1. The filtered fixed-point theorem

A <u>filtered set</u> is defined to be a set X equipped with a sequence $\sim_n$, $n \geq 0$, of equivalence relations and satisfying the following properties

(1.1) $x \sim_0 x'$ for all $x, x' \in X$

(1.2) $x \sim_{n+1} x'$ implies $x \sim_n x'$

(1.3) $x \sim_n x'$ for all $n \geq 0$ implies $x = x'$

(1.4) Given a sequence $\{x_n \mid n \geq 0\}$ in X such that $x_n \sim_n x_{n+1}$ for $n \geq 0$, there exists an element $x \in X$ such that $x \sim_n x_n$ for all $n \geq 0$

It follows from (1.3) that the element x asserted in (1.4) is unique. A sequence $\{x_n \mid n \geq 0\}$ as in (1.4) will be called an ascending sequence and we shall write

$$x = \lim x_n$$

Let X and Y be filtered sets and let $f : X \to Y$ be a function. We shall say that f preserves filtration if

$$x \sim_n x' \quad \text{implies} \quad xf \sim_n x'f$$

We shall say that f is <u>shrinking</u> if

$$x \sim_n x' \quad \text{implies} \quad xf \sim_{n+1} xf.$$

Proposition 1.1. If $f : X \to Y$ preserves filtration and $\{x_n \mid n \geq 0\}$ is an ascending sequence in X, then $\{x_n f \mid n \geq 0\}$ is an ascending sequence in Y and $\lim (x_n f) = (\lim x_n) f$.

Proof. Since $x_n \sim_n x_{n+1}$ we have $x_n f \sim_n x_{n+1} f$ so that $\{x_n f \mid n \geq 0\}$ is ascending. Let $x = \lim x_n$. Then $x \sim_n x_n$ and therefore $xf \sim_n x_n f$. Thus $xf = \lim (x_n f)$ ∎

Theorem 1.2. Let $f : X \to X$ be a shrinking transformation of a filtered set X into itself. For any $x \in X$, the sequence $\{xf^n \mid n \geq 0\}$ is an ascending sequence and $\lim xf^n$ is a fixed point for f. This fixed point is independent of the choice of x and is the only fixed point of f.

Proof. Since $x \sim_0 xf$ and f is shrinking it follows that $xf \sim_1 xf^2$ and by induction $xf^n \sim_n xf^{n+1}$. Thus the sequence $\{xf^n / n \geq 0\}$ is ascending. Let $\bar{x} = \lim(xf^n)$. Then by Proposition 1.1, $\bar{x}f = \lim(xf^{n+1}) = \bar{x}$. If x' is another fixed point for f, then $\bar{x} \sim_0 x'$ and this implies

$$\bar{x} = \bar{x}f \sim_1 x'f = x'$$

and by induction $\bar{x} \sim_n x'$ for all $n \geq 0$. Thus $\bar{x} = x'$ and the fixed point is unique ∎

Theorem 1.2 is actually a special case of a theorem in general topology that is widely used and is attributed to Banach. The theorem deals with a complete metric space X and a transformation $f : X \to X$ satisfying

(1.5) $$(xf, x'f)\varrho \leq k\,(x, x')\varrho$$

for some $0 \leq k < 1$. Here $(x, x')\varrho$ denotes the distance between x and x'. The conclusion is that f has a unique fixed-point and that for any $x \in X$, the sequence $\{xf^n\}$ converges to this fixed-point.

To deduce Theorem 1.2 from the above one introduces a distance into the filtered set X by setting

$$(x, x')\varrho = \frac{1}{2^n} \quad \text{if } x \sim_n x' \text{ but not } x \sim_{n+1} x'$$

$$(x, x)\varrho = 0$$

It take very little work to show that the resulting metric space is complete, thanks to condition (1.4). Any shrinking transformation $f : X \to X$ satisfies (1.5) with $k = \frac{1}{2}$. This insures the existence and uniqueness of the fixed-point etc.

If X and Y are filtered sets then a filtration is defined in $X \times Y$ by setting:

$$(x, y) \sim_n (x', y') \text{ iff } x \sim_n x' \text{ and } y \sim_n y'$$

If Z is another filtered set, then a function $f : Z \to X \times Y$ is filtration preserving (or shrinking) if each of the two components $Z \to X$, $Z \times Y$ is filtration preserving (or shrinking).

