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Free Probability and Combinatorics

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Abstract

A combinatorial approach to free probability theory has been developped by Roland Speicher, based on the notion of noncrossing cumulants, a free analogue of the classical theory of cumulants in probability theory. We review this theory, and explain the connections between free probability theory and random matrices. We relate noncrossing cumulants to classical cumulants and also to characters of large symmetric groups. Finally we give applications to the asymptotics of representations of symmetric groups, specifically to the Littlewood-Richardson rule.

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1. Introduction

Free probability has been introduced by D. Voiculescu [21] as a means of studying the group von Neumann algebras of free groups, using probabilistic techniques. His theory has become very successful when he discovered a deep relation with the theory of random matrices, and solved some old questions in operator algebra, see [4], [7], [24] for an overview. A purely combinatorial approach to Voiculescu's definition of freeness has been given by R. Speicher [19], [20], building on G. C. Rota's [16] approach to classical probability. It is based on the notion of noncrossing partitions, also known as "planar diagrams" in quantum field theory, and provides unifying concepts for many computations in free probability. Noncrossing partitions turn out to be connected with the geometry of the symmetric group, and this leads to some new understanding of the asymptotic behaviour of the characters and representations of large symmetric groups. Our aim is to survey these results, we shall start with the basic definition of freeness, then explain its connection to random matrix theory. In the third section we review Speicher's theory. In the fourth section we show how noncrossing cumulants arise naturally in connection

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with classical cumulants associated with random matrices, and with characters of symmetric groups. Finally in section 5 we explain the asymptotic behaviour of representations of symmetric groups in terms of free probability concepts.

2. Freeness and random matrices

The usual framework for free probability is a von Neumann algebra A, equipped with a faithful, tracial, normal state τ. To any self-adjoint element $X \in A$ one can associate its distribution, the probability measure on the real line, uniquely determined by the identity $\tau(X^n) = \int_{\mathbf{R}} x^n \mu(dx)$ for all $n \geq 1$. This makes it natural to think of the elements of A as noncommutative random variables, and of τ as an expectation map, and one usually calls noncommutative probability space such a pair (A, τ). Although a great deal of the theory, especially the combinatorial side, can be developped in a purely algebraic way, assuming only that A is a complex algebra with unit, and τ a complex linear functional, we shall stick to the von Neumann framework in the present exposition.

Given (A, τ), one considers a family $\{A_i; i \in I\}$ of von Neumann subalgebras. This family is called a *free family* if the following holds: for any $k \geq 1$ and k-tuple $a_1, \ldots, a_k \in A$ such that

- each a_j belongs to some algebra A_{i_j}, with $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$,
- $\tau(a_j) = 0$ for all j,

one has $\tau(a_1 \ldots a_k) = 0$.

Moreover, a family of elements of A is called free if the von Neumann algebras each of them generates form a free family. Freeness is a noncommutative notion analogous to the independence of σ-fields in probability theory, but which incorporates also the notion of algebraic independence.

Observe that if a_1 and a_2 are free elements in (A, τ), and one defines the centered elements $\hat{a}_i = a_i - \tau(a_i)1$ then one can conpute

$$\tau(a_1 a_2) = \tau(\hat{a}_1 \hat{a}_2) + \tau(a_1)\tau(a_2) = \tau(a_1)\tau(a_2)$$

where the freeness condition has been used to get $\tau(\hat{a}_1 \hat{a}_2) = 0$. Actually, if $\{A_i; i \in I\}$ is a free family, it is not dificult to see that one can compute the value of τ on any product of the form $a_1 \ldots a_k$, where each a_j belongs to some of the $A_i's$, in terms of the quantities $\tau(a_{j_1} \ldots a_{j_l})$ where all the elements $a_{j_1}, \ldots, a_{j_l}$ belong to the same subalgebra. This implies that the value of τ on the algebra generated by the family $\{A_i; i \in I\}$ is completely determined by the restrictions of τ to each of these subalgebras. However the problem of finding an explicit formula is nontrivial, and this is where combinatorics comes in. We shall describe Speicher's theory of noncrossing cumulants, which solves this problem, in the next section, but before that we explain how free probability is relevant to understand large random matrices.

