## 0.1 Representations of planar graphs

### 0.1.1 Tutte's barycentric embedding theorem

**Theorem 0.1** (Tutte's theorem, 1963)**.** *Let $G = (V, E)$ be a graph without loops or multiple edges. Assume $G$ has an embedding in $\mathbb{R}^2$ such that every face is a triangle (including the outer face). Let $V_e = \{v_1, v_2, v_3\}$ be the vertices of the outer face. Assign unique positions $f(v)$ in $\mathbb{R}^2$ for each vertex $v$, such that*

- *the $f(v_i)$, $i = 1, 2, 3$ are mapped in the vertices of a triangle in the plane;*
- *the image of every vertex $v$ different from the $v_i$'s is a barycenter with strictly positive coefficients of the images of its neighbors in $G$.*

*Then drawing straight-line edges between the image points gives an embedding of $G$.*

We refer to the triangles of the original embedding as *triangles* of $G$. Given the existence of such a graph embedding, it is always possible to achieve the conditions of the theorem. For example, choose the barycentric coefficients to be all equal to one. The barycentric condition yields an affine system, which is solvable by an argument of "dominant diagonal". Equivalently, one may view the edges as "springs" with the same rigidity, and the interior vertices as being free to move. The equilibrium of this physical system is met when the "energy" is minimized.

We refer to the vertices in $V_e$ as *exterior vertices*, and to the other ones as *interior vertices*. Let $v$ be an interior vertex of $G$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be an affine form vanishing on $f(v)$. If all the neighbors of $v$ lie on $h^{-1}(0)$, we say $v$ is *$h$-inactive*. Otherwise, $v$ is *$h$-active*. In this case, $v$ has neighbors in both $h^{-1}((0, \infty))$ and $h^{-1}((-\infty, 0))$. In particular one can find a *falling path* from $v$ to an exterior vertex: a path whose value of $h$ strictly decreases.

**Proposition 0.2.** *The image of every interior vertex of $G$ is in the interior of the triangle $f(V_e)$.*

*Proof.* Let $h$ be an affine form vanishing on two vertices of $V_e$ such that the third vertex lies in $h^{-1}((0, \infty))$. If there is a vertex whose image in

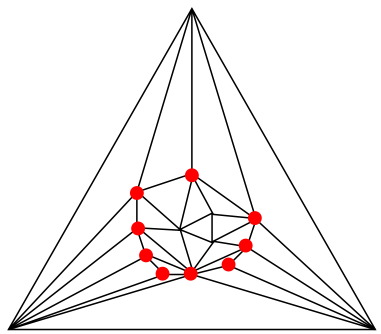

**Figure 1.** All the interior vertices are connected: Indeed, every interior vertex can be connected to one incident with an exterior vertex; all such vertices are connected.

$h^{-1}((-\infty, 0))$, then, taking one that has minimum value of $h$, since it is barycenter with positive coefficients of its neighbors, all its neighbors must have the same value of $h$. By connectivity of $G$, some exterior vertex must have that value of $h$, which is not possible. Therefore each interior vertex lies in the interior or on the boundary of the triangle $f(V_e)$.

Let $v$ be an interior vertex; assume $v$ lies on an edge of the outer polygon, whose supporting line is $h^{-1}(0)$. Then all the neighbors of $v$ are $h$-inactive. Thus, all interior vertices that can be reached from $v$ by a path using only interior vertices lie on $h^{-1}(0)$. However, all the interior vertices are connected (Figure 1). Therefore, all vertices are on $h^{-1}(0)$; but looking at one that is incident with the vertex of $V_e$ in $h^{-1}((0, \infty))$, we get a contradiction. □

**Proposition 0.3.** *Let $uvy$ and $uvz$ be two triangles of $G$ sharing the edge $uv$. Let $h$ be an affine form vanishing on $f(u)$ and $f(v)$. Then $h(f(y))h(f(z)) < 0$.*

The proof relies on the following lemma.

**Lemma 0.4.** *If $h(f(y)) > 0$, then $h(f(z)) < 0$.*

*Proof.* By assumption $u$, $v$, and $y$ are $h$-active. Find strictly falling paths going from $u$ and $v$ to an exterior vertex. The falling paths may share a

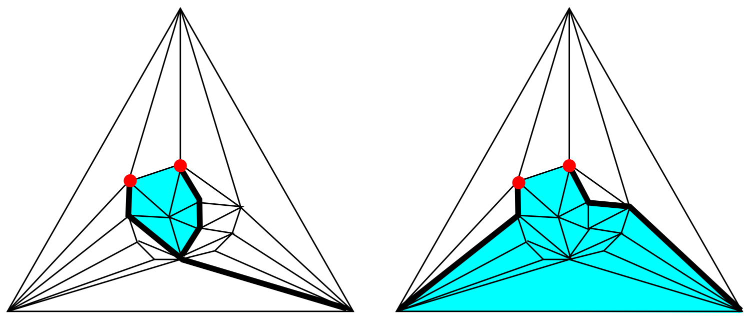

**Figure 2.** Two cases for the proof of Lemma 0.4. Vertices $u$ and $v$, together with falling paths, are highlighted.

vertex. In any case, we get a simple circuit $C$ in $G$ using $uv$ whose set of vertices are on the strictly negative side of $h$, except $u$ and $v$. (If the falling paths do not share a vertex, we may need to connect their endpoints with an exterior edge.) See Figure 2. Therefore, by the Jordan curve theorem, $C$ bounds a disk.