2. Polynomials and paranomials

Let S be a loose semigroup and let K be a commutative semiring. We consider the set K^S of all K-subsets A of S. Thus each A is a function $A : S \to K$. For $A, B \in K^S$ we define $A+B$ and AB as follows

$$s(A+B) = sA + sB$$

$$s(AB) = \sum_{s=uv} (uA)(vB)$$

Note that the last summation is finite since s has only a finite number of factorizations $s = s_1 s_2$. For $k \in K$ we also define $kA \in K^S$ by setting

$$s(kA) = k(sA)$$

These operations satisfy all the axioms of a K-algebra except for the existence of a unit element.

We introduce a filtration in K^S by defining $A \sim_n B$ iff $sA = sB$

for all $s \in S$ such that $|s| \leq n$. The verification of axioms $(1.1)-(1.3)$ is immediate. To verify (2.4) assume that $\{A_n, n \geq 0\}$ is ascending, i.e. that $A_n \sim_n A_{n+1}$. Define A by setting

$$sA = sA_{|s|}$$

Then $A_n \sim_n A$ for all $n \geq 0$.

Proposition 2.1. The function

$$K^S \times K^S \to K^S$$

given by addition is filtration preserving, while the function defined by multiplication is shrinking.

Proof. Let $A \sim_n A'$ and $B \sim_n B'$. We must prove

$$A + B \sim_n A' + B'$$

$$AB \sim_{n+1} A'B'$$

The first relation is clear. To prove the second one assume $|s| \leq n+1$. Then

$$s(AB) = \sum_{s=uv} (uA)(vB)$$

However $s = uv$ implies $|u| \leq n$ and $|v| \leq n$. Thus $uA = uA'$, $uB = uB'$ and consequently $s(AB) = s(A'B')$ ∎

As usual each element $s \in S$ will also be viewed as an element of K^S by identifying s with the function $S \to K$ which assumes the value 1 on s and 0 elsewhere. The elements of K^S of the form ks with $k \in K$, $s \in S$ are called <u>monomials</u>. Any finite sum of monomials is called a <u>polynomial</u>. The limit of an ascending sequence of polynomials is called a paranomial. The set of all paranomials is denoted by $\langle S, K \rangle$.

Proposition 2.2. For any $A \in S^K$ the following conditions are equivalent

(i) $A \in \langle S, K \rangle$

(ii) For each $n \geq 0$ there exists a polynomial P_n such that $A \sim_n P_n$

(iii) For each $n \geq 0$ the set

$$\{ s \mid sA \neq 0, \ |s| \leq n \}$$

is finite.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. Assume (iii) and define $P_n \in K^S$ by setting

$$sP_n = \begin{cases} sA & \text{if } |s| \leq n \\ 0 & \text{if } |s| > n \end{cases}$$

Then by (iii) P_n is a polynomial. Further, $\{P_n, n \geq 0\}$ is an ascending sequence and $A = \lim P_n$. Thus $A \in \langle S, K \rangle$ ∎

~~Proposition 4.3. Let $\{A_n, n \geq 0\}$ be an ascending sequence in K^S and let~~

Proposition 2.3. Let $A = \lim A_n$ be the limit of an ascending sequence in K^S. If $A_n \in \langle S, K \rangle$ for all $n \geq 0$, then $A \in \langle S, K \rangle$.

Proof. Define P_n as follows

$$s\,P_n = \begin{cases} sA & \text{if } |s| \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then $P_n \sim_n A \sim_n A_n$. Since $A_n \in \langle S, K \rangle$ it follows that P_n is a polynomial. Since $A = \lim P_n$ it follows that $A \in \langle S, K \rangle$ ∎

Clearly $\langle S, K \rangle$ is closed under the algebraic operations that we defined in K^S. Henceforth we shall only be concerned with the set $\langle S, K \rangle$ together with its algebraic operations and filtration.

3. Polynomial and paranomial transformations.

Let S be a loose semigroup and let $\Xi = \{\xi_1, \dots, \xi_n\}$. We consider the loose semigroup $S[\Xi]$. Given $w \in S[\Xi]$ we define a function

$$\hat{w} : \langle S, K \rangle^n \longrightarrow \langle S, K \rangle$$

as follows: for each $X \in \langle S, K\rangle^n$

$$X\hat{s} = s \quad \text{for } s \in S$$

(3.1) $$X\hat{\xi}_i = X_i$$

$$X(\widehat{uv}) = (X\hat{u})(X\hat{v})$$

Thus $X\hat{w}$ is obtained from w by replacing each ξ_i in w by X_i and carrying out the multiplication. We claim

(3.2) $$s(X\hat{w}) = 0 \quad \text{whenever } |s| < |w|$$

Indeed assume $|w| = p$. Then $w = u_1 \cdots u_p$ and $Xw = B_1 \cdots B_p$ with $B_i = X\hat{u}_i$. If $|s| < p$ then necessarily $s(B_1 \cdots B_p) = 0$.

