Transitive factorizations of free partially commutative monoids and Lie algebras

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Résumé

Soit $\mathbb{M}(A,\theta)$ un monoïde partiellement commutatif libre. Nous donnons une condition nécessaire et suffisante sur un sous alphabet $B \subset A$ pour que le facteur droit d'une bisection de la forme $\mathbb{M}(A,\theta) = \mathbb{M}(B,\theta_B).T$ soit partiellement commutatif libre. Ceci nous permet d'étendre strictement et de façon optimale la théorie (classique) de l'élimination avec commutations partielles et de construire de nouvelles factorisations de $\mathbb{M}(A,\theta)$ ainsi que les bases de $L_K(A,\theta)$ associées.

1 Introduction

A factorization of a monoid is a direct decomposition

$$M = \prod_{i\in I}^{\leftarrow} M_i$$

where M and the M_i are monoids and I is totally ordered. This notion is due to Schützenberger (see [16,17] where the link with the free Lie algebra is studied). Then, in his Ph. D. [19], Viennot showed how combinatorial bases of the free Lie algebra could be constructed by composition of bisections (i.e. $|I| = 2$) obtained by elimination of generators (ideas initiated by Lazard [13] and Shirshov [18]). One of the authors with D. Krob found similar decompositions for the free partially commutative monoid into free factors and studied the link with Lie algebras and groups [6]. This works generalizes the completely free case, but is restricted to the situation where the outgoing factors are also free.

Here, we study the general problem of eliminating generators in these structures and first remark that, in any (set theoretical) direct decomposition

$$M(A,\theta) = M(B,\theta_B).T$$

(with $B \subset A$, a subalphabet), the complement is a monoid. We get a criterion to characterize the case when T is free partially commutative and construct bases of the associated Lie algebras. The case of the group is also mentionned.

2 Definitions and background

We recall that the free partially commutative monoid is defined by generators and relations as

$$\mathbb{M}(A,\theta) = \langle A | ab = ba, (a,b) \in \theta \rangle_{Mon},$$

where A is an alphabet and $\theta \subset A \times A$ is an antireflexive (i.e. without loops) and symmetric graph on A (θ is called an independence relation). Thus, $\mathbb{M}(A,\theta)$ is the quotient $A^*/_{\equiv_\theta}$ where $\equiv_\theta$ is the congruence generated by the set $\{(ab,ba) | (a,b) \in \theta\}$.

Definition 1 *If X is a subset of $\mathbb{M}(A,\theta)$, we set*

$$\theta_X = \{(x_1,x_2) \in X^2 | Alph(x_1) \times Alph(x_2) \subset \theta\}.$$

Note that $(x_1,x_2) \in \theta_X$ implies $\mathrm{Alph}(x_1) \cap \mathrm{Alph}(x_2) = \emptyset$, similarly we set $\theta_{\mathbb{M}} = \theta_{\mathbb{M}(A,\theta)}$.
As in [7], we denote $IA(t) = \{z \in A | t = zw\}$ and $TA(t) = \{z \in A | t = wz\}$.
If X is a subset of $\mathbb{M}(A,\theta)$, we denote $\langle X \rangle$ the submonoid generated by X.
In [3] and [4], Choffrut introduces the partially commutative codes as some generating sets of free partially commutative submonoids. Let X be a set, we can prove easily that this definition is equivalent to the fact that each trace $t \in \langle X \rangle$ admits a unique decomposition on X up to the commutations (i.e. (X,θ_X) is the independence alphabet of $\langle X \rangle$ the submonoid generated by X).

Example 1 *(i) Each subalphabet B of A is a partially commutative code.*
(ii) Let $(A,\theta) = a{-}b \quad c$. The set $\{c, cb, ca\}$ is a code but not the set $\{b, a, ca, cb\}$.

3 Transitive bisections

3.1 Generalities

We recall the definition of a factorization in the sense of Schützenberger (cf. Viennot in [19] and [20]), this notion will be reused extensively at the end of the paper.

Definition 2 *(i) Let $\mathbb{M}$ be a monoid and $(\mathbb{M}_i)_{i\in J}$ an ordered family of submonoids (the total ordering on J will be denoted $<$). The family $(M_i)_{i\in J}$ will be called a* factorization *of $\mathbb{M}$ if and only if every $m \in \mathbb{M}^+ = \mathbb{M} - \{1\}$ has a unique decomposition*

$$m = m_{i_1} m_{i_2} \dots m_{i_n}$$

with $i_1 > i_2 > \dots > i_k$ and for each $k \in [1..n]$, $m_{i_k} \in \mathbb{M}^+_{i_k}$.
(ii) In the case of a free partially commutative monoid, a factorization will be denoted by the sequence of the minimal generating sets of its components.

In the maximal case (each monoid has a unique generator), the factorization is called *complete*.

Example 2 *(Complete factorizations in free and free partially commutative monoids.)*
In the free monoid, it exists many complete factorizations. The most famous of this kind being the Lyndon factorization (defined as the set of primitive words minimal in their conjugacy classes) is an example of a complete factorization [14–16]. Hall sets defined in [17] give us a wider example.
The set of Lyndon traces (i.e. the generalization of Lyndon words to the partially commutative case, defined by Lalonde in [11]) endowed with the lexicographic ordering is a complete factorization of the free partially commutative monoid.

In the smallest case ($|J| = 2$), the factorization is called a *bisection*. Let M be a monoid, then (M_1, M_2) is a bisection of M if and only if the mapping

$$M_1 \times M_2 \to M$$

$$(m_1, m_2) \to m_1 m_2$$

is one to one.

Remark 1 *Not every submonoid is a left (right) factor of a bisection. If $M = M_1 M_2$ is a bisection then M_1 satisfies $(u, uv \in M_1) \Rightarrow (v \in M_1)$ (see [5]), however, this condition is not sufficient as shown by $M_1 = 2\mathbb{Z} \subset \mathbb{Z} = M$.*

In case $M = \mathbb{M}(A, \theta)$, one can prove the following property.