Since $h(f(y)) > 0$, vertex $y$ does not belong to this disk. Since $uv$ is on the boundary of this disk, $z$ belongs to this disk. The vertices in the interior of this disk are also barycenters with strictly positive coefficients of their neighbors. Therefore, by the same argument as in the proof of Proposition 0.2, we have $h(f(z)) < 0$. □

*Proof of Proposition 0.3.* Lemma 0.4 shows that, whenever one triangle is non-degenerate, its incident triangles are non-degenerate. Necessarily, any triangle having one exterior edge is non-degenerate (Proposition 0.2). So every triangle is non-degenerate. The result follows. □

*Proof of Theorem 0.1.* Since the triangles are non-degenerate by Proposition 0.3, it suffices to prove that the interiors of two distinct triangles are disjoint. For the sake of a contradiction, let $a$ be a point of $\mathbb{R}^2$ in the interior of two triangles $t$ and $t'$. Shoot a ray from $a$ to the boundary of the exterior triangle avoiding the image of every vertex. Whenever the ray leaves $t$, by Proposition 0.3, it enters another triangle. So we get a sequence of triangles on the ray starting with $t$ and finishing with $t''$, the unique triangle incident to the boundary edge that is on the end of the ray. Similary, we get a sequence of triangles on the ray starting with $t'$ and finishing with $t''$. Going back in both sequences from $t''$, we pass from a triangle to an unambiguously defined preceding triangle. Since we start with the same triangle, we get $t = t'$. □

**Corollary 0.5** (Fáry–Stein–Wagner's theorem)**.** *Every planar graph can be embedded in the plane with straight-line edges.*

*Proof.* We extend $G$ into a triangulation, by adding edges in the initial embedding, and then apply Theorem 0.1. Finally, we remove the artificial edges. □

### 0.1.2 Steinitz' theorem

Every convex polyhedron in $\mathbb{R}^3$ has a set of *vertices* (extremal points) and *edges*. This is called the *1-skeleton* of the polyhedron.

**Theorem 0.6** (Steinitz' theorem (1922))**.** *Let $G = (V, E)$ be a graph without loops or multiple edges. Assume $G$ has an embedding in $\mathbb{R}^2$ such that every face is a triangle (including the outer face). Then $G$ can be embedded in $\mathbb{R}^3$ such that it is the 1-skeleton of a convex polyhedron in $\mathbb{R}^3$.*

Let $\omega : E \to (0, \infty)$ be a function from the (undirected) edges of $G$ to a set of strictly positive coefficients. (In the sequel, we could take $\omega$ to be constant, equal to one.)

Let $f : V \to \mathbb{R}^2$ be the corresponding Tutte equilibrium given by Theorem 0.1, where every vertex $v$ is barycenter with coefficients $\omega_{uw_1}, \ldots, \omega_{uw_m}$ of its neighbors $w_1, \ldots, w_m$. We actually assume that $f : V \to \mathbb{R}^3$ maps the vertices into the plane $z = 1$ of $\mathbb{R}^3$.

To every *interior* triangle $t$ of $G$ we associate a vector $q_t$ in $\mathbb{R}^3$. We choose an arbitrary interior triangle $t_0$, for which $q_{t_0} = 0$. The other $q_t$'s are defined by the following formula: For every interior edge $uv$ with left triangle $uvy$ and right triangle $uvz$, we define

$$q_{uvz} = \omega_{uv}(f(u) \times f(v)) + q_{uvy} \tag{1}$$

where $\times$ denotes the cross-product in $\mathbb{R}^3$.

**Lemma 0.7.** *The vectors $q_t$ are well-defined.*

*Proof.* First note that exchanging $u$ and $v$ and $y$ and $z$ gives $q_{vuy} = \omega_{vu}(f(v) \times f(u)) + q_{vuz}$, which rewrites $q_{uvy} = \omega_{uv}(-f(u) \times f(v)) + q_{uvz}$; this is exactly Equation (1).