Now, given a paranomial

$$\Lambda = A \in \langle S[\Xi], K \rangle$$

we define the parcinomial transformation

$$(3.3) \qquad \hat{A} : \langle S, K\rangle^n \longrightarrow \langle S, K\rangle$$

by setting for $X \in \langle S, K\rangle^n$

$$(3.4) \qquad s(X\hat{A}) = \sum_{w \in S[\Xi]} (wA)\, s(X\hat{w}) \cancel{\neq \sum_{\substack{w \in S[\Xi] \\ |w| \leq |s|}} (wA)\, s(X\hat{w})}$$

Because of (3.2) the summation may be restricted by the condition $|w| \leq |s|$ and therefore the summation is finite. If A is a polynomial, then (3.3) is called a polynomial transformation. Whenever there is no danger of confusion, we shall write w and A instead of $\hat{w}$ and $\hat{A}$.

Proposition 3.1. For all $A \in \langle S[\Xi], K\rangle$, the transformation (3.3) is filtration preserving. If further, $\xi_i A = 0$ for all $1 \leq i \leq n$ then (3.3) is shrinking.

Proof. Let $X \sim_n Y$ i.e. $X_i \sim_n Y_i$ for all $1 \leq i \leq n$. To prove that $X\hat{A} \sim_n Y\hat{A}$ (or $X\hat{A} \sim_{n+1} Y\hat{A}$ it suffices by formula

13

(3.4), to show that $X\hat{w} \sim_n Y\hat{w}$ (or $Xw \sim_{n+1} Yw$) for all $w \in S[\Xi]$ such that $wA \neq 0$. Thus it suffices to prove

Proposition 3.2. For each $w \in S[\Xi]$ the transformation

$$(3.5) \qquad w : \langle S, K\rangle^n \longrightarrow \langle S, K\rangle$$

is filtration preserving, and is shrinking if $w \neq \xi_i$.

Proof. First consider the case $|w| = 1$. Then either $w = \xi_i$ or $w = s \in S$ and $|s| = 1$. If $w = \xi_i$,

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then $Xw = X_i$ for all $X \in \langle S, K \rangle^n$ and thus (3.5) is filtration preserving. If $w = s \in S$ then $Xw = s$ for all $X \in \langle S, K \rangle^n$, so that (3.5) is shrinking.

Now assume ~~[illegible]~~ the the conclusion of Proposition 3.1 has already been established for all words $w \in S[\Xi]$ with $|w| < n$ $(n > 1)$ and consider the case $|w| = n$. Then $w = uv$ with $|u| < n$, $|v| < n$. Then (3.5) is the composition

$$\langle S, K \rangle^n \xrightarrow{f} \langle S, K \rangle^2 \xrightarrow{g} \langle S, K \rangle$$

with

$$Xf = (Xu, Xv)$$
$$(A, B)g = AB$$

Since u and v define filtration preserving transformations, f is filtration preserving. However, by Proposition 2.1, g is shrinking. Thus fg is shrinking and so is (3.5) ∎

4. The Main Theorem.

We consider a positive (S, K)-grammar $G = (\Xi, R)$ with $\Xi = \{\xi_1, \dots, \xi_n\}$. Thus S is a loose semigroup and K is a commutative semiring. The grammar G has no specified start state, and we denote by L_i the language defined by the grammar (Ξ, ξ_i, R).

We first show that

$$L_i \in \langle S, K\rangle \quad \text{for } 1 \leq i \leq n.$$

By Proposition 2.2 we must verify that for each $p \geq 0$, the set

(4.1) $\quad \{ s \mid sL_i \neq 0 , |s| \leq p \}$

is finite. Indeed, assume $sL_i \neq 0$. There is then a derivation $d : \xi_i \to s$. From Proposition I,4.1 we deduce $|d| \leq 2|s| \leq 2p$. Since there is only a finite number of such derivations, the finiteness of (4.1) follows.

