Exercises on embedded graphs, Lecture 2

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Words in ***blue*** are words that I had forgotten in the printed version (sorry!).

Exercise 1: Prove that simple planar graphs are 5-colorable. *Hint: look at paths connecting non-adjacent neighbors of a degree 5 vertex.*

Exercise 2: An independent set in a graph G is a set of vertices no two of which are adjacent. Show that there exist constants $\alpha > 0$ and $\beta > 0$ so that any ***simple*** planar graph has an independent set of size at least αn in which every vertex has degree at most β. Provide a linear-time algorithm to find such an independent set.

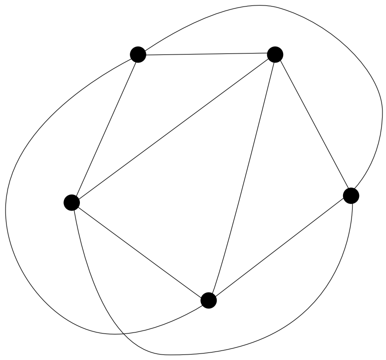
Solution: We first prove the existence of the independent set and will provide a linear-time algorithm in a second step.

Since the exercise does not ask us to optimize the constants, I will not do so. Let us consider the set V_{1000} of vertices in G of degree at least 1000. If more than one third of the vertices of G belong to V_{1000}, then there are at least $1/2 \cdot |V|/3 \cdot 1000$ edges, which contradicts the sparsity inequality $|E| \leq 3|V|$. So if we remove all the vertices in V_{1000} and all their adjacent edges, at least two thirds of the vertices remain. Now we can pick an arbitrary vertex, put it in our target independent set, remove it from the graph as well as all the adjacent vertices and recurse until the graph is empty. It is clear that we obtain an independent set since no two adjacent vertices can be picked in this process. At each recursion step we remove at most 1000 vertices, so there are at least $2|V|/3 \cdot 1/1000$ recursion steps, and therefore the independent set we obtain has size at least $2|V|/3000$, and every vertex in it has degree at most 1000.

Now, in order to do this algorithmically, we can go through the graph a first time and initialize an array listing all the vertices of degree less than 1000, marking the other ones and their adjacent edges as deleted. Then we take the first element of the array, put it in our target independent set and mark it and all the adjacent vertices as deleted. Then we consider sequentially the next elements in the array and do the same for the first that has not been marked for deletion yet, and so on. Overall, the first pass through the graph takes linear time, and the array has linear size and is only traversed once. Furthermore, each update operation (marking vertices as deleted) only taking $O(1)$ time since it is local: it only involves a vertex and its $O(1)$ neighbors. So the overall algorithm takes linear time.

Exercise 3: Let G be a plane connected graph, where every face is given with a real weight. Let T be a spanning tree of G (a subgraph of G with the same vertex set as G that is a tree). Give a linear-time algorithm that, for all edges e of G not lying in T, computes the sum of the weights of all the faces inside the cycle formed by e and the unique path in T joining its endpoints.

Exercise 4: A **1-planar graph** is a graph that can be drawn in the plane so that each edge is involved in at most one crossing, where it crosses a single, distinct, edge. The corresponding drawing is called a **1-plane drawing**. Here is an example of a 1-plane drawing of K_5, the complete graph on 5 vertices.



1. Prove that K_6, the complete graph on 6 vertices is 1-planar.

2. Let G be a 1-planar graph. Show that there exists a 1-plane drawing of G where no two adjacent edges (two edges incident to the same vertex) cross. *Hint: Given a drawing where there is a crossing with adjacent edges, provide another drawing where that crossing has been removed.*

In the subsequent questions, all the 1-plane drawings are assumed to have this property.

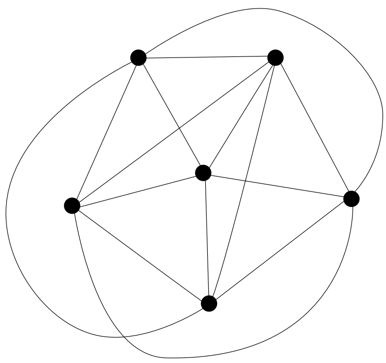
3. Let $G = (V, E)$ be a 1-planar graph with a 1-plane drawing that is *maximal*, i.e., no edges can be added to the drawing while preserving the property that it is a 1-plane drawing. Define a plane graph $G' = (V', E')$ by replacing each crossing with a vertex of degree 4. Prove that G' has no faces of degree 1 and 2, and that it is a triangulation (all the faces are adjacent to exactly three edges).