Proposition 3 *Let (A, θ) be an independence relation and $B \subset A$. Then $\mathbb{M}(B, \theta_B)$ is the left (resp. right) factor of a bisection of $\mathbb{M}(A, \theta)$.*

Proof We treat, here, only the left case, the right case being symmetrical. It is clear that $M = \{t \in \mathbb{M}(A,\theta) | IA(t) \subset A - B\}$ is always a monoid and that we have the (set theoretical) equality $\mathbb{M}(A,\theta) = \mathbb{M}(B,\theta_B).M$. It suffices to prove the unicity of the decomposition of a trace. Let $w, w' \in \mathbb{M}(B,\theta_B)$ and $t, t' \in M$ such that $wz = w'z'$. Using Levi's lemma, we find four traces p, q, r, s such that $w = ps$, $t = rq$, $w' = pr$ and $t' = sq$. But, by definition of M, we have $uv \in M$ implies $u \in M$, then $r, s \in M \cap \mathbb{M}(B,\theta_B) = \{1\}$. It follows $w = w'$ and $t = t'$, which gives the result.

□

In the sequel, we denote $Z = A - B$.
In the left case, the right submonoid above has

$$\beta_Z(B) = \{zw/z \in Z, w \in \mathbb{M}(B,\theta_B), IA(zw) = \{z\}\}$$

as minimal generating subset.

Remark 2 *The monoid $\langle\beta_Z(B)\rangle$ may not be free partially commutative. For example, if $A = \{a, b, c\}$,*

$$\theta : a - b \quad c$$

and $B = \{c\}$ then $a, b, ac, bc \in \beta_Z(B)$ and $a.bc = b.ac$.

3.2 Transitively factorizing subalphabet

Here we discuss a criterium for the complement $\langle\beta_Z(B)\rangle$ to be a free partially commutative submonoid.

Definition 4 *Let $B \subset A$, we say that B is a* transitively factorizing subalphabet *(TFSA) if and only $\beta_Z(B)$ is a partially commutative code.*

We prove the following theorem.

Theorem 5 *Let $B \subset A$. The following assertions are equivalent.*

(i) The subalphabet B is a TFSA.
(ii) The subalphabet B satisfies the following condition.
For each $z_1 \neq z_2 \in Z$ and $w_1, w_2, w'_1, w'_2 \in \mathbb{M}(A,\theta)$ such that $IA(z_1w_1) = IA(z_1w'_1) = \{z_1\}$ and $IA(z_1w_2) = IA(z_2w'_2) = \{z_2\}$ we have

$$z_1w_1z_2w_2 = z_2w'_2z_1w'_1 \Rightarrow w_1 = w'_1, w_2 = w'_2.$$

(iii) For each $(z,z') \in Z^2 \cap \theta$, the dependence[1] *graph has no partial graph*[2] *like*

$$z - b_1 - \ldots - b_n - z'.$$

with $b_1, \ldots, b_n \in B$.

Proof It is easy to see that (i)⇒(ii) : by contraposition, if B does not satisfy (ii) we can find $z_1w_1, z_2w_2, z_1w'_1, z_2w'_2 \in \beta_Z(B)$ such that $z_1w_1.z_2w_2 = z_2w'_2.z_1w'_1$ with $w_1 \neq w'_1$ or $w_2 \neq w'_2$ and this implies obviously that $\beta_Z(B)$ is not a partially commutative code.
Let us prove that (ii)⇒(iii). Suppose that

$$z - b_1 - \ldots - b - n - z'$$

is a partial graph of the dependence graph, then it exists a subgraph of the dependence graph of the form

$$z - c_1 - \ldots - c_m - z'$$

with $c_i \in B$. Consider the smallest integer k such that $(c_{k+1}, z') \notin \theta$. Then we have $zc_1 \ldots c_k.z'c_{k+1} \ldots c_m = z'.zc_1 \ldots c_m$, which proves that B does not satisfy (ii).

Finally, we prove that (iii)⇒ (i). For each $z \in Z$, we define B_z the set of letters of B having in the dependence graph a path leading to z and all inner points belonging to B. Clearly the assertion (iii) is equivalent to the fact that $(z,z') \in \theta_Z$ implies $(\{z\} \cup B_z) \times (\{z'\} \cup B_{z'}) \subseteq \theta$. It follows that $\beta_z(B) \times \beta_{z'}(B) \subset \theta_{\mathbb{M}}$. Consider the mapping μ from Z into $K\langle\langle A, \theta\rangle\rangle$ defined by $\mu(z) = \underline{\beta_z(B)}$. As $(z,z') \in \theta_Z \Rightarrow [\mu(z), \mu(z')] = [\underline{\beta_z(B)}, \underline{\beta_{z'}(B)}] = 0$ and $\langle \mu(z), 1\rangle = \overline{0}$[3], we can extend μ as a continuous morphism from $K\langle\langle Z, \theta_Z\rangle\rangle$ in $K\langle\langle A, \theta\rangle\rangle$. Let s be the morphism from $\langle\beta_z(B)\rangle$ in $\mathbb{M}(Z, \theta_Z)$ defined by $s(zw) = z$ for each $zw \in \beta_Z(B)$. We have

$$\begin{aligned} \underline{\langle\beta_Z(B)\rangle} = \; & s^{-1}(\underline{\mathbb{M}(Z,\theta_Z)}) \; = \textstyle\sum_{w \in \mathbb{M}(Z,\theta_Z)} s^{-1}(w) \\ & = \textstyle\sum_{w \in \mathbb{M}(Z,\theta_Z)} \mu(w) = \quad \mu(\underline{\mathbb{M}(Z,\theta_Z)}) \end{aligned}$$

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[1] The dependence graph is defined by $A \times A - \Delta - \theta$ where $\Delta = \{(a,a)/a \in A\}$.
[2] We repeat here the notion of partial graph. A graph $G' = (S', A')$ is a partial graph of $G = (S, A)$ if and only if $S' \subset S$ and $A' \subset A \cap S' \times S'$ (G' is a subgraph of G when equality $S = S'$ occurs).
[3] Here, for a series $S = \sum \alpha_u u$, we denote $\langle S, w\rangle = \alpha_w$.