Consider n random $N \times N$ matrices $X_1^{(N)}, \ldots, X_n^{(N)}$, of the form

$$X_j^{(N)} = U_j D_j^{(N)} U_j^* \tag{2.1}$$

where $D_j^{(N)}; j = 1, \dots, n$ are diagonal, hermitian, nonrandom matrices and U_j are independent unitary random matrices, each distributed with the Haar measure on the unitary group $\mathbf{U}(N)$. In other words we have fixed the spectra of the $X_i^{(N)}$ but their eigenvectors are chosen at random. The n-tuple $X_1^{(N)}, \dots, X_n^{(N)}$ can be recovered, up to a global unitary conjugation $X_i^{(N)} \mapsto UX_i^{(N)}U^*$, (where U does not depend on i), from its mixed moments, i.e. the set of complex numbers $\frac{1}{N}Tr(X_{i_1}^{(N)} \dots X_{i_k}^{(N)})$ where $i_1, \dots i_k$ are arbitrary sequences of indices in $\{1, \dots, n\}$. In particular the spectrum of any noncommutative polynomial of the $X_i^{(N)}$ can be recovered from these data. A most remarkable fact is that if we assume that the individual moments $\frac{1}{N}Tr((X_i^{(N)})^k)$ converge as N tends to infinity, then the mixed moments $\frac{1}{N}Tr(X_{i_1}^{(N)} \dots X_{i_k}^{(N)})$ converge in probability, and their limit is obtained by the prescriptions of free probability.

Theorem 1. *Let (A, τ) be a noncommutative probability space with free self-adjoint elements $X_1, \dots, X_n$, satisfying $\tau(X_i^k) = \lim_{N\to\infty} \frac{1}{N}Tr((X_i^{(N)})^k)$, for all i and k, then, in probability, $\frac{1}{N}Tr(X_{i_1}^{(N)} \dots X_{i_k}^{(N)}) \to_{N\to\infty} \tau(X_{i_1} \dots X_{i_k})$, for all $i_1, \dots, i_k$.*

This striking result was first proved by D. Voiculescu [23], and has lead to the resolution of many open problems about von Neumann algebras, upon which we shall not touch here.

3. Noncrossing partitions and cumulants

A partition of the set $\{1, \dots, n\}$ is said to have a crossing if there exists a quadruple (i, j, k, l), with $1 \le i < j < k < l \le n$, such that i and k belong to some class of the partition and j and l belong to another class. If a partition has no crossing, it is called noncrossing. The set of all noncrossing partitions of $\{1, \dots, n\}$ is denoted by $NC(n)$. It is a lattice for the refinement order, which seems to have been first systematically investigated in [10].

Let (A, τ) be a non-commutative probability space, then we shall define a family $R^{(n)}$ of n-multilinear forms on A, for $n \ge 1$, by the following formula

$$\tau(a_1 \dots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \dots, a_n) \tag{3.1}$$

Here, for $\pi \in NC(n)$, one has defined

$$R[\pi](a_1, \dots, a_n) = \prod_{V \in \pi} R^{(|V|)}(a_V)$$

where $a_V = (a_{j_1}, \dots, a_{j_k})$ if $V = \{j_1, \dots, j_k\}$ is a class of the partition π, with $j_1 < j_2 < \dots < j_k$ and $|V| = k$ is the number of elements of V. In particular $R[1_n] = R^{(n)}$ if 1_n is the partition with only one class. Thus one has, for $n = 3$,

$$\begin{aligned}\tau(a_1 a_2 a_3) \quad &= \quad R^{(3)}(a_1, a_2, a_3) + R^{(2)}(a_1, a_2)R^{(1)}(a_3) + R^{(2)}(a_1, a_3)R^{(1)}(a_2) \\ &\quad + R^{(2)}(a_2, a_3)R^{(1)}(a_1) + R^{(1)}(a_1)R^{(1)}(a_2)R^{(1)}(a_3)\end{aligned}$$

Observe that

$$\tau(a_1 \dots a_n) = R^{(n)}(a_1, \dots, a_n) + \text{terms involving } R^{(k)} \text{ for } k < n$$

so that the $R^{(n)}$ are well defined by (3.1) and can be computed by induction on n. They are called the noncrossing (or sometimes free) cumulant functionals on A.

