The D_1 example. (1)

Let D_1 be the Dyck set on one letter. Thus $D_1 \subset \Sigma^{\square}$ where $\Sigma^{\square} = \{\bar{\sigma}, \sigma\}^*$. We propose to prove that the set

$$A = \{ \tau_1^p \, \xi^{p+q} \, \tau_2^q \mid p \geq 0, q \geq 0 \}$$

is not in the principal cone $\mathcal{C} D_1$.

This follows from the following

Ω^* **Proposition .1.** Let $f : \Sigma^{\square} \to \Xi^*$ be a ~~[illegible]~~ rational relation where $\Omega = \tau_1 \cup \tau_2 \cup \Xi$, Ξ is a finite alphabet and τ_1 and τ_2 are two distinct letters not in Ξ. Then one of the following two possibilities prevails

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1) Pour Eilenberg, D_1 est le semi-Dyck sur 1 lettre, ce que nous notons habituellement $D_1'^*$ –

(i) There exist an integer $k \geq 0$ such that if

$$\tau_1^{m_1} \times \tau_2^{m_2} \in f D_1 \quad , \quad x \in \Xi^*$$

then either $m_1 \leq k$ or $m_2 \leq k$.

(ii) There exist integers $m_1, m_2, \ell_1, \ell_2 \in N$, $\ell_1 + \ell_2 > 0$, and an element $x \in \Xi$ such that

$$\tau_1^{m_1 + \ell_1 n} \times \tau_2^{m_2 + \ell_2 n} \in f D_1$$

for all $n > 0$.

Proof. Let $\mathfrak{A} = (Q, I, T)$ be a $\bar{\sigma} \vee \sigma \vee \Omega$ automaton computing the relation f. Let p and q be a pair of states of $\mathfrak{A}$. We define the automata $\mathfrak{A}_p$, ${}_p\mathfrak{A}_q$, ${}_q\mathfrak{A}$ as follows

$$\mathfrak{A}_p = (Q, I, p) \quad , \quad {}_q\mathfrak{A} = (Q, q, T)$$

$${}_p\mathfrak{A}_q = (Q, p, q)$$

Further in $\mathfrak{A}_p$ all edges starting at p as well as all edges carrying a label $\xi \neq \xi_1$ are removed. Thus $\mathfrak{A}_p$ is a $\{\bar{\sigma}, \sigma, \xi_1\}$-automaton with p as exit. Similarly in ${}_q\mathfrak{A}$ all edges terminating at q and all edges carrying a label $\xi \neq \xi_2$ are removed. Thus ${}_q\mathfrak{A}$ is a $\{\bar{\sigma}, \sigma, \xi_2\}$-automaton with q as entry. In ${}_p\mathfrak{A}_q$ all edges with labels τ_1 or τ_2 are removed. Thus ${}_p\mathfrak{A}_q$ is a $\bar{\sigma}, \sigma \cup \Xi$ -automaton.

Consider the composite automaton

(.1) $(\mathfrak{A}_p)({}_p\mathfrak{A}_q)({}_q\mathfrak{A})$

and let

$$f_{pq} : \Sigma^{\square} \longrightarrow \Omega^{*}$$

be the relation that it computes.

The following assertion is clear

$$\tau_1^* \Xi^* \tau_2^* \cap f D_1 = \bigcup f_{pq} D_i$$

the union extended over all pairs (p.q) of states of $\mathfrak{A}$. In view of this it suffices to prove the conclusion for each of the relations f_{pq}.. Thus we may assume that $f = f_{pq}$ and that $\mathfrak{A}$ is the composite automaton (.).

Let D_1' be the set of all initial segments of D_1 and consider the set

$$A_1 = (D_1' \sqcup \tau_1^*) \cap |\mathfrak{A}_p|$$

Since D_1' is algebraic the set A_1 is algebraic. Define the "norm"

$$\delta_1 : \{\bar{\sigma}, \sigma, \tau_1\}^* \longrightarrow N$$

by $\delta_1\bar{\sigma} = \delta_1\sigma = 0$, $\delta\tau_1 = 1$. By the iteration ..... there exists an integer $k_1 \geqq 0$ such that if $a_1 \in A_1$ and $\delta_1 a_1 > k_1$ then a_1 admits a factorization

(.2) $\quad a_1 = b_1 c_1 d_1 e_1 f_1 \ , \quad \delta_1 c_1 + \delta_1 e_1 > 0$

such that

(.3) $\quad b_1 c_1^n d_1 e_1^n f_1 \in A_1$

for all $n > 0$.