~~We shall be interested in the language vector~~

The result just established yields the inclusion

(4.2) $$K\mathrm{Alg}_S \subset \langle S, K \rangle.$$

Returning to the grammar G, we shall be interested in the <u>language vector</u>.

$$L = (L_1, \ldots L_n) \in \langle S, K \rangle^n$$

The set of rules R, viewed as a K-subset of $\Xi \times S[\Xi]$ may be writtend down as follows

$$R = \xi_1 \times P_1 + \cdots + \xi_n \times P_n$$

where

$$P_i = \sum_{\bar{r} = \xi_i} (r\mu)\, \underline{r} \in \langle S, K \rangle$$

The polynomials $P_1, \ldots, P_n$ determine R completely. Each of the polynomials P_i defines a transformation

(4.3) $$\underline{P}_i : \langle S, K \rangle^n \longrightarrow \langle S, K \rangle$$

Since the grammar G is assumed to

be positive, we have $\xi \underline{P}_i = 0$ for all $\xi \in \Xi$ and thus the transformation (4.3) is shrinking. There results a shrinking transformation

$$\underline{P} : \langle S, K \rangle^n \longrightarrow \langle S, K \rangle^n$$

$$X \underline{P} = (X \underline{P}_1, \ldots, X \underline{P}_n)$$

This is the polynomial transformation associated with the grammar G. Since this transformation is shrinking, Theorem 1.2 implies that $\underline{P}$ has a unique fixed point that we shall denote by $\underline{P}^{\#}$.

Theorem 4.1. The language vector $L \in \langle S, K \rangle^n$ determined by the positive (S, K)-grammar $G = (\Xi, R)$ with $\Xi = \{\xi_1, \ldots, \xi_n\}$ is the unique fixed point $\underline{P}^{\#}$ of the polynomial transformation

$$P : \langle S, K \rangle^n \longrightarrow \langle S, K \rangle^n$$

defined by the grammar.

Proof. We must prove

$$L = L P$$

or equivalently

$$L_i = L P_i \quad \text{for } 1 \le i \le n.$$

We have

$$s L_i = \sum_{\xi_i e = s} e\mu$$

the summation extending over all derivation

$$e : \xi_i \longrightarrow s$$

Each such derivation admits a unique factorization $e = rd$ where

$$\xi_i \xrightarrow{r} \underline{r} \xrightarrow{d} s$$

It follows that

$$s L_i = \sum (r\mu)(s L_{\underline{r}})$$

where for any $w \in S[\Xi]$ we define

$$(4.4) \qquad s L_w = \sum_{wd = s} d\mu$$

Since

$$s(L P_i) = \sum_{\bar{r} = \xi_i} (r\mu)\, s(L \underline{r})$$

the required equality $L_i = L P_i$ will follow if we prove $L_{\underline{\Gamma}} = L\Gamma$ or more generally $L_w = Lw$ for every $w \in S[\Xi]$.

If $w = t \in S$, then by (3.13) $Lw = t$. Formula (4.4) yields $L_w = t$ so that $L_w = Lw$ in this case. If $w = \xi_j \in \Xi$ then by (3.13) $Lw = L_j$. Formula (4.4) yields $L_w = L_j$ so that $Lw = L_w$ also in this case. Suppose now that $w = uv$ and assume the equalities $Lu = L_u$, $Lv = L_v$. Then by (3.13)

$$Lw = L(uv) = (Lu)(Lv) = L_u L_v$$

To conclude the proof it therefore suffices to establish $L_u L_v = L_{uv}$. This is a direct consequence of the definition (4.4) and of Proposition I, 1.1.

Corollary 4.2. The sequence $\underline{O}\underline{P}^{\delta}$ is ascending and

$$L = \lim_{\delta \to \infty} \underline{O}\underline{P}^{\delta}.$$

This follows from Theorem 1.2 since

$\underline{P}^{\#} = \lim \underline{O}\underline{P}^{\delta}$ ∎

Exercise 4.1. Let Σ be any ~~finite~~ alphabet. Show that the equation

$$X = X^2$$

for $X \in \mathcal{N}^{\Sigma^*}$ has exactly the following solutions

$$X = \emptyset$$
$$X = 1 + \infty S$$
$$X = \infty 1 + \infty S$$

where S is an arbitrary subsemigroup of Σ^+ (including $S = \emptyset$ and $S = \Sigma^+$). ~~This shows that the condition in Corollary 4.4 is essential for the uniqueness of solutions.~~

The only solution of the above equation with $X \in \mathcal{N}^{\Sigma^+}$ is $X = \emptyset$.