The formula (3.1) can be inverted to yield

$$R^{(n)}(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} Moeb([\pi, 1_n])\tau[\pi](a_1, \dots, a_n)$$

Here $\tau[\pi](a_1, \dots, a_n) = \prod_{V \in \pi} \tau(a_{j_1} \dots a_{j_k})$ where $V = \{j_1, \dots, j_k\}$ are the classes of π, and $Moeb$ is the Möbius function of the lattice $NC(n)$, see [20].

For example, one has

$$\begin{aligned} R^{(1)}(a_1) &= \tau(a_1); \qquad R^{(2)}(a_1, a_2) = \tau(a_1 a_2) - \tau(a_1)\tau(a_2) \\ R^{(3)}(a_1, a_2, a_3) &= \tau(a_1 a_2 a_3) - \tau(a_1)\tau(a_2 a_3) - \tau(a_2)\tau(a_1 a_3) \\ &\quad -\tau(a_3)\tau(a_1 a_2) + 2\tau(a_1)\tau(a_2)\tau(a_3) \end{aligned}$$

Note that when the lattice of all partitions is used instead of noncrossing partitions, then one gets the usual family of cumulants (see Rota [16]), with another Möbius function.

The connection between noncrossing cumulants and freeness is the following result from section 4 of [19].

Theorem 2. *Let $\{A_i; i \in I\}$ be a free family of subalgebras of (A, τ), and $a_1, \dots, a_n \in A$ be such that a_j belongs to some A_{i_j} for each $j \in \{1, 2, \dots, n\}$. Then one has $R^{(n)}(a_1, \dots, a_n) = 0$ if there exists some j and k with $i_j \neq i_k$.*

This result leads to an explicit expression for $\tau(a_1 \dots a_n)$, where $a_1, \dots, a_n$ is an arbitrary sequence in A, such that each a_j belongs to one of the algebras $A_i; i \in I$. By Theorem 2, in the right hand side of (3.1), the terms corresponding to partitions π having a class containing two elements j, k such that a_j and a_k belong to distinct algebras give a zero contribution. Thus we have to sum over partitions in which all $j's$ belonging to a certain block of the partition are such that a_j belongs to the same algebra. Since we can express noncrossing cumulants in terms of moments we get the formula for $\tau(a_1 \dots a_n)$ in terms of the restrictions of τ to each of the subalgebras A_i. Noncrossing cumulants are a powerful tool for making computations in free probability, see [11], [12], [13], [14], [18], for some applications. We give a simple illustration below.

Let X_1 and X_2 be two self-adjoint elements which are free, then the distribution of $X_1 + X_2$, depends only on the distributions of X_1 and X_2 and can be computed as follows. Let $R^{(n)}(X_1, \dots, X_1)$ and $R^{(n)}(X_2, \dots, X_2)$, for $n \geq 1$, be the noncrossing cumulants of X_1 and X_2, then one can expand $R^{(n)}(X_1 + X_2, \dots, X_1 + X_2)$ by multilinearity as $\sum_{i_1, \dots, i_n} R^{(n)}(X_{i_1}, \dots, X_{i_n})$ where the sum is over all sequences of 1 and 2. By Theorem 2, all terms vanish except $R^{(n)}(X_1, \dots, X_1)$ and $R^{(n)}(X_2, \dots, X_2)$. It follows that

$$R^{(n)}(X_1 + X_2, \dots, X_1 + X_2) = R^{(n)}(X_1, \dots, X_1) + R^{(n)}(X_2, \dots, X_2)$$

allowing the computation of the moments of $X_1 + X_2$, hence its distribution, in terms of the distributions of X_1 and X_2. It remains to give a compact form to the relation between moments and noncrossing cumulants. For any self-adjoint element X with distribution μ, let