Similarly let D_1'' be the set of all terminal segments of D_1, and let

$$A_2 = (D_2'' \sqcup \tau_2^*) \cap |_g \Omega|$$

Then A_2 is algebraic. Using the norm $\delta_2 : \{\bar{\sigma}, \sigma, \tau_2\}^* \to N$ given by $\delta_2 \bar{\sigma} = \delta_2 \sigma = \delta_2 \tau_2 = 1$, we obtain an integer $k_2 \geqq 0$ such that if $a_2 \in A_2$ and $\delta_2 a_2 > k_2$ then

(.4) $\quad a_2 = f_2 e_2 d_2 c_2 b_2 \ , \quad \delta_2 e_2 + \delta_2 c_2 > 0$

such that

(.5) $\quad f_2 e_2^n d_2 c_2^n b_2 \in A_2$

for all $n > 0$.

Let $k = \sup(k_1, k_2)$, and suppose that (contrary to (2)) there is an element

$$\tau_1^{m_1} x \tau_2^{m_2} \in f D_1 \quad , \quad x \in \Xi^*$$

with $k < m_1$ and $k < m_2$. Then there exist elements

$$a_1 \in A_1, \quad a \in |_p \mathfrak{O}_g| \quad , \quad a_2 \in A_2$$

such that

$$a_1 a a_2 \in D_1 \sqcup \tau_1^{m_1} x \tau_2^{m_2}$$

Consequently

$$\delta_1 a_1 = m_1 > k_1 \quad , \quad \delta_2 a_2 = m_2 > k_2$$

Since

$$a_1 \in A_1 \quad , \quad a_2 \in A_2$$

we have factorizations (.2) and (.4 satisfying (.3) and (.5).

Now extend the function $\gamma : \Sigma^\square \to Z$ to $\gamma : \Sigma^\square \sqcup \Omega^* \longrightarrow Z$ by setting $\gamma \tau_1 = \gamma \tau_2 = \gamma \xi = 0$ for all $\xi \in \Xi$.

Condition (.3) implies

$$\gamma c_1 \geqq 0 \quad , \quad \gamma c_1 + \gamma e_1 \geqq 0$$

Assume $\gamma c_1 + \gamma e_1 = 0$. Then

$$\gamma a_1 = \gamma (b_1 c_1^n d_1 e_1^n f_1)$$

for all $n > 0$. It follows that

$$b_1 c_1^n d_1 e_1^n f_1 a a_1$$

is in $|O\mathcal{L}|$ and also in $D_1 \sqcup \tau_1^* \equiv^* \tau_2^*$

Consequently

$$\tau_1^{m_1 + \ell_1 n} \times \tau_2^{m_2} \in f D_1$$

for all $n > 0$ where $\ell_1 = \delta_1 c_1 + \delta_1 e_1 > 0$;

thus (22) holds. We may therefore

assume

$$\gamma c_1 + \gamma c_1 = r_1 > 0 .$$

Similarly condition (.3) implies

$$\gamma c_2 \leqq 0 \quad , \quad \gamma c_2 + \gamma e_2 \leqq 0$$

and by an argument dual to the

above we may assume

$$\gamma c_2 + \gamma e_2 = - r_2 < 0$$

The elements

$$b_1 c_1^{r_2 n} d_1 e_1^{r_2 n} f_1 a f_2 e_2^{r_1 n} d_2 c_2^{r_2 n} b_2$$

are then in $|0 2|$ and also in

$D_1 \sqcup \tau_1^* \Xi^* \tau_2^*$. Consequently

$$\tau_1^{m_1 + \ell_1 n} \times \tau^{m_2 + \ell_2 n} \in f D_1$$

for all $n > 0$ where

$$\ell_1 = r_2(\delta_1 c_1 + \delta_1 e_1), \quad \ell_2' = r_1(\delta_2 c_2 + \delta_2 e_2)$$