Let $P(\theta_Z)$ be the polynomial such that

$$\underline{\mathbb{M}(Z, \theta_Z)} = \frac{1}{P(\theta_Z)}.$$

As μ is a continuous morphism, we have

$$\underline{\langle \beta_Z(B) \rangle} = \frac{1}{\mu(P(\theta_Z))} = \frac{1}{P(\theta_{\beta_Z(B)})}$$

which is the characteristic series of $\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})$.

$\square$

Remark 3 *(i) Elimination in [7] deals with the particular case when $A - B$ is totally non commutative. In this case B is a TFSA of A.*
(ii) As an example of other case, consider the independence alphabet due to the graph

$$\theta = a - b - c.$$

The monoid $\langle \beta_{a,b}(c) \rangle$ is free partially commutative, its alphabet is $\beta_{a,b}(c) = \{b\} \cup \{ac^n / n \geq 0\}$, its independence graph is

$$\theta_{\beta_{a,b}(c)} = \begin{array}{ccccc} & & ac & & \\ & & | & \ddots & \\ a & - & b & - & ac^n \\ & & & & \vdots \end{array}$$

and

$$\underline{< \beta_Z(B) >} = \frac{1}{1 - \left(b + \sum_{n \geq 0} ac^n\right) + \sum_{n \geq 0} abc^n}.$$

4 Factorizations and bases of free partially commutative Lie algebra

4.1 Transitive factorizations

We recall some definitions given by Viennot in [19].

Definition 6 *Let $\mathbb{M}$ be a monoid, $\mathbb{M}'$ a submonoid of $\mathbb{M}$ and $\mathbb{F} = (\mathbb{M}_i)_{i\in J}$ a factorization of $\mathbb{M}$. We denote $\mathbb{F}|_{\mathbb{M}'} = (\mathbb{M}_{i_k})_{k\in K}$ where $K = \{k \in J | \mathbb{M}_k \subseteq \mathbb{M}'\}$ (in the general case it is not a factorization).*

Definition 7 *Let $\prec$ be the partial order on the set of all the factorizations of a monoid $\mathbb{M}$ defined by $\mathbb{F} = (\mathbb{M}_i)_{i\in J} \prec \mathbb{F}' = (\mathbb{M}'_i)_{i\in J'}$ ($\mathbb{F}'$ is finer than $\mathbb{F}$) if and only if J' admits a decomposition $J' = \sum_{i\in J} J_i$ as an ordered sum of intervals such that for each $i \in J$, $(\mathbb{M}'_j)_{j\in J_i}$ is a factorization of $\mathbb{M}_i$.*

The following property is straightforward.

Proposition 8 *Let $\mathbb{F} = (\mathbb{M}_i)_{i\in I}$ be a factorization and $\mathbb{F}'$ be a factorization such that $\mathbb{F} \preceq \mathbb{F}'$ then for each $i \in I$, $\mathbb{F}'|_{\mathbb{M}_i}$ is a factorization of $\mathbb{M}_i$.*

Definition 9 *Let $\mathbb{B} = (B_1, B_2)$ be a bisection and $\mathbb{F} = (Y_i)_{i\in J}$ a factorization. We say that Y_i is* cut *by $\mathbb{B}$ if and only if $\mathbb{L}_i(\mathbb{B}) = \langle B_1\rangle \cap \langle Y_i\rangle$ and $\mathbb{R}_i(\mathbb{B}) = \langle B_2\rangle \cap \langle Y_i\rangle$ are not trivial (i.e. not $\{1\}$).*

We need the following lemma.

Lemma 10 *Let $\mathbb{B} = (B_1, B_2)$ be a bisection of $\mathbb{M}(A, \theta)$ and $\mathbb{F} = (Y_i)_{i\in[1,n]}$ a factorization with $n > 1$, such that it exists a factorization $\mathbb{G} = (G_k)_{k\in K}$ with $\mathbb{B}, \mathbb{F} \preceq \mathbb{G}$ then $\mathbb{B} \preceq \mathbb{F}$ if and only if no Y_i is cut by $\mathbb{B}$.*

Proof We use the decomposition of K as an ordered sum of intervals $K = J_1 + J_2 = \sum_{i\in[1,n]} I_i$ as in definition 7. The assertion (ii) implies the existence of an integer $k \in [1, n]$ such that $J_1 = \sum_{i\in[1,k]} I_i$ and $J_2 = \sum_{i\in[k+1,n]} I_i$. This allows us to conclude.

□

Note 1 *In the preceding lemma, the existence of a common bound $\mathbb{G}$ is essential as shown by the following example (with $\mathbb{M}(A, \emptyset) = \{a, b, c\}^*$ and the rational expressions written as in [1])*

$$\mathbb{B} = (a, ba^* \cup ca^*) \text{ and } \mathbb{F} = (b, a, ab^+a^* \cup cb^*a^*)$$

No factor of $\mathbb{F}$ is cut by $\mathbb{B}$ and the two factorizations admit no common upper bound.

In the sequel, we use the notion of a composition of factorizations as it is defined by Viennot in [19]. We recall it here.

Definition 11 *Let $\mathbb{F} = (\mathbb{M}_i)_{i\in I}$ be a factorization of a monoid $\mathbb{M}$ and for some $k \in I$, $\mathbb{F}' = (\mathbb{M}'_i)_{i\in I'}$ a factorization of $\mathbb{M}_k$. The* composition *of $\mathbb{F}$ and $\mathbb{F}'$ is the factorization $\mathbb{F}' \circ \mathbb{F} = (\mathbb{M}''_i)_{i\in I''}$ where $I'' = I \cup I' - \{k\}$ is ordered by $i < j$ if and only if*

(i) $(i, j \in I$ and $i <_I j)$ or $(i, j \in I'$ and $i <_{I'} j))$
(ii) $i \in I$, $i <_I k$ and $j \in I'$
(iii) $i \in I'$, $j >_I k$ and $j \in I$

and

$$\mathbb{M}''_i = \begin{cases} \mathbb{M}_i \text{ if } i \in I \\ \mathbb{M}'_i \text{ if } i \in I' \end{cases}$$