$$G_X(z) = \frac{1}{z} + \sum_{k=1}^{\infty} z^{-k-1}\tau(X^k) = \int_{\mathbf{R}} \frac{1}{z-x}\mu(dx)$$

be its Cauchy transform, and let

$$K(z) = \frac{1}{z} + \sum_{k=0}^{\infty} R_k z^k$$

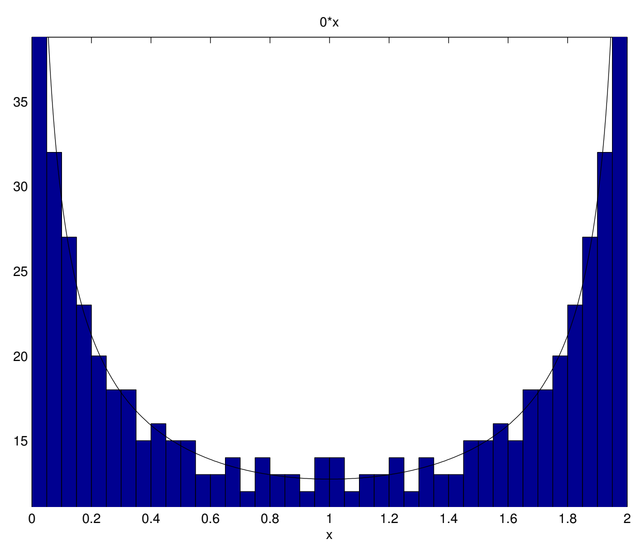
be the inverse series for composition.

Theorem 3. [19]

One has $\quad R_k = R^{(k)}(X, \dots, X) \quad$ *for all* k.

The operation which associates to the two distributions of X_1 and X_2 the distribution of their sum is called the free convolution of measures on the real line, and was introduced by D. Voiculescu, who first considered the coefficients R_k and proved the formula for the free convolution of two measures, using very different methods [22].

Combining theorems 1 and 2, given two large random matrices of known spectra one can predict the spectral distribution of their sum, with a good accuracy and probability close to 1. It is illuminating to look at the following example. The histogram below is made of the 800 eigenvalues of a random matrix of the form $\Pi_1 + \Pi_2$ where Π_1 and Π_2 are two orthogonal projections onto some random subpaces of dimension 400 in $\mathbf{C}^{800}$, chosen independently. The curve $y = \frac{40}{\pi\sqrt{x(2-x)}}$ which corresponds to the large N limit predicted by free probability has been drawn.



4. Noncrossing cumulants, random matrices and characters of symmetric groups

Besides free probability theory, noncrossing partitions appear in several areas of mathematics. We indicate some relevant connections. The first is with the theory of map enumeration initiated by investigations of theoretical physicists in two-dimensional quantum field theory. The noncrossing partitions appear there under the guise of planar diagrams, the Feynman diagrams which dominate the matrix integrals in the large N limit. This is of course related to the fact that large matrices model free probability. We shall not discuss this further here, but refer to [26] for an accessible introduction. Another place where noncrossing partitions play a role, which is closely related to the preceding, is the geometry of the symmetric group, more precisely of its Cayley graph. Consider the (unoriented) graph whose vertex set is the symmetric group Σ_n, and such that $\{\sigma_1, \sigma_2\}$ is an edge if and only if $\sigma_1^{-1}\sigma_2$ is a transposition, i.e. this is the Cayley graph of Σ_n with respect to the generating set of all transpositions. The distance on the graph is given by

$$d(\sigma_1, \sigma_2) = n - \text{number of orbits of } \sigma_1^{-1}\sigma_2 := |\sigma_1^{-1}\sigma_2|$$

The lattice of noncrossing partitions can be imbedded in Σ_n in the following way [10], given a noncrossing partition of $\{1, \dots, n\}$, its image is the permutation σ such that $\sigma(i)$ is the element in the same class as i, which follows i in the cyclic order $12...n$. One can check [1] that the image of $NC(n)$ is the set of all permutations satisfying $|\sigma| + |\sigma^{-1}c| = |c|$ where c is the cyclic permutation $c(i) = i + 1 \, mod(n)$, in other words, this set consists of all permutations which lie on a geodesic from the identity to c in the Cayley graph. These facts are at the heart of the connections between free probability, random matrices and symmetric groups. As an illustration we shall see how free cumulants arise from asymptotics of both random matrix theory and symmetric group representation theory.