4. ***Still assuming maximality***, denoting by $|C|$ the number of crossings in the 1-plane drawing of G, and by $|F|'$ the number of faces in G', show that $4|C| \leq |F|'$ and that $4|C| \leq |E|'$.

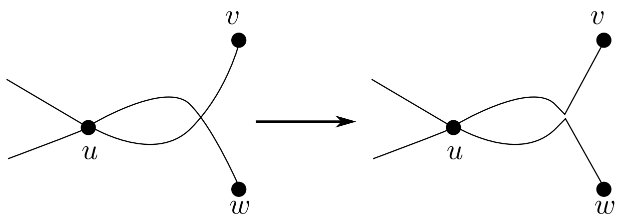
5. Prove that for any 1-planar graph $G = (V, E)$ (not necessarily maximal), $|E| \leq 4|V| - 8$. Deduce that K_7 is not 1-planar.

Solution:

1.



2. Let G be a 1-planar graph with a 1-plane drawing where two adjacent edges, say uv and uw cross. By the definition of 1-plane drawing, these two edges do not cross any other edges. Thus we can redraw the drawing as pictured just below: this new drawing still has edges connecting u to v and u to w, and the crossing has been removed. Iterating over all the adjacent crossings, all of those get removed.



3. The graph G' is obtained by subdividing some edges of G. Since G is simple, G' has no loops. Furthermore, multiple edges in G' would correspond multiple edges of G (which do not happen) or adjacent crossings in G, which we can assume do not happen by the previous point. So G' is simple. Now we consider a face of the graph G', and suppose for the sake of contradiction that it is not a triangle, i.e., that it is incident to at least four vertices. We claim that it is impossible for two consecutive vertices of that face to be new vertices of G', or equivalently, crossings of G. Indeed, if that were the case, the edge between them would see two crossings, which contradicts the hypothesis of a 1-plane drawing.

   So our hypothetical face F with at least four vertices has at least two vertices of G which are not consecutive. Adding an edge between these two vertices does not break the property of being 1-planar, and therefore we have a contradiction* with the maximality hypothesis. We conclude that all the faces of G' are triangles.

4. By the previous question, every face of G' is a triangle, and every such triangle is incident to at most one crossing. So there are at most four faces per crossing yielding the first inequality. For the second inequality, it is clear that each crossing is associated in a unique way to four edges from E', but these are only two edges from E. However, looking at the boundary of the diamond formed by the four triangles around a crossing, there are four additional edges of G which we can attempt to charge to the crossing. Since each of these edges is adjacent to at most two such diamonds, this results in $4 \cdot \frac{1}{2} = 2$ additional edges that can be charged to each crossing, and thus we get the second inequality.

5. The graph G can be assumed to be maximal: otherwise just add edges which only makes our inequality stronger. Thus we can assume that G' is triangulated. Applying the sparsity inequality to G', we get that

$$|E'| \leq 3(|V| + |C|) - 6,$$

   and when going from G to G', we have subdivided exactly $|C|$ edges twice, so we have $|E'| = |E| + 2|C|$. Therefore

$$|E| \leq 3|V| + |C| - 6,$$

   and plugging in $4|C| \leq |E|$ gives the desired result. The conclusion for K_7 is immediate since it satisfies $|V| = 7$ and $|E| = 21$ and $20 < 21$.

*: There is an annoying subtlety here which I had missed when I wrote this exercise (thanks for asking me to write a correction!). It could be that there was already an edge connecting these two vertices, passing outside the face F. Then the simplicity constraint forbids us from adding another edge. One way to handle this issue is to still add this edge, which makes G and G' non-simple. It is still fine for the rest, in particular for the question 5: even though G' is not simple, it has no faces of degree 1 or 2 which is all that is needed to prove that it satisfies $|E'| \leq 3|V'| - 6$.