Definition 12 *A* transitive factorization *is a factorization which is composed of transitive bisections (in finite number).*

Lemma 13 *Let $\mathbb{F} = (Y_i)_{i\in[1,p]}$ be a transitive factorization and $\mathbb{B} = (B, \beta_Z(B))$ be a transitive bisection such that it exists a factorization $\mathbb{G}$ finer that $\mathbb{B}$ and $\mathbb{F}$. Then it exists at most one Y_i cut by $\mathbb{B}$ and for a such i we have*

(i) The subset $T = Y_i \cap \mathbb{M}(B, \theta_B)$ is a TFSA of Y_i and $\mathbb{R}_i(\mathbb{B})$ is the right monoid of the associated bisection.
(ii) The sequence $(Y_1, \ldots, Y_{i-1}, T)$ is a transitive factorization of $\mathbb{M}(B, \theta_B)$.
(iii) The sequence $(\beta_{Y_i-T}(T), Y_{i+1}, \ldots, Y_p)$ is a transitive factorization of $\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})$

Sketch of the proof First it suffices to remark that, if $i > j$ are two indices such that Y_i and Y_j are cut by $\mathbb{B}$ then $\mathbb{L}_j(\mathbb{B}) \subseteq \mathbb{M}(B, \theta_B) \cap \mathbb{M}(\beta_Z(B), \theta_Z(B)) = \{1\}$ and this contradicts our hypothesis, hence $i = j$.
Let us prove assertion (i).
1) First, we remark that

$$\underline{\mathbb{M}(Y_i, \theta_{Y_i})} = \underline{\mathbb{L}_i(\mathbb{B})}.\underline{\mathbb{R}_i(\mathbb{B})}$$

and using the equality $\mathbb{L}_i(\mathbb{B}) = \mathbb{M}(T, \theta_T)$ we prove that $\mathbb{R}_i(\mathbb{B}) = \langle \beta_{Y_i-T}(Y_i)\rangle$.
2) We show that if T is not a TFSA of Y_i then B is not a TFSA of A and this implies the result.

Let us prove (ii) and (iii) by induction on p. If $p = 1$ the result is trivial. If $p > 1$, we can write $\mathbb{F}$ under the form $\mathbb{F} = \mathbb{F}_1 \circ \mathbb{F}_2 \circ \mathbb{B}'$ where $\mathbb{B}' = (B', \beta_{Z'}(B'))$ is a transitive bisection, $\mathbb{F}_1 = (Y_1, \ldots, Y_k)$ a transitive factorization of $\mathbb{M}(B', \theta_{B'})$ and $\mathbb{F}_2 = (Y_{k+1}, \ldots, Y_p)$ a transitive factorization of the monoid $\mathbb{M}(\beta_{Z'}(B'), \theta_{\beta_{Z'}(B')})$. If $\mathbb{B} = \mathbb{B}'$ the result is trivial. If $\mathbb{B} \neq \mathbb{B}'$, we have necessarily $B \subset B'$ or $B' \subset B$. We suppose that $B' \subset B$ (the other case is symmetric), and we consider the transitive trisection $(B', \beta_{B-B'}(B'), \beta_Z(B))$. Using the induction hypothesis we find that

$$(Y_k, \ldots, Y_{i-1}, T) \text{ and } (\beta_{Y_i - T}(T), Y_{i+1}, \ldots, Y_p)$$

are transitive factorizations (respectively of the monoid $\mathbb{M}(\beta_{B-B'}, \theta_{\beta_{B-B'}(B)})$ and $\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})$). And then

$$(Y_1, \ldots, Y_{i-1}, T) = \mathbb{F}_1 \circ (Y_k, \ldots, Y_{i-1}, T) \circ (B', \beta_{B-B'}(B))$$

is a transitive factorization.

□

Lemma 14 *Let $\mathbb{B} = (B, \beta_Z(B))$ be a transitive bisection and $\mathbb{F} = (Y_i)_{i\in[1,n]}$ be a transitive factorization such that $\mathbb{B} \preceq \mathbb{F}$. Then the factorizations $\mathbb{F}|_{\mathbb{M}(B,\theta_B)}$ and $\mathbb{F}|_{\mathbb{M}(\beta_Z(B),\theta_{\beta_Z(B)})}$ are transitive.*

Proof We can prove the result by induction on n.

□

Proposition 15 *Let $\mathbb{F} = (Y_i)_{i\in J}$ and $\mathbb{F}' = (Y'_j)_{j\in J'}$ be two finite transitive factorizations such that it exists a factorization $\mathbb{G}$ with $\mathbb{F}, \mathbb{F}' \preceq \mathbb{G}$ then it exists a transitive finite factorization $\mathbb{G}'$ such that*

(i) $\mathbb{F}, \mathbb{F}' \preceq \mathbb{G}' \preceq \mathbb{G}$
(ii) For each $j \in J$, $\mathbb{G}'|_{\mathbb{M}(Y_j,\theta_{Y_j})}$ is a transitive finite factorization.
(iii) For each $j \in J'$, $\mathbb{G}'|_{\mathbb{M}(Y'_j,\theta_{Y'_j})}$ is a transitive finite factorization.