Recall that cumulants (also called semi-invariants, see e.g. [17]) of a random variable X with moments of all orders, are the coefficients in the Taylor expansion of the logarithm of its characteristic function, i.e.

$$\log E[e^{itX}] = \sum_{n=0}^{\infty} (it)^n \frac{C_n(X)}{n!}$$

We shall consider random variables of the following form $Y^{(N)} = NX_{1,1}^{(N)}$ where $X^{(N)} = UD^{(N)}U^*$ is a random matrix chosen as in (2.1) and $X_{1,1}^{(N)}$ is its upper left coefficient. Assume now that the moments of $X^{(N)}$ converge

$$\frac{1}{N} Tr((X^{(N)})^k) \to_{N \to \infty} \int_{\mathbf{R}} x^k \mu(dx)$$

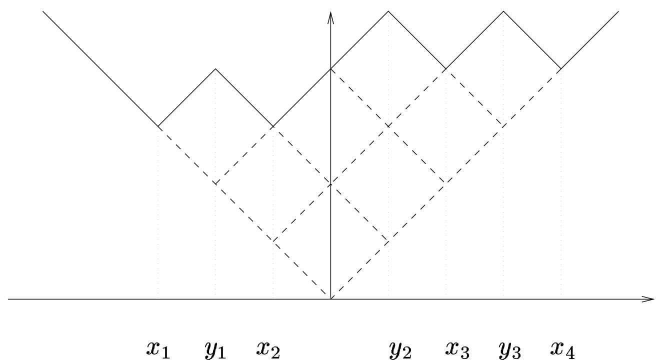
for some probability measure μ on $\mathbf{R}$, with noncrossing cumulants $R_n(\mu)$, then one has

$$\lim_{N \to \infty} \frac{1}{N^2} C_n(Y^{(N)}) = \frac{1}{n} R_n(\mu).$$

This was first observed by P. Zinn-Justin [25], a proof using representation theory has been found by B. Collins [6].

We have related noncrossing cumulants to usual cumulants via random matrix theory, we shall see that that noncrossing cumulants are also useful in evaluating characters of symmetric groups. The precise relation however is not obvious at first sight.

Let us recall a few facts about irreducible representations of symmetric groups. It is well known that they can be parametrized by Young diagrams. In the following it will be convenient to represent a Young diagram by a function $\omega : \mathbf{R} \to \mathbf{R}$ such that $\omega(x) = |x|$ for $|x|$ large enough, and ω is a piecewise affine function, with slopes ± 1, see the following picture which shows the Young diagram corresponding to the partition $8 = 3 + 2 + 2 + 1$.



Alternatively we can encode the Young diagram using the local minima and local maxima of the function ω, denoted by $x_1, \dots, x_k$ and $y_1, \dots, y_{k-1}$ respectively, which form two interlacing sequences of integers. These are (-3,-1,2,4) and (-2,1,3) respectively in the above picture. Associated with the Young diagram there is a unique probability measure m_ω on the real line, such that

$$\int_{\mathbf{R}} \frac{1}{z-x} m_\omega(dx) = \frac{\prod_{i=1}^{k-1}(z-y_i)}{\prod_{i=1}^{k}(z-x_i)} \quad \text{for all } z \in \mathbf{C} \setminus \mathbf{R}$$

This probability measure is supported by the set $\{x_1, \dots, x_k\}$ and is called the transition measure of the diagram, see [8]. Let σ denote the conjugacy class in Σ_n of a permutation with k_2 cycles of length 2, k_3 of length 3, etc.. Here $k_2, k_3, \dots$ are fixed while we let $n \to \infty$. Denote by χ_ω the normalized character of Σ_n associated with the Young diagram ω, then the following asymptotic evaluation holds uniformly on the set of A-balanced Young diagrams, i.e. those whose longest row and longest column are less that $A\sqrt{n}$ (where A is some constant > 0),

$$\chi_\omega(\sigma) = \prod_{j=2}^{\infty} n^{-jk_j} R_{j+1}^{k_j}(\omega) + O(n^{-1-|\sigma|/2}) \tag{4.1}$$

