Exercises on embedded graphs, Lecture 6

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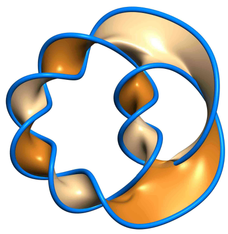
Exercise 1:

Identify the surfaces described by these polygonal schemes:

1. $abc\bar{b}c\bar{a}$,
2. $a_1a_2\dots a_n\bar{a_1}\bar{a_2}\dots\bar{a_n}$,

and the surfaces in these pictures (the third one is a sculpture by Bathseba Grossman, see her website and shop here!):



Solution: The important thing here is to *not* apply the cut-and-pasting tools that we saw in the proof of the classification of surfaces, because they are bulky, prone to error and slow to apply. Instead one can just trust the theorem, which implies that in order to recognize a surface it suffices to compute its Euler characteristic and its orientability. The first surface is non-orientable since c appears twice positively. It has one face and three edges, and tracking edge identifications, we obtain that it has two vertices. Since $v-e+f=2-g$ for non-orientable surfaces, we directly obtain that this has genus 2: it is a Klein bottle. The second surface is orientable, has one face and n edges. Following edge identifications shows that it has one vertex if n is even and two vertices if n is odd. It follows, using the formula $v-e+f=2-2g$ (do note that the formula differs by a factor two depending on whether the surface is orientable or not), that this is an orientable surface of genus $\lfloor n/2 \rfloor$.

We did not identify the surfaces in the pictures in class. I have mostly given them so that you realize that in the presence of boundaries, even when the surface is orientable (the first one clearly is as you can tell from the two coloring shades), it is not easy to "just count" the handles. Instead, one should draw a nice cellularly embedded graph on them and compute the Euler characteristic.

Exercise 2:

1. Let G be a graph embedded on an orientable surface of genus g, not necessarily cellularly. Prove that $v-e+f \geq 2-2g$, where v, e and f denote respectively the number of vertices, edges and faces of the graph embedding.

2. Let G be a simple graph cellularly embedded on an orientable surface of genus g, with the properties that (1) all the faces have degree three (i.e., are incident to three edges), and (2) each cycle of length 3 in the graph bounds a face. The set of such (triangular) faces is denoted by T. Use the previous question to show that in any embedding of G on an orientable surface of genus g, the number of faces is $|T|$. Deduce that the embedding of G on an orientable surface of genus g is unique up to homeomorphism.

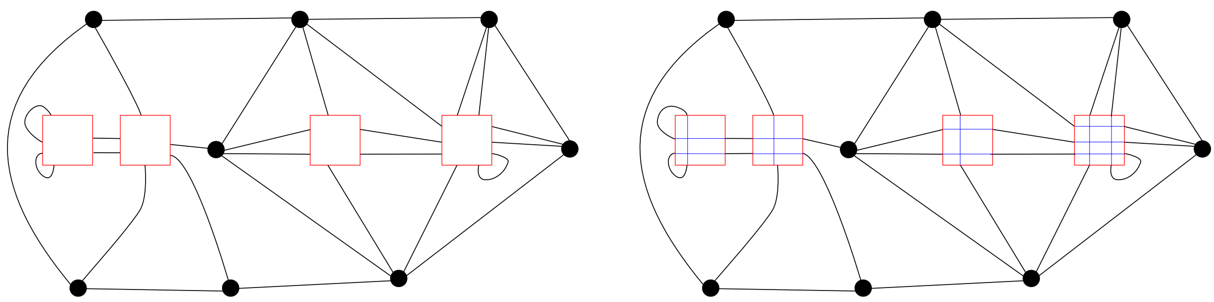
Solution sketch:

- You can take the problem from either of the two ends. If a face is not cellular, one can always add edges to make it cellular, without increasing the number of faces nor vertices. Then one can apply the valid Euler characteristic. Alternatively, if a face is not cellular, one can make it cellular by cutting along non-contractible curves within the face (wait for next week to see a proper definition of contractibility in the lecture notes). This either increases the number of faces (think of cutting a cylinder) or decreases the genus. In both cases this yields the desired inequality.
- The Euler characteristic from the cellularly embedded graph G gives that $V - E + T = 2 - 2g$, and the standard double counting argument on triangles gives that $3T = 2E$. Now, any other embedding of G on the same surface would have the same number of vertices and edges but perhaps a different number of faces F. By the previous question, we have $V - E + F \geq 2 - 2g$ and thus $F \geq T$. But in a simple graph, faces always have degree at least 3 so $\sum_F deg(F) \geq 3F$ and thus by double counting $3F \leq 2E = 3T$, so $F = T$. Therefore we have equality in the inequality $\sum_F deg(F) \geq 3F$ and thus all the faces actually have degree 3, and thus the facial walks are 3-cycles. Since all the 3-cycles of G are triangular faces in G and $T = F$, this means that the set of faces in the second embedding is the same as in the first embedding. Therefore the embedding is unique.

Be wary about what this exercise does *not* prove: it is not true that if a graph triangulates a surface, then it admits a unique embedding on that surface. Complete graphs in general can triangulate surfaces in exponentially many ways. For the argument to work it is crucial to also have the hypothesis that all the 3-cycles are facial.

Exercise 3:

We consider the following way of representing non-planar graphs with boxes. There are k disjoint squares called **boxes** drawn in the plane, and each side acts as a teleporter to the same point on the opposite side. A graph is embedded in the plane with k boxes if it is drawn without crossings in the plane when the edges are allowed to use these teleporters: when an edge intersects a point on the box, it continues on the same point on the opposite side. Note that each edge is allowed to use the same box any number of times. For example, here is a picture of a graph embedded in the plane with four boxes (left picture). Equivalently, a box is a way to hide a grid of crossings (see the right picture).



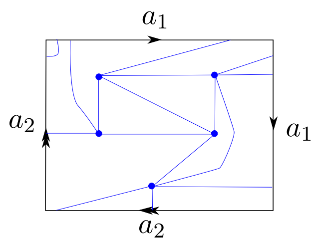
1. Provide an embedding of $K_{3,3}$ in the plane with a single box.

2. Prove that a graph can be embedded in the plane with g boxes if and only if it can be embedded on a surface of genus g.

3. Let G be a graph embedded on a surface of genus g. By the previous question, G can be embedded in the plane with g boxes. Find a function $f(g)$ so that the following strengthening holds (and prove it): G can be embedded in the plane with g boxes so that each edge of G crosses at most $f(g)$ boxes (counted with multiplicity). Any function (even non-polynomial) will do, but the smaller ones are better!

Exercise 4:

Recall that a cellular embedding is an embedding where all the faces are disks, and that a non-orientable surface of genus g is a surface with polygonal scheme $a_1a_1a_2a_2\dots a_ga_g$. A convenient way to represent a graph on a non-orientable surface is to draw it on top of this polygonal scheme. For example, here is a cellular embedding of K_5 on a non-orientable surface of genus two.



1. Provide an explicit cellular embedding of K_4 on a non-orientable surface of genus 3.

2. Let G be a simple graph with v vertices, e edges cellularly embedded on a non-orientable surface of genus g. Prove that $g \leq e - v + 1$.

3. Let G be a simple graph with v vertices and e edges, and let g_1 be the smallest genus of a non-orientable surface on which G embeds. Prove that for any g such that $g_1 \leq g \leq e - v + 1$, G can be cellularly embedded on a non-orientable surface of genus g.

4. In particular, G can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.