1 one by one so that they coincide with the respectives vertices of G_2. This can be done by choosing, for every vertex v in G, a polyogonal path p_v between its position v_1 in G_1 and its position v_2 in G_2. This path can clearly be chosen so that it avoids vertices of G_1 and G_2 and so that it intersects their edges transversely and outside of existing crossings. Then we move v_1 along p, dragging all the incident edges along the way, as pictured in Figure 3, top.

In a second step, we inductively straigthen the edges in G_1 and in G_2: we focus on G_1, the case of G_2 being identical. Each edge of G_1 is a polygonal path $e = (q_1, \ldots, q_k)$, where the points q_i and q_{i+1} are connected with straight segments. We can straighten such a polygonal path by replacing the triangle q_1, q_2, q_3 by the straight segment q_1q_3: this is done by moving one tip of the triangle to the opposite edge. Here again, one can do so along a path that avoids all the vertices of G_1 and G_2 and intersects their edges transversely and outside of existing crossings, see Figure 3, middle.. We then go on inductively so that every edge in G_1 is a straight line between its endpoints, and likewise in G_2. Since the vertices coincide, the drawings are now identical.

Let us now analyze how the crossings evolve as we do these deformations. By our choice of deformation paths, such evolutions only happen at discrete moments: in the first step this will be when a vertex passes through an edge (Figure 3, (a)), and in the second step this will be when a point q_i crosses an edge (Figure 3, (b)), when one of the two incident straight lines

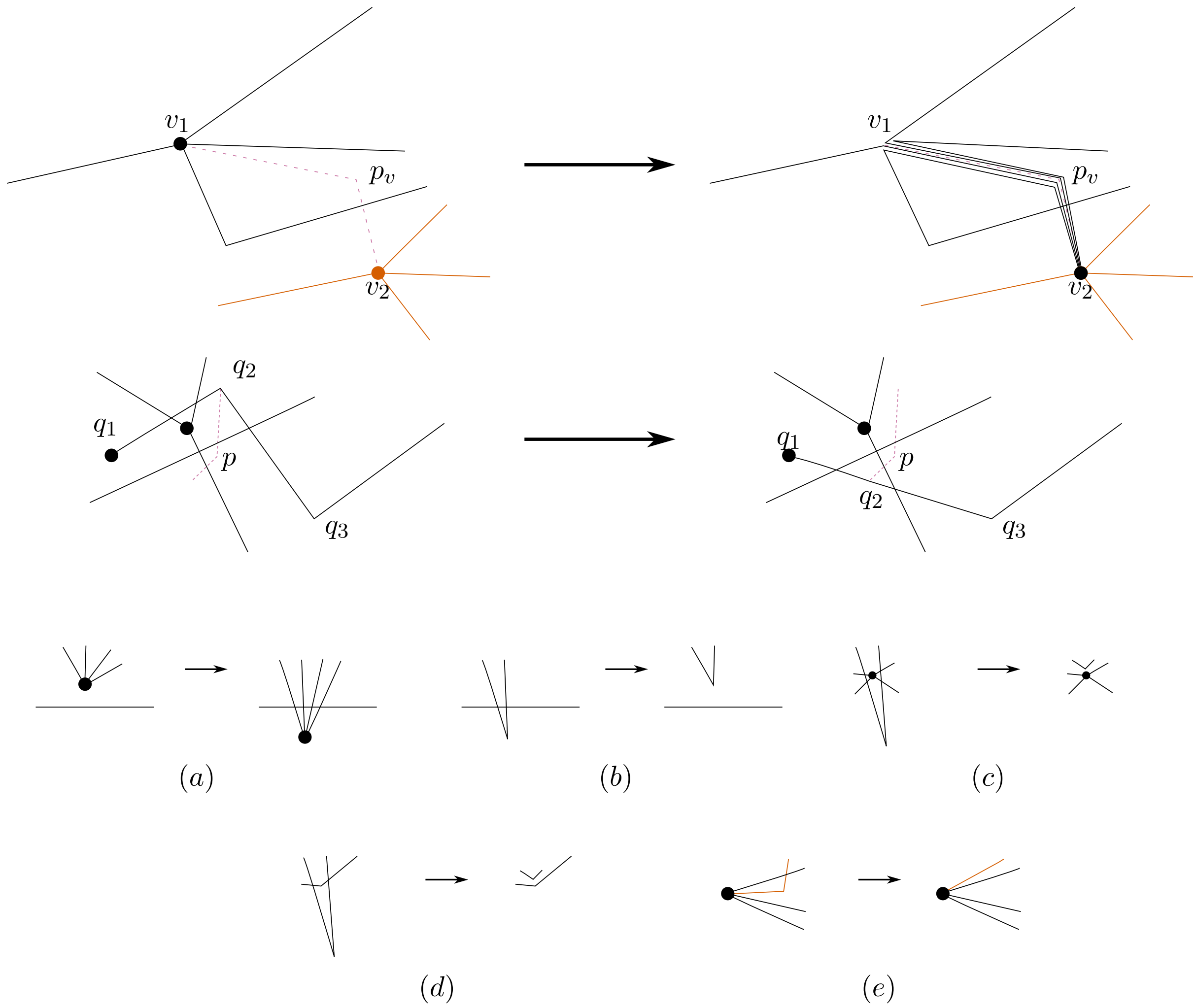

Figure 3: Deforming drawings.

