It is highly recommended that you work on these exercises. Optionally, you can give me a sheet with your answers, in which case this homework (prepared without external help) will be graded. If you choose to do so, the strict deadline is October 9 at 12:45 (at the beginning of the course). Late homeworks will not be graded.

Vincent Cohen-Addad will also propose a similar exercise sheet, which you can also have graded. These homeworks could give you bonus points for the final grade (up to 2 points over 20 in total).

Vous pouvez écrire ce devoir en français ! You can write this homework in English!

Exercise 1

Let $G$ be a graph cellularly embedded in the sphere. Prove that $G$ has either a face of degree at most two, or a vertex of degree at most five.

Exercise 2

Let $G$ be a graph cellularly embedded in the plane. Assume that each face of $G$ has a real weight. Let $T$ be a spanning tree of $G$. For each edge $e$ in $G - T$, let $\tau(e)$ be the cycle obtained by concatenating $e$ with the unique path in $T$ connecting the endpoints of $e$.

Give an efficient algorithm that computes, for each edge $e$ of $G - T$, the sum of the weights of the faces of $G$ enclosed by $\tau(e)$.

Exercise 3

Let $G$ be a cellular embedding of a graph on the sphere $S^2$; let $G^*$ be its dual. Finally, let $G^+$ be the graph embedding obtained by overlaying $G$ and $G^*$:

- each vertex of $G^+$ is of one of the three following types: (i) a vertex of $G$, (ii) a vertex of $G^*$, or (iii) an intersection between an edge of $G$ and the corresponding dual edge in $G^*$;
- each edge of $G^+$ is of one of the two following types: (i) half of an edge of $G$, or (ii) half of an edge of $G^*$.

Figure 1 illustrates the construction.

![Figure 1: Construction of $G^+$ (right) from $G$ (left).](image)

Assume that one is given the combinatorial map of the graph $G^+$, without specifying which vertices or edges are of which type. Give an efficient algorithm to determine two combinatorial maps, one corresponding to the embedding $G$ and the other corresponding to the embedding of $G^*$. Estimate its running time. (Note: The algorithm cannot determine which combinatorial map is that of $G$ and which one is that of $G^*$!)
Exercise 4

Let $G$ be a planar straight-line triangulation: a graph cellularly embedded in the plane, represented by its combinatorial map, such that each face has degree three and every edge is drawn as a straight-line segment. The combinatorial map also stores the coordinates of the vertices. Let $n$ be the number of vertices of $G$.

1. Compute the number of edges and faces of $G$ as a function of $n$.

2. An independent set in a graph is a subset of pairwise non-adjacent vertices. Prove that there exist two constants $c > 0$ and $d$ such that $G$ has an independent set of size $c \cdot n$, made of vertices of degree at most $d$. \textit{Hint:} Use Exercise 1.

3. Let $P$ be a cycle with $k$ vertices and edges, straight-line embedded in the plane. Give an algorithm of complexity $f(k)$, where $f$ is an arbitrary function (we really don’t care), to add edges (no vertices) in the interior of $P$ so as to obtain a straight-line embedding of a graph in which every face, except possibly the outer face, is a triangle.

4. Prove that, in $O(n)$ time, one can build a data structure for $G$ that allows to answer in $O(\log n)$ time any query of the following form: Given a point $p \in \mathbb{R}^2$, compute a triangle of $G$ (if any) containing $p$. \textit{Hint:} Build a hierarchy of increasingly simplified triangulations, starting from $G$. 