Sketch of the proof We set $J = [1, n]$, $J' = [1, n']$ and we prove the result by induction on n. If $n = 1$ the result is trivial. If $n = 2$, lemmas 10, 13 and 14 give us the proof. If $n \geq 2$, we set $\mathbb{F} = \mathbb{F}_1 \circ \mathbb{F}_2 \circ \mathbb{B}$ where $\mathbb{B} = (B, \beta_Z(B))$ is a transitive bisection of $\mathbb{M}(A, \theta)$, $\mathbb{F}_1$ a transitive factorization of $\mathbb{M}(B, \theta_B)$ and $\mathbb{F}_2$ a transitive factorization of $\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})$. Using lemmas 10, 13 and 14 we define a factorization

$$\mathbb{F}'' = \begin{cases} \mathbb{F}' & \text{If } B \underline{\prec} \mathbb{F}' \\ (Y'_1, \dots, Y'_{i-1}, T, \beta_Z(T), Y'_{i+1}, \dots, Y'_{n'}) & \text{Otherwise} \end{cases}$$

such that $\mathbb{F}', \mathbb{B} \underline{\prec} \mathbb{F}'' \underline{\prec} \mathbb{G}$, $\mathbb{F}''|_{\mathbb{M}(Y'_j, \theta_{Y'_j})}$ is transitive for each $j \in [1, n]$ (in fact this factorization is trivial for all $j \in [1, n]$ but at most one for which it is a transitive bisection), $\mathbb{F}''|_{\mathbb{M}(B, \theta_B)}$ and $\mathbb{F}''|_{\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})}$ are transitive. Using the induction hypothesis we can construct two factorizations $\mathbb{F}''_1$ and $\mathbb{F}''_2$ such that

$$\mathbb{F}_1, \mathbb{F}''|_{\mathbb{M}(B, \theta_B)} \underline{\prec} \mathbb{F}''_2 \underline{\prec} \mathbb{G}|_{\mathbb{M}(B, \theta_B)}$$

and

$$\mathbb{F}_2, \mathbb{F}''|_{\mathbb{M}(\beta_Z(B), \theta_{\beta_Z(B)})} \underline{\prec} \mathbb{F}''_2 \underline{\prec} \mathbb{G}|_{\ M(\beta_Z(B), \theta_{\beta_Z(B)})}$$

and satisfying (ii) and (iii). We set $\mathbb{G}' = \mathbb{F}''_1 \circ \mathbb{F}''_2 \circ \mathbb{B}$, then $\mathbb{F}, \mathbb{F}' \underline{\prec} \mathbb{G}' \underline{\prec} \mathbb{G}$ and the induction hypothesis, the construction of $\mathbb{F}''$ and lemma 14 allow us to conclude.

□

Corollary 16 *Let $\mathbb{F} = (Y_i)_{i \in I} \underline{\prec} \mathbb{F}'$ be two transitive finite factorizations then for each $i \in I, \mathbb{F}'|_{\mathbb{M}(Y_i, I_{Y_i})}$ is a transitive finite factorization.*

Proof It suffices to use proposition 15 with $\mathbb{F}, \mathbb{F}' \underline{\prec} \mathbb{F}'$.

□

The following definition is an adaptation to partial commutations of a definition given by Viennot in [19].

Definition 17 *A factorization $(Y_i)_{i \in I}$ of $\mathbb{M}(A, \theta)$ has* locally the property $\mathfrak{P}$ *if and only if for each finite subalphabet $B \subset A$ and $n \geq 0$ it exists a factorization $(Y'_i)_{i \in I'}$ with the property $\mathfrak{P}$ such that there is an strictly increasing mapping $\phi : I' \to I$ satisfying*

$$Y'_i \cap B^{\leq n} = Y_{\phi(i)} \cap B^{\leq n} \text{ and } Y_j \cap B^{\leq n} = \emptyset \text{ if } j \notin \phi(I')$$

Definition 18 *We denote $CLTF(A, \theta)$ the set of the complete locally transitive finite factorizations.*

Example 3 *Consider the following independence graph*

$$a - b - c - d.$$

We construct a complete locally transitive finite factorization $\mathbb{F}$ as follow. We eliminate successively the traces c, ac^2, b, d, ac and a. So we have

$$M(A, \theta) = c^*.(ac^2)^*.b^*.d^*.(ac)^*.a^*.M$$

where M is a (non-commutative) free monoid. It suffices to take a Lazard factorization on M to construct a complete locally transitive finite factorization of $\mathbb{M}(A, \theta)$.
We can remark that one can not obtain this factorization using only transitive bisections with a non commutative right member. Examining all the transitive bisections of this kind

1. $\mathbb{B}_1 = (\{a, c\}, \beta_{b,d}(a, c))$
2. $\mathbb{B}_2 = (\{b, c\}, \beta_{a,d}(b, c))$
3. $\mathbb{B}_3 = (\{b, d\}, \beta_{a,c}(b, d))$
4. $\mathbb{B}_4 = (\{a, b, c\}, \beta_d(a, b, c))$
5. $\mathbb{B}_5 = (\{a, c, d\}, \beta_b(a, c, d))$
6. $\mathbb{B}_6 = (\{b, c, d\}, \beta_a(b, c, d))$
7. $\mathbb{B}_7 = (\{a, b, d\}, \beta_c(a, b, d))$,

we can easily prove that $\mathbb{F}$ could not be written like $\mathbb{F} = \mathbb{F}_1 \circ \mathbb{F}_2 \circ \mathbb{B}_i$ with $i \in \{1, 2, \dots, 7\}$.

4.2 Transitive elimination in $L_K(A, \theta)$

The algebra of trace polynomials $K\langle A, \theta\rangle = K[\mathbb{M}(A, \theta)]$ endowed with the classical Lie bracket is a Lie algebra ([2,6,14] when $\theta = \emptyset$). The free partially commutative Lie algebra is at first defined as the free object with respect to the given commutations [6]. Here we will use its realisation as the sub-Lie algebra of $K\langle A, \theta\rangle$ generated by the letters [12]. We will denote it $L_K(A, \theta)$. One can show that this definition is equivalent to $L_K(A, \theta) = L_K(A)/_{I_\theta}$ where $L_K(A)$ is the free Lie algebra and I_θ is the Lie ideal of $L_K(A)$ generated by the polynomials $[a, b]$ with $(a, b) \in \theta$. The following theorem proves that elimination in $L_K(A, \theta)$ and transitive factorization of $\mathbb{M}(A, \theta)$ occur under the same condition.

Theorem 19 *Let (B, Z) be a partition of A*

(i) We have the decomposition

$$L_K(A, \theta) = L_K(B, \theta_B) \oplus J$$

where J is the Lie ideal generated (as a Lie algebra) by

$$\tau_Z(B) = \{[\dots[z, b_1], \dots b_n] \quad | \quad zb_1 \dots b_n \in \beta_Z(B)\}.$$

(ii) The subalgebra J is a free partially commutative Lie algebra if B is a TFSA of A.
(iii) Conversely if J is a free partially commutative Lie algebra with code $\tau_Z(B)$ then B is a TFSA.

