1 Exercise 1

1. A simple bipartite cellularly embedded planar graph is bibipartite if its dual graph is simple and also bipartite. Give a complete list of all bibipartite planar graphs and prove that it is complete. *Hint: it is non-empty!*

**Solution:** In a bipartite graph, faces have even degree. So in a bibipartite graph, vertices have even degree. Any vertex of degree two yields a multiple dual edge between the two adjacent faces, contradicting bibipartiteness. If there is a vertex of degree 0, it is unique because the graph is cellularly embedded. Otherwise, all the vertices have degree at least four. Dually, all the faces have degree at least four. As we have seen in class, this is not possible due to the Euler formula.

2. Let $G$ be a simple planar graph, and suppose we arbitrarily color each edge of $G$ either blue or red. Prove that for any embedding of $G$ in the plane, there exists a vertex around which the incident red edges are consecutive.

**Solution:** Let us assume otherwise, then for each red edge incident to a vertex $v$, there exists another red edge incident to $v$ so that there are blue edges on both sides of the path formed by these two edges. So we can find a red cycle $C$ with the property that for every vertex on $C$ except maybe one, there is at least one blue edge incident to that vertex that is inside the disk $D$ bounded by $C$ (which exists by the Jordan-Schoenflies theorem). The vertex where this might not happen is the one where we close the cycle, where we have no control, we call it the *closing vertex*. We call such a cycle a *balanced* red cycle. Let us pick such a balanced red cycle $C$ so that the disk $D$ it bounds is minimal with respect to containment: there is no other balanced red cycle within $D$.

We claim that this implies that there is no red edge in the interior of $D$, as otherwise we could follow it and either find a smaller disk if it closes into a disk, or find a red path within the red disk. If this red path closes itself, we have a smaller balanced cycle. Otherwise, one of the two endpoints of this red path is incident to a vertex that is not a closing vertex, and then it bounds a balanced cycle with one of the two red paths in $C$, as the four possible cases in Figure 1 picture, contradicting minimality. So all the edges within $D$ are blue. If there is a vertex within $D$, all its incident edges are blue, a contradiction. Otherwise, we get a polygon with some edges inside, but any such polygon has at least two vertices of degree two. This contradicts the fact that in a balanced cycle, all the vertices except maybe one have an incident blue edge within the disk $D$, and finishes the proof.

3. Find universal constants $\alpha, \beta$ and $\gamma$ (not depending on $n$ or $g$) such that the following holds:

For all integers $n$ and $g$ such that $n \geq \gamma g$, every simple $n$-vertex graph embedded on a surface of genus $g$ has an independent set$^1$ of size $n/\alpha$, in which every vertex has degree at most $\beta$.

$^1$An independent set in a graph $G$ is a subset of the vertices of $G$, no two of which are connected by an edge in $G$. 
Solution: Let us pick $\beta = 24$, $\gamma = 2$ and $\alpha = 49$, which have not been optimized at all. Let us denote by $n$, $e$ and $f$ the number of vertices, edges and faces of a graph embedded on a surface of genus $g$. Without loss of generality, by reducing $g$ if necessary, we can assume that the graph is cellurally embedded. Then the Euler formula gives $v - e + f = 2 - 2g$. We claim that for $n \geq \gamma g$, there are at most $n/2$ vertices of degree higher than $\beta$. Otherwise we would have $\beta n/2 \leq 2e$, and since the graph is simple, faces have degree at least 3, so $3f \leq 2e$. So we get $2 - 2g \leq n - e/3 \leq n(1 - \beta/12)$ which is a contradiction for our choice of $\beta$ and $\gamma$.

So if we remove all the vertices of degree higher than $\beta$, we still have at least $n/2$ vertices. Now we can pick any of those, remove it and all its neighbors, and do it again. This will give us an independent set of size $\frac{n}{2(\beta+1)}$.

4. Describe an algorithm to find such an independent set in $O(n)$ time.

Solution: We first search through the graph in linear time to remove all vertices of degree bigger than $\beta$. Then we pick any vertex, and remove it and its neighbors, and we induct. Each of these removal steps takes constant time (because all the vertices have constant degree), so the whole procedure takes linear time.

2 Exercise 2

A cycle $C$ on a graph $G$ is nonseparating if $G \setminus C$ is connected.

1. Prove that any $n$-vertex triangulation of an orientable surface $S$ of positive genus contains a non-separating cycle $C$ of length at most $2\sqrt{n}$. Hint: cut $S$ along $C$, yielding two copies $C_1$ and $C_2$ of $C$ on the boundary. How many independent paths are there from $C_1$ to $C_2$ and how long are they?