Note that R_k is scaled by λ^k if we scale the diagram ω by a factor λ, therefore the first term in the right hand side is of order $O(n^{\sum_j (j+1)k_j/2 - \sum_j jk_j}) = O(n^{-|\sigma|/2})$, this gives the order of magnitude of the character of a fixed conjugacy group for an A-balanced diagram.

In [2] a proof of (4.1) has been given, using in an essential way the Jucys-Murphy operators. Another proof, leading to an exact formula for characters of cycles due to S. Kerov [9], was shown to me later by A. Okounkov [15], see [5].

5. Representations of large symmetric groups

The asymptotic formula (4.1) shows in particular that irreducible characters of symmetric groups become asymptotically multiplicative i.e. for permutations with disjoint supports σ_1 and σ_2, one has

$$\chi_\omega(\sigma_1\sigma_2) = \chi_\omega(\sigma_1)\chi_\omega(\sigma_2) + O(n^{-1-|\sigma_1\sigma_2|/2}) \tag{5.1}$$

uniformly on A-balanced diagrams. Conversely, given a central, normalized, positive definite function on Σ_n, a factorization property such as (5.1) implies that the positive function is essentially an irreducible character [3]. More precisely, recall that a central normalized positive definite function ψ on Σ_n is a convex combination of normalized characters, and as such it defines a probability measure on the set of Young diagrams. For any $\varepsilon, \delta > 0$, for all n large enough, if an approximate factorization such as (5.1) holds for ψ, then there exists a curve ω, such that the measure on Young diagrams associated with ψ puts a mass larger than $1 - \delta$ on Young diagrams which lie in a neighbourhood of this curve, of width $\varepsilon\sqrt{n}$. Therefore one can say that condition (5.1) on a positive definite function implies that the representation associated with this function is approximately isotypical, i.e. almost all Young diagrams occuring in the decomposition have a shape close to a certain definite curve.

Using this fact it is possible to understand the asymptotic behaviour of several operations in representation theory. Consider for example the operation of induction. One starts with two irreducible representations of symmetric groups $\Sigma_{n_1}, \Sigma_{n_2}$, corresponding to two Young diagrams ω_1 and ω_2. One can then induce the product representation $\omega_1 \otimes \omega_2$ of $\Sigma_{n_1} \times \Sigma_{n_2}$ to $\Sigma_{n_1+n_2}$. This new representation is reducible and the multiplicities of irreducible representations can be computed using a combinatorial device, the Littlewood-Richardson rule. This rule however gives little light on the asymptotic behaviour of the multiplicities. Using the factorization-concentration result, one can prove that when n_1 and n_2 are very large, but of the same order of magnitude, then there exists a curve, which depends on ω_1 and ω_2, and such that the typical Young diagram occuring in the decomposition of the induced representation, is close to this curve. As we saw in section 4, one can associate a probability measure on the real line to any Young diagram. The description of the typical shape of Young diagram which occurs in the decomposition of the induced representation is easier if we use this correspondance between probability measures and Young diagrams, indeed the probability measure associated with the shape of the typical Young diagram corresponds to the free convolution of the two probability measures [2].

There are analogous results for the restriction of representations from large symmetric groups to smaller ones. There the corresponding operation on probability measure is called the free compression, it corresponds at the level of the large matrix approximation, to taking a random matrix with prescribed eigenvalue distribution, as in section 2, and extracting a square submatrix. Finally there are also results for Kronecker tensor products of representations. Here a central role is played by the well known Kerov-Vershik limit shape, whose associated probability measure is the semi-circle distribution with density $\frac{1}{2\pi}\sqrt{4-x^2}$ on the interval $[-2,2]$, see [2].

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