MPRI Course 2-38-1: Algorithms and Combinatorics for Geometric Graphs
Exercise Sheet 1

This exercise sheet is to be returned on October 22nd, either via email to `arnaud.demesmay@u-pem.fr`, or manually at the end of Vincent Pilaud's lecture. The language can be either English or French. This sheet is optional, and will provide bonus points to the final exam. The harder questions are denoted by (*). Don't hesitate to ask me questions.

1 Exercise 1

1. A simple bipartite cellularly embedded planar graph is bibipartite if its dual graph is simple and also bipartite. Give a complete list of all bibipartite planar graphs and prove that it is complete. *Hint: it is non-empty!*.

2. Let G be a simple planar graph, and suppose we arbitrarily color each edge of G either blue or red. Prove that for any embedding of G in the plane, there exists a vertex around which the incident red edges are consecutive.

3. Find universal constants α, β and γ (not depending on n or g) such that the following holds: For all integers n and g such that $n \geq \gamma g$, every simple n-vertex graph embedded on a surface of genus g has an independent set[1] of size n/α, in which every vertex has degree at most β.

4. Describe an algorithm to find such an independent set in $O(n)$ time.

2 Exercise 2

A cycle C on a graph G is *nonseparating* if $G \setminus C$ is connected.

1. Prove that any n-vertex triangulation of an orientable surface S of positive genus contains a non-separating cycle C of length at most $2\sqrt{n}$. *Hint: cut S along C, yielding two copies C_1 and C_2 of C on the boundary. How many independent paths are there from C_1 to C_2 and how long are they?*

2. Deduce that any n-vertex graph on an orientable surface of genus g has a 2/3-separator S of size $O(g\sqrt{n})$, and such that each component of $G \setminus S$ is planar.

[1] An independent set in a graph G is a subset of the vertices of G, no two of which are connected by an edge in G.

3 Exercise 3

Let G be a planar graph, and let G_1 and G_2 be two isomorphic[2] straight-line embeddings of G, where each face, including the outer face, is a triangle. A *morphing step* between G_1 and G_2 is a straight-line continuous transformation of one into the other, such that the graph stays planar at all times: for each vertex v of G, we denote by $S(v)$ the segment connecting v_1, the embedding of v in G_1 to v_2, the embedding of v in G_2, and we slide v from v_1 to v_2 at uniform speed along this segment. At a time $t \in [1, 2]$, we denote by v_t the position of v, and for any edge (uv) in E, we connect v_t to u_t with a straight segment. This defines a family of drawings $(G_t)_{t\in[1,2]}$, and this is a morphing step if all these drawings are planar embeddings. A *morphing* from G_1 to G_2 is a sequence of morphing steps $G_1 \to G' \to G^{(2)} \ldots \to G^{(k)} = G_2$, where the graphs $G^{(i)}$ are all straight-line embeddings of G. The integer k is the *complexity* of the morphing.

1. Provide an example of a planar graph G and two straight-line embeddings (not necessarily triangulated) that are not connected by a single morphing step.

The rest of the exercise aims at proving that for any two straight-line embeddings G_1 and G_2 with the above conditions, there always exists a morphing of finite complexity between G_1 and G_2. The proof is by induction.

2. Prove the base case of induction for $n = 4$.

The *visibility kernel* of a polygon is the set of points inside or on the polygon that can be "seen" from any vertex of the polygon, i.e., the set of points p such that for any vertex v of the polygon the segment pv does not cross the polygon.

3. Prove that for any polygon with at most 5 vertices, one of the vertices is contained in its visibility kernel.

The *link* $L(v)$ of a vertex v of G_1 or G_2 is the polygon defined by the neighbors of v.

4. Prove that there exists a vertex v of G so that both in G_1 and in G_2, the link $L(v)$ contains a vertex u that is in the visibility kernel of $L(v)$. (Note that u might be different in G_1 and G_2.

We first assume that there are no edges in G connecting non-adjacent vertices of $L(v)$.

5. (*) Prove that there exists a straight-line embedding G' of $G \setminus v$ so that $L(v)$ is convex.
6. What is the visibility kernel of $L(v)$ in G'? Assuming the induction hypothesis (every two straight-line triangulations with $n - 1$ vertices can be morphed one into the other), prove that one can morph G_1 into G_2. *Hint: contract an edge, and use the induction to morph into G'.*

We now remove the additional assumption.

7. (*) Prove the induction step in the general case. *Hint: without the assumption, there is no hope of finding a straight-line embedding where $L(v)$ is convex, but we can still find an embedding G' where all the vertices of $L(v)$ except the non-adjacent ones which are joined by an edge of G are in the visibility kernel of $L(v)$.*

[2] This means here that G_1 and G_2 have the same outer face and the same combinatorics *as an embedded graph*: same set of facial walks when turning clockwise.