Proof (i) We have the classical Lazard bisection

$$L_K(A) = L_K(B) \oplus L_K(T_Z(B))$$

where $T_Z(B) = \{[\ldots[z, b_1], \ldots], b - n] \quad | \quad z \in Z, b_1, \ldots, b_n \in B\}$. Then using the natural mapping $L_K(A) \to L_K(A, \theta)$ (as $[\ldots[z, b_1], \ldots], b_n]$ maps to 0 if $zb_1 \ldots b_n \notin \beta_Z(B)$) we get the claim.
(ii) The proof goes as in [7], due to the fact that, for a TFSA, (c) below still holds, we sketch the proof.
Define a mapping ∂_b from $\beta_Z(B)$ to $L_K(\beta_Z(B), \theta_{\beta_Z(B)})$ by

$$\partial_b = \begin{cases} zwb \text{ if } zwb \in \beta_Z(B), \\ 0 \quad \text{ otherwise.} \end{cases}$$

a) We prove that if B is TFSA, ∂_b can be extended as a derivation of the Lie algebra $L_K(\beta_Z(B), \theta_{\beta_Z(B)})$.
b) We define ∂ a mapping from B to $Der(L_K(\beta_Z(B), \theta_{\beta_Z(B)}))$ by $\partial(b) = \partial_b$ and we extend it as a Lie morphism from $L_K(B, \theta_B)$ into $Der(L_K(\beta_Z(B), \theta_{\beta_Z(B)}))$.
c) We prove that the semi-direct product $L_K(B, \theta_B) \propto_\partial L_K(\beta_Z(B), \theta_{\beta_Z(B)})$ and the Lie algebra $L_K(A, \theta)$ are isomorphic using the universal property of the latter. Hence, J is a free partially commutative Lie algebra isomorphic to $L_K(\beta_Z(B), \theta_{\beta_Z(B)})$.
(iii) If the dependence graph admits the following subgraph

$$z - b_1 - \ldots - b_n - z'$$

with $b_i \in B$ and $z, z' \in Z$ we have the identity

$$[z, [[\ldots[z', b_n] \ldots, b_2], b_1]] = [\ldots[z', b_n] \ldots b_2], [z, b_1]].$$

Which implies that $\tau_Z(B)$ is not a code for J.

$\square$

4.3 Construction of bases of $L_K(A, \theta)$

In this section, we define a class of bases which contains the bases found by Duchamp and Krob in [6], [7] and [4] using chromatic partitions and the partially commutative Lyndon bases found by Lalonde (see Lalonde [11], Krob and Lalonde [10]).

Definition 20 *Let $\mathbb{F} = (Y_i)_{i\in[1,n+1]}$ be a finite transitive factorization. We denote $\widetilde{\mathbb{F}}$, the set of the n-uplets $(\mathbb{B}_1, \ldots, \mathbb{B}_n)$ of transitive bisections such that $\mathbb{F} = \mathbb{B}_n \circ \ldots \circ \mathbb{B}_1$.*
Let $\mathbb{F}$ be a transitive factorization and $\mathfrak{f} = (\mathbb{B}_1, \ldots, \mathbb{B}_n) \in \widetilde{\mathbb{F}}$, we denote $\mathfrak{f}\mathbb{B}_n^{-1} = (\mathbb{B}_1, \ldots, \mathbb{B}_{n-1})$.

In the following, if $\mathbb{F}$ is the sequence $(Y_i)_{i\in J}$, we denote $\mathrm{Cont}\ \mathbb{F} = \bigcup_{i\in J} Y_i$ as in [19].

Definition 21 *Let $\mathbb{F} = (Y_i)_{i\in[1,n+1]}$ be a finite transitive factorization. A bracketing of $\mathbb{F}$ along $\mathfrak{f} \in \widetilde{\mathbb{F}}$ is a mapping $\Pi_\mathfrak{f}$ from $\mathrm{Cont}\ \mathbb{F}$ to $L_K(A, \theta)$ inductively defined as follows. If $n = 1$, then $\mathfrak{f}$ is a sequence of length 1 under the form $((B, \beta_Z(B)))$ and*

$$\Pi_\mathfrak{f}(w) = \begin{cases} w \text{ if } w \in B, \\ [\ldots[z, b_1]\ldots b_k] \text{ if } w = zb_1\ldots b_k \in \beta_Z(B) \text{ and } z \in Z. \end{cases}$$

If $n > 1$, let $\mathfrak{f} = (\mathbb{B}_1, \ldots, \mathbb{B}_n) \in \widetilde{\mathbb{F}}$. We set $\mathbb{B}_{n-1} \circ \cdots \circ \mathbb{B}_1 = (Y'_i)_{i\in[1,n]}$ and $j \in [1, n]$ such that $\mathbb{B}_n = (Y''_j, \beta_{Y'_j - Y''_j}(Y''_j))$ (remark that in this case, one has $\mathrm{Cont}\ \mathbb{F} - \mathrm{Cont}\ \mathbb{B}_{n-1} \circ \cdots \circ \mathbb{B}_1 = \beta_{Y'_j - Y''_j}(Y''_j)$). And

$$\Pi_\mathfrak{f}(w) = \begin{cases} \Pi_{\mathfrak{f}\mathbb{B}_n^{-1}}(w) \text{ if } w \in \mathrm{Cont}\ \mathbb{B}_{n-1} \circ \cdots \circ \mathbb{B}_1, \\ [\ldots[\Pi_{\mathfrak{f}\mathbb{B}_n^{-1}}(y_1), \Pi_{\mathfrak{f}\mathbb{B}_n^{-1}}(v_1)], \ldots \Pi_{\mathfrak{f}\mathbb{B}_n^{-1}}(v_k)] \quad \text{if } w = y_1v_1\ldots v_k, \\ \qquad w \in \beta_{Y'_j - Y''_j}(Y''_j), \\ \qquad y_1 \in Y'_j - Y''_j \\ \qquad \text{and } v_1, \ldots, v_k \in Y''_j. \end{cases}$$

Using theorem 19 in an induction on n we prove the following proposition.