Let $v$ be a vertex of $G$, and $w_1, \ldots, w_m$ be its neighbors. We get:

$$\sum_{i=1}^{m} \omega_{vw_i}(f(v) \times f(w_i)) = f(v) \times \left( \sum_{i=1}^{m} \omega_{vw_i}(f(w_i) - f(v)) \right) = 0.$$

Therefore, Equation (1) gives consistent vectors $q_t$ for all triangles around an interior vertex. Now, starting from the initial triangle $t_0$, we may define the value of $q_t$ by choosing an arbitrary sequence of triangles from $t_0$. (Incidentally, this shows that the function $q$, if it exists, is unique.) There remains to show that any two paths of triangles from $t_0$ to $t$ give the same value of $q_t$. We may assume that these two paths use distinct sets of triangles except at $t_0$ and $t$. Then the result is proven by induction on the number of vertices of $G$ enclosed by the two paths: The case of one vertex is the previous paragraph. For the induction step, build one path of triangles that enclose one less vertex. See Figure 3. □

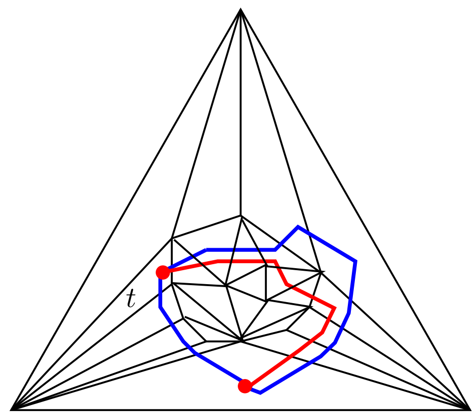


**Figure 3.** Illustration of the proof of Lemma 0.7. We wish to prove that the definition of $q_t$ is the same, whichever of the two exterior paths we choose. This is done by induction on the number of enclosed vertices, and by choosing an "intermediate" path that contains less vertices on both sides.

We define a piecewise linear function $g$ from the union of the interior triangles to $\mathbb{R}$ by setting, for every point $x$ in a triangle $t$, $g(x) = \langle x \,|\, q_t \rangle$.

**Lemma 0.8.** *This map $g$ is well-defined.*

*Proof.* We only need to prove that, whenever $x$ belongs to an edge $uv$ with adjacent vertices $y$ and $z$, the value of $g(x)$ is the same, whichever triangle $uvy$ or $uvz$ we choose for the computation; in other words, $\langle x \,|\, q_{uvy} \rangle = \langle x \,|\, q_{uvz} \rangle$. By linearity it suffices to prove the result for $x = f(u)$ and $x = f(v)$.

$$\langle f(v) \,|\, q_{uvz} \rangle = \langle f(v) \,|\, \omega_{uv}(f(u) \times f(v)) + q_{uvy} \rangle = \langle f(v) \,|\, q_{uvy} \rangle \,.$$

A similar computation holds for $f(v)$. □

**Lemma 0.9.** *Let $uv$ be an edge with left triangle $uvy$ and right triangle $uvz$. Assume $x$ is in $f(u)f(v)f(y)$. Then $\langle x \,|\, q_{uvy} \rangle < \langle x \,|\, q_{uvz} \rangle$.*

*Proof.*

$$\langle x \,|\, q_{uvz} \rangle - \langle x \,|\, q_{uvy} \rangle = \omega_{uv} \langle x \,|\, f(u) \times f(v) \rangle = \omega_{uv} \det(x, f(u), f(v)) > 0,$$

by our orientation convention (recall that the last coordinate of the points $x$, $f(u)$, and $f(v)$ is one) and the fact that $\omega_{uv} > 0$. □

*Sketch of proof of Theorem 0.6.* Recall that the position of a vertex $v$ is $f(v)$ in $\mathbb{R}^3$, actually in the plane $z = 1$. We just move vertically $f(v)$ to height $g(f(v))$. Let $F(v)$ be the new position. Let $P$ be the convex hull of the $F(v)$. Lemma 0.9 implies that every interior edge $uv$ is an edge of $P$, because every such edge is a "valley"; the same clearly holds for the exterior edges. □

## 0.2 Notes

The contents of this section is taken from Edelsbrunner and Harer [3] and Richter-Gebert [7]. In addition to the original paper proving Tutte's theorem [10], there are many other proofs [1, 7, 2, 4, 9, 5].

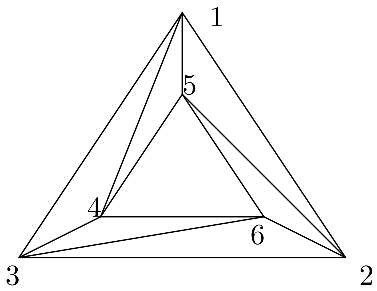


**Figure 4.** An embedding that cannot be lifted to a convex polyhedron. Indeed, assume every interior edge is an edge on the "bottom" of the convex polyhedron. We can suppose, by adding a suitable affine form to all the $z_i$'s, that $z_4 = z_5 = z_6 = 0$. Then $z_1 > z_2 > z_3 > z_1$, which is impossible.

Both Tutte's and Steinitz' [8] theorems hold actually in the more general case of *3-connected* graphs, that is, graphs that remain connected after the removal of zero, one, or two vertices. Both proofs are slightly more technical in this general case. Actually Steinitz' theorem states that a graph is the 1-skeleton of a convex polyhedron in $\mathbb{R}^3$ *if and only if* it is planar and 3-connected (the "only if" direction is easy). The correspondence between Tutte embeddings where every vertex is barycenter of its neighbors and the height function $g$ is the *Maxwell-Cremona correspondence* (see for example Hopcroft and Kahn [6]).

There are some straight-line graph embeddings that cannot be "lifted" to a convex polyhedron (Figure 4).

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