Solution: This solution assumes that the reader is familiar with the setup for non-separating cycles that is introduced at the end of the lecture notes, but that we skipped in class. One can argue without it but it makes the arguments cleaner.
Any orientable surface has a non-separating simple cycle (this is proved at the end of the lecture notes), and we pick a minimal one $C$, and cut along it. By Menger’s theorem, the number of independent paths from $C_1$ to $C_2$ equals the size of the smallest vertex cut $S$ separating $C_1$ from $C_2$. The induced subgraph $G[S]$ contains a cycle that is homologous to $C$ and thus non-separating. Therefore it must have size at least that of $C$. Each of these independent paths must have length at least $|C|/2$, as otherwise one would find another non-separating cycle by taking it and taking the smaller of the subpaths of $C$, yielding a shorter non-separating cycle than $C$. Therefore there are at least $|C| \times |C|/2$ vertices in $G$, and thus there is a non-separating cycle of length as most $\sqrt{2n}$. (Yes there was a typo in the exercise and the 2 should have been under the square root (but it was still correct.).)

2. Deduce that any $n$-vertex graph on an orientable surface of genus $g$ has a 2/3-separator $S$ of size $O(g\sqrt{n})$, and such that each component of $G \setminus S$ is planar.

Solution: Each time we remove a nonseparating cycle of the graph and all the adjacent edges, the graph can be embedded on a surface of lower genus. So after applying $g$ times the inequality in the first question, we have removed $O(g\sqrt{n})$ vertices and the remaining graph is planar. We can now find a separator for planar graphs of size $O(\sqrt{n})$. Taking the union of the planar separator and the removed vertices yields the surface separator.

3 Exercise 3

Let $G$ be a planar graph, and let $G_1$ and $G_2$ be two isomorphic straight-line embeddings of $G$, where each face, including the outer face, is a triangle. A morphing step between $G_1$ and $G_2$ is a straight-line continuous transformation of one into the other, such that the graph stays planar at all times: for each vertex $v$ of $G$, we denote by $S(v)$ the segment connecting $v_1$, the embedding of $v$ in $G_1$, to $v_2$, the embedding of $v$ in $G_2$, and we slide $v$ from $v_1$ to $v_2$ at uniform speed along this segment. At a time $t \in [1,2]$, we denote by $v_t$ the position of $v$, and for any edge $(uv)$ in $E$, we connect $v_1$ to $u_t$ with a straight segment. This defines a family of drawings $(G_t)_{t \in [1,2]}$, and this is a morphing step if all these drawings are planar embeddings. A morphing from $G_1$ to $G_2$ is a sequence of morphing steps $G_1 \rightarrow G' \rightarrow G' \cdots \rightarrow G'(k) = G_2$, where the graphs $G'(i)$ are all straight-line embeddings of $G$. The integer $k$ is the complexity of the morphing.

1. Provide an example of a planar graph $G$ and two straight-line embeddings (not necessarily triangulated) that are not connected by a single morphing step.

Solution: See Figure 2

The rest of the exercise aims at proving that for any two straight-line embeddings $G_1$ and $G_2$ with the above conditions, there always exists a morphing of finite complexity between $G_1$ and $G_2$. The proof is by induction.

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2This means here that $G_1$ and $G_2$ have the same outer face and the same combinatorics as an embedded graph: same set of facial walks when turning clockwise.
2. Prove the base case of induction for $n = 4$.

Solution: Even for $n = 4$, one cannot in general do it in a single step: think about two $K_4$, one of which has been rotated by a 180 compared to the first one. But one can first move everything so that the middle vertex coincide, then rotate, and then a single morphing step will work.

The visibility kernel of a polygon is the set of points inside or on the polygon that can be “seen” from any vertex of the polygon, i.e., the set of points $p$ such that for any vertex $v$ of the polygon the segment $pv$ does not cross the polygon.

3. Prove that for any polygon with at most 5 vertices, one of the vertices is contained in its visibility kernel.

Solution: For convex polygons, and thus for triangles, the visibility kernel is the whole polygon. For a concave quadrilateral, the concave vertex is in the visibility kernel. For pentagons, either the two concave vertices are consecutive, or not. In the first case, the vertex opposite to the consecutive vertices sees everything. In the second case, the vertex inbetween the consecutive vertices sees everything.

The link $L(v)$ of a vertex $v$ of $G_1$ or $G_2$ is the polygon defined by the neighbors of $v$.