Proposition 22 *Let $\mathbb{F} = (Y_i)_{i\in[1,n]}$ be a transitive factorization. For each $\mathfrak{f} \in$*

$\widetilde{\mathbb{F}}$*, we have the following decomposition*

$$L_K(A,\theta) = \bigoplus_{i\in[1,n-1]} L_K(\Pi_{\mathfrak{f}}(Y_i),\theta_i)$$

where

$$\theta_i = \{(\Pi_{\mathfrak{f}}(y_1),\Pi_{\mathfrak{f}}(y_2))|(y_1,y_2)\in\theta_{\mathbb{M}}\}.$$

Definition 23 *Let $\mathbb{F} = (Y_i)_{i\in J}$ be a locally transitive finite factorization, a bracketing of $\mathbb{F}$ is a mapping Π from $\bigcup_{i\in J} Y_i$ to $L_K(A,\theta)$ such that for each finite subalphabet $B \subset A$ and each integer $n \geq 0$, it exists a transitive finite factorization $\mathbb{F}_{n,B} = (Y_i^{n,B})_{i\in J_{n,B}}$ and $\mathfrak{f}_{n,B} \in \widetilde{\mathbb{F}}_{n,B}$ such that for each $t \in \mathrm{Cont}\ \mathbb{F}_{n,B} \cap B^{\leq n}$, $\Pi(t) = \Pi_{\mathfrak{f}_{n,B}}(t)$.*

Lemma 24 *Let $\mathbb{F} = (Y_i)_{i\in J} \preceq \mathbb{F}'$ be two finite transitive factorizations. Then, for each $\mathfrak{f}\in\widetilde{\mathbb{F}}$, it exists $\mathfrak{f}'\in\widetilde{\mathbb{F}}'$ such that for each $t\in \mathrm{Cont}\ \mathbb{F}$, $\Pi_{\mathfrak{f}}(t) = \Pi_{\mathfrak{f}'}(t)$.*

Proof It is a direct consequence of corollary 16.

□

Theorem 25 *Let (A,θ) be an independence alphabet. Each locally finite transitive factorization of $\mathbb{M}(A,\theta)$ admits a bracketing.*

Proof Let $\mathbb{F} = (Y_i)_{i\in J}$ be a locally finite transitive factorization. Using proposition 15, one can construct a sequence of finite transitive factorizations $(F_{n,B})_{\substack{n\in\mathbb{N},B\subset A\\ Card B<\infty}}$ such that

(1) if $n \leq n'$ and $B \subset B'$ then

$$\mathbb{F}_{n,B} \preceq \mathbb{F}_{n',B'},$$

(2) for each $n\geq 0$ and $B\subset A$

$$\mathbb{F}_{n,B} \preceq \mathbb{F},$$

(3) for each $n \geq 0$ and each finite subalphabet B, if we set $\mathbb{F}_{n,B} = (Y^{n,B})_{i\in[1,k_{n,B}]}$, it exists a strictly increasing mapping $\phi_{n,B}$ from $[1,k_{n,B}]$ to J verifying

$$\mathbb{M}(Y_i^{n,B},\theta_{Y_i^{n,B}}) \cap B^{\leq n} = \mathbb{M}(Y_{\phi_{n,B}(i)},\theta_{Y_{\phi_{n,B}(i)}})$$

and

$$j\notin\phi_{n,B}([1,k_{b,B}]) \Rightarrow \mathbb{M}(Y_j,\theta_{Y_j})\cap B^{\leq n} = \{1\}.$$

By lemma 24, we can define for each $n > 0$ and each finite subalphabet B of A a sequence $\mathfrak{f}_{n,B}\in\tilde{\mathbb{F}}_{n,B}$ such that for each $m < n$, $B' \subset B$ and

$t \in \mathrm{Cont}\ \mathbb{F}_{m,B'} \cap B'^{\leq n}$ we have $\Pi_{\mathfrak{f}_{m,B'}} t = \Pi_{\mathfrak{f}_{n,B}} t$.
Thus, we can define Π as the mapping from Cont $\mathbb{F}$ into $L_K(A,\theta)$ such that $\Pi t = \Pi_{\mathfrak{f}_{|t|,\mathrm{Alph}(t)}} t$.

□

We have easily the following result.

Proposition 26 *Let $\mathbb{F} = (\{l_i\})_{i\in I} \in CLTF(A,\theta)$ and Π be a bracketing of $\mathbb{F}$ then the family $(\Pi(l_i))_{i\in I}$ is a basis of $L_K(A,\theta)$ as K-module.*

Example 4 *We set $A = \{a,b,c,d\}$ and $\theta = a-b-c-d$. We construct locally (for $n \leq 3$) the following basis.*

[[a,d],b], [[a,d],d], [[a,d],a], [a,d], [a,[a,c]], a, [a,c], [[a,c],c], [[a,d],c], [b,d], [[b,d[,b[, [[b,d],d], b, c, d.