passes through a vertex or a crossing (Figure 3, (c) and (d)), or when two edges switch their order around a vertex (Figure 3, (e)). Note that outside of these discrete events, the crossings do not change: actually the homeomorphism class of the drawing (viewed as a plane graph) does not change. Also note that the events of type (b), (d) and (e) do change the drawing but do not change the parity of crossings of independent edges.

There remains events (a) and (c) which both feature, in slightly different ways, a vertex v passing through an edge e. If v is incident to e , the changes in crossings only involve non-independent edges and are thus irrelevant to our count. If $G = K_{3,3}$, the degree of every vertex is 3, but for any vertex v not incident to e, exactly two of the edges incident to v are independent with e (look at Figure 4). So the count of independent crossings changes by 2, an even amount. If $G = K_5$, every vertex v in K_5 has degree exactly four, and for any fixed e not incident to v, exactly two of the edges incident to v are independent with e (look again at Figure 4). So here again the count of independent crossings changes by 2, an even amount. ⌟

We conclude the proof by exhbiting drawings of K_5 and $K_{3,3}$ where this sum is odd, as done in Figure 4 (note that the proof implies that *any* drawing will work). □

This theorem and its proof provide a polynomial-time algorithm to test whether a graph is planar. It is not as efficient as other more standard approaches, but is simple conceptually and, with a lot of work, can be generalized to higher dimensions. Let G be a graph of which

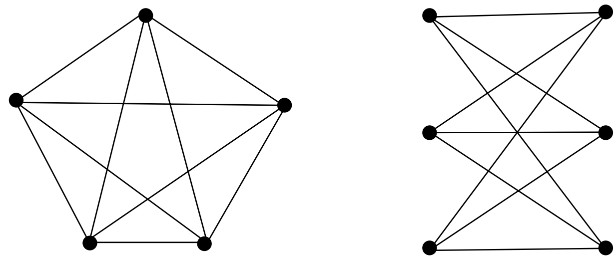

Figure 4: The graphs K_5 and $K_{3,3}$.

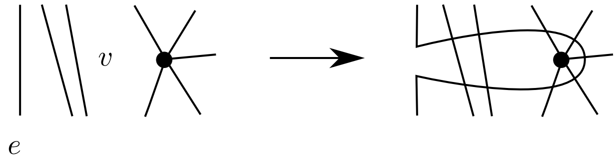

Figure 5: A finger move for a pair $p = (e, v)$.

we want to test planarity. Let us consider the vector space S over the field with 2 elements $\mathbb{F}_2$ whose basis consists of pairs of independent edges (e, f) in G.

1. Start with *any* drawing of the graph G. Write down the vector $x(D) \in S$ of the number of crossings mod 2 of independent edges.
2. For every pair $p = (v, e)$ in the graph G, define a vector $u(p) = \sum_{f \text{ incident to } v} (f, e) \in S$: it contains a 1 for each pair of edges (f, e) such that f is incident to e, and 0 otherwise.
3. Denote by U the subspace spanned by the family $u(p)$. Test whether $x(D)$ belongs to U. If yes, output that the graph is planar, otherwise, output that it is not planar.

Lemma 1.7. *This algorithm is correct and has polynomial complexity.*

Proof. The reason why this algorithm is correct follows from (the proof of) Theorem 1.5. Indeed, starting from any drawing D of G, one can change its vector $x(D)$ by exactly $u(p)$ by applying a ***finger move*** (see Figure 5) from the edge e around the vertex p. If $x(D)$ belongs to U, there exists a sequence of finger moves that brings the vector of crossings $x(D)$ to zero mod 2. Then by Theorem 1.5, the graph is planar. Conversely, if the graph is planar, starting from any drawing we can deform it similarly to the proof of Theorem 1.5 until there are no crossings remaining, and this morphing gives a sequence of finger moves, and thus a family of vectors $u(p)$ showing that $x(D)$ belongs to U.

The third step can be solved by Gaussian elimination in cubic time (or there are faster algorithms, even more so since the underlying field is $\mathbb{F}_2$). Since S has size $O(|E|^2) = O(|V|^2)$ (if the graph is not sparse, we can reject it straight away), this yields a complexity $O(|V|^6)$. $\square$

For the algebraic-minded reader, this algorithm actually amounts to computing an obstruction class in the equivariant cohomology of the deleted product (ask me in class if you are interested about what these words mean).