4. Prove that there exists a vertex $v$ of $G$ so that both in $G_1$ and in $G_2$, the link $L(v)$ contains a vertex $u$ that is in the visibility kernel of $L(v)$. (Note that $u$ might be different in $G_1$ and $G_2$).

Solution: Since $G$ is planar, at least one vertex has degree at most 5. Its link in $G_1$ is a polygon, which therefore has a vertex contained in the visibility kernel of the link by the previous question. Likewise in $G_2$.

We first assume that there are no edges in $G$ connecting non-adjacent vertices of $L(v)$.

5. (*) Prove that there exists a straight-line embedding $G'$ of $G \setminus v$ so that $L(v)$ is convex.

Solution: One can do it by hand (see reference at the end of the document). But we take a simpler way and prove that $G'$ is 3-connected, and the result will follow by using Tutte’s theorem on 3-connected graphs. Note that $G'$ is triangulated except for possibly the face $L(v)$, and $L(v)$ is a cycle and in this face there are not edges connecting non-adjacent vertices of $L(v)$. Let us call such a graph almost triangulated. In an almost-triangulated graph, if $e$ is an edge belonging to exactly two triangles, or to exactly one triangle and the face $L(v)$, then contracting that edge $e$ yields another almost triangulated graph (note that this is not true without the assumption). Let us assume that there are two vertices $u$ and $w$ disconnecting $G'$, into at least two components $X$ and $Y$. We claim that there is always an edge different from $uw$ that belongs to two triangles or to exactly one triangle and the face $L(v)$: pick any edge, and if it belongs to too many triangles, one of them is separating. Then we pick another edge inside that separating triangle. The separating triangles between nested, this process stops and we have found our edge, proving the claim. Now, we repeatedly contract one edge different from $uw$ that belongs to two triangles or to exactly one triangle and the face $L(v)$, keeping at least one vertex in $X$ and in $Y$. At this stage, there are exactly four vertices in the graph: $u$, $w$ and one from $X$ and one from $Y$. The graph is still almost-triangulated, thus and if $u$ and $w$ are on the face $L(v)$, and it has degree more than 3 (thus degree four), then $u$ and $w$ are adjacent. Therefore, the graph is either $K_4$, or $K_4$ minus an edge with $u$ and $w$ adjacent. Furthermore, $u$ and $w$ disconnect that graph, which is not the case. Hence we have a contradiction, finishing the proof.
6. What is the visibility kernel of $L(v)$ in $G'$? Assuming the induction hypothesis (every two straight-line triangulations with $n-1$ vertices can be morphed one into the other), prove that one can morph $G_1$ into $G_2$. **Hint:** contract an edge, and use the induction to morph into $G'$.

**Solution:** Since $L(v)$ is convex, its visibility kernel is $L(v)$. By Question 4., there exists a vertex $v$ so that there is a vertex $u_1$ in the visibility kernel of $L(v)$ in $G_1$, and a vertex $u_2$ in the visibility kernel of $L(v)$ in $G_2$. In $G_1$, we send the vertex $v$ to $u_1$ using a single morphing step. Actually, we send it very close to $u_1$, with the idea that any move of $u_1$ will tell us how to move $v$. This is a tad painful to justify accurately (the keyword is pseudomorphs, you can look it up online). Likewise, in $G_2$ we send $v$ to $u_2$ using a single morphing step. Now $G_1$ and $G_2$ have become $G'_1$ and $G'_2$ which are supergraphs of $G'$ (they have more edges in the face $L(v)$). They both have one less vertex than $G$. By the induction hypothesis and question 5., $G'_1$ can be morphed into a straight-line embedding of $G'$ where the face $L(v)$ is convex. Likewise, $G'_2$ can be morphed into the same (except for the edge within $L(v)$) straight-line embedding. Remembering that we put $v$ just next to $u_1$ (respectively $u_2$), now there just remains to move the vertex $v$ between those two embeddings, which can be done using a single morphing step since $L(v)$ is convex, see Figure 3.

We now remove the additional assumption.

7. (*) Prove the induction step in the general case. **Hint:** without the assumption, there is no hope of finding a straight-line embedding where $L(v)$ is convex, but we can still find an embedding $G'$ where all the vertices of $L(v)$ except the non-adjacent ones which are joined by an edge of $G$ are in the visibility kernel of $L(v)$.

**Solution:** Since nobody even attempted this question, I will just refer to the original article that inspired this exercise, and is quite readable: Cairns, S.: Deformations of plane rectilinear complexes. The American Mathematical Monthly 51(5), 247252 (1944).