5 The case of the group

The free partially commutative group [9] can be defined by the presentation

$$\mathbb{F}(A,\theta) =< A; \{ab = ba\}_{(a,b)\in\theta} >_{gr} .$$

To extend the elimination process, we need the alphabet of the inverse letters. Recall that one can construct the free partially commutative group using "reduced" traces [**?**,8]. If A is an alphabet, we define $\tilde{A} = A \cup \overline{A}$ where $\overline{A} = \{\overline{a}\}_{a\in A}$ is a disjoint copy of A. The set $\tilde{A}$ is then provided with the involution $x \to \overline{x}$ such that $\overline{\overline{x}} = x$. Thus, θ is extended by

$$\tilde{\theta} = \{(x,y) \in \tilde{A}^2 | \{(x,y),(\overline{x},y),(x,\overline{y}),(\overline{x},\overline{y})\} \cap \theta \neq \emptyset\}.$$

We define the natural mapping $s_0 : \tilde{A} \to \mathbb{F}(A,\theta)$ such that $s_0(a) = a$ and $s_0(\overline{a}) = a^{-1}$ for each letter $a \in A$.
As s_0 is compatible wih the commutations of $\tilde{\theta}$ (i.e. if $(x,y) \in \tilde{\theta}$ then $s_0(x)s_0(y) = s_0(y)s_0(x)$), one has the factorization

$$\begin{tikzcd} \tilde{A} \arrow[r, "s_0"] \arrow[d] & \mathbb{F}(A,\theta) \\ \mathbb{M}(\tilde{A},\tilde{\theta}) \arrow[ur, "s"'] & \end{tikzcd} .$$

The mapping s is onto. For each $g \in \mathbb{F}(A,\theta)$, it exists an unique preimage with minimal length in $s^{-1}(g)$, this element is called "reduced expression" of

g (the subset of these traces will be denoted by $red(\tilde{A},\tilde{\theta})$). The links with the bisections $(B,\beta_Z(B))$ is given by the following.

Lemma 27 *Let $B \subset A$, $z \notin B$ and $w \in red(\tilde{B},\tilde{\theta}_{\tilde{B}})$. The following assertions are equivalent.*

(1) $\overline{w}zw$ is a reduced trace (i.e. $\overline{w}zw \in red(\tilde{A},\tilde{\theta})$).
(2) $zw \in \beta_z(\tilde{B})$.

Proof Straightforward, using the criterion given in [8]:
Let $t = a_1a_2\ldots a_n \in \mathbb{M}(\tilde{A},\tilde{\theta})$, t is not a reduced trace if and only if it exists $1 \le i < j \le n$ with $a_i = \overline{a}_j$ and such that for each k, $i < k < j$, $(a_k, a_i) \in \tilde{\theta}$.
□

We denote $\beta_Z^R(\tilde{B})$ the set $\beta_Z(\tilde{B}) \cap red(\tilde{A},\tilde{\theta})$ with the commutation $\tilde{\theta}_{\beta_Z^R(\tilde{B})}$ provided by the definition 1. One has an analogue of the theorem 19.

Proposition 28 *Let (B,Z) be a partition of A.*

(i) One has the decomposition as the semi direct product

$$\mathbb{F}(A,\theta) = \mathbb{F}(B,\theta_B) \ltimes H_Z$$

where H_Z is the normal subgroup generated by Z. It is the subgroup generated by

$$\rho_Z(B) = \{w^{-1}zw | zw \in \beta_z^R(\tilde{B})\}$$

(ii) The subgroup H_Z is free partially commutative for the code $\rho_Z(B)$ and the commutations

$$\hat{\theta}_\rho := \{(t,t') \in \rho_Z(B)^2 | tt' = t't \text{ and } t \neq t'\}.$$

(iii) The natural mapping $\alpha : \mathbb{F}(\beta_Z^R(\tilde{B}), \tilde{\theta}_{\beta_Z^R(\tilde{B})}) \to H_Z$ is one to one if and only if B is TFSA.

Proof (i) The decomposition given by (i) is the image of the non commutative Lazard elimination in the free group. The unicity of the decomposition with respect to the semidirect product can be obtain (as in the classical case) by sending all the element of Z to one.
(ii) Let $\hat{\rho} = \{a_t\}_{t\in\rho_Z(B)}$ be an alphabet and $\hat{\theta}$ be the commutation relation defined by $(a_t, a_{t'}) \in \hat{\theta}$ if and only if $t \neq t'$ and $tt' = t't$.
For each $b \in B$, we define the mapping $\sigma_b : \hat{\rho} \to \hat{\rho}$ by $\sigma_b(a_t) = a_{b^{-1}tb}$. Remarking that $b^{-1}tb$ belongs to $\rho_Z(B)$ and then $(a_t, a_{t'}) \in \hat{\theta}$ implies $(\sigma_b(a_t), \sigma_b(a_{t'})) \in \hat{\theta}$, this mapping can be extended in an automorphism σ_b of $\mathbb{F}(\hat{\rho},\hat{\theta})$. Let σ be the mapping from B to $Aut(\mathbb{F}(\hat{\rho},\hat{\theta}))$ defined by $\sigma(b) = \sigma_b$. As $\sigma_b\sigma_{b'} = \sigma_{b'}\sigma_b$ when

$(b, b') \in \theta_B$, σ can be extended as a morphism from $\mathbb{F}(B, \theta_B)$ in $Aut(\mathbb{F}(\hat{\rho}, \hat{\theta}))$. Using the same proof than in theorem 19, we find that the semidirect product $\mathbb{F}(B, \theta_B) \propto_\sigma \mathbb{F}(\hat{\rho}, \hat{\theta})$ and $\mathbb{F}(A, \theta)$ are isomorphic.

(iii) Suppose that B is not TFSA then it exists a $(z_1, z_2) \in \theta_Z$ and a minimal path in the non commutation graph

$$z_1 - a_1 - \cdots - a_k - c - b_l - \cdots - b_1 - z_2.$$

Let $r_1 = z_1 a_1 \cdots a_k$ and $r_2 = z_2 b_1 \cdots b_l$. Due to the fact that the chain is of minimal length, one has $(r_1, r_2) \in \tilde{\theta}_{\beta_Z^R(\tilde{B})}$ and $\alpha(r_1)\alpha(r_2) = \alpha(r_2)\alpha(r_1)$. But $r_1 c$ and $r_2 c$ do not commute and their images $\alpha(r_1 c) = c^{-1}\alpha(r_1)c$ and $\alpha(r_2 c) = c^{-1}\alpha(r_1)c$ do. This proves that α is not one to one.
The converse follows from the fact that, when B is a TFSA, the commutation graph $(\beta_Z^R(\tilde{B}), \tilde{\theta}_{\beta_Z^R(\tilde{B})})$ and $(\rho_Z(B), \hat{\theta}_\rho)$ are obviously isomorphic.

□

Note 2 *In general α is into.*

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