Pfaffian and Hafnian Identities in Shuffle Algebras

Jean-Gabriel LUQUE and Jean-Yves THIBON

Abstract

Chen's lemma on iterated integrals implies that certain identities involving multiple integrals, such as the de Bruijn and Wick formulas, amount to combinatorial identities for Pfaffians and hafnians in shuffle algebras. We provide direct algebraic proofs of such shuffle identities, and obtain various generalizations. We also discuss some Pfaffian identities due to Sundquist and Ishikawa-Wakayama, and a Cauchy formula for anticommutative symmetric functions. Finally, we extend some of the previous considerations to hyperpfaffians and hyperhafnians.

1 Introduction

The free associative algebra $\mathbb{Z}\langle A\rangle$ over an alphabet A is naturally endowed with a commutative operation, the shuffle product $\sqcup\!\sqcup$ which can be recursively defined by

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)$$

where $a, b \in A$ and $u, v \in A^*$.

This operation has at least two interesting interpretations. First, it is dual to the comultiplication which admits the Lie polynomials as primitive elements (Friedrich's theorem, see, *e.g.*, [14]). Second, it describes the multiplication of iterated integrals (Chen's lemma [7]). It is this last property which will be of interest to us. To state it in a convenient way, let us fix some vector space $\mathcal{H}$ of suitably integrable functions over an interval (a, b). For $f_1, \ldots, f_n \in \mathcal{H}$, we identify the tensor product $F = f_1 \otimes \cdots \otimes f_n$ with the function of n variables $F(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$. Let $\langle\ \rangle$ be the linear form defined on each $\mathcal{H}^{\otimes n}$ by

$$\langle f_1 \otimes \cdots \otimes f_n\rangle = \int_a^b dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n f_1(x_1)f_2(x_2)\cdots f_n(x_n)\,. \quad (1)$$

We now choose a family of functions $(\phi_i)_{i\in I}$, labelled by some alphabet I, and for a word $w = i_1 i_2 \cdots i_n \in I^*$, we set

$$\langle w\rangle = \langle \phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_n}\rangle . \tag{2}$$

Then, Chen's lemma states that

$$\langle u\rangle\langle v\rangle = \langle u \sqcup\!\sqcup v\rangle . \tag{3}$$

Iterated integrals occur for instance in the power series solutions of linear differential equations with variable coefficients. An example of current interest is provided by the multiple zeta values [20], and an interesting application of the shuffle product appears in [3], where a conjecture of Zagier [20] is proved by reduction to a simple shuffle identity.

In certain cases, iterated integrals can be evaluated in closed form by means of de Bruijn's formulas [4]

$$\int \cdots \int_{a\le x_1<\cdots<x_{2n}\le b} \det\left(\phi_i(x_j)\right) dx_1 \cdots dx_{2n} = \mathrm{Pf}\ \left(P_{ij}\right)_{1\le i,j\le 2n} \tag{4}$$

where $\mathrm{Pf}\,(P)$ denotes the Pfaffian of the antisymmetric matrix P,

$$P_{ij} = \int\int_{a\le x<y\le b} [\phi_i(x)\phi_j(y) - \phi_j(x)\phi_i(y)]dxdy$$

and

$$\int \cdots \int_{a\le x_1<\cdots<x_n\le b} \det\left(\phi_i(x_j)|\psi_i(x_j)\right) dx_1 \cdots dx_n = \mathrm{Pf}\ \left(Q_{ij}\right)_{1\le i,j\le 2n} \tag{5}$$

where $Q_{ij} = \int_a^b [\phi_i(x)\psi_j(x) - \phi_j(x)\psi_i(x)]dx$ and $(\phi_i(x_j)|\psi_i(x_j))$ denotes the matrix whose ith row is $[\phi_i(x_1), \psi_i(x_1), \phi_i(x_2), \psi_i(x_2), \ldots, \phi_i(x_n), \psi_i(x_n)]$.

There is also a version of formula (4) for determinants of odd order,

$$\int \cdots \int_{a\le x_1<\cdots<x_{2n+1}\le b} \det\left(\phi_i(x_j)\right) \quad dx_1 \cdots dx_{2n+1}$$
$$= \sum_{p=1}^{2n+1} (-1)^{p+1} \int_a^b \phi_p(x)dx\ \mathrm{Pf}\ \left(P_{ij}\right)_{\substack{1\le i,j\le 2n \\ i,j\ne p}} .$$

The foundations of the theory of random matrices can be developed from these identities [18].

2 The Wick formulas

The above formulas are reminescent of identities which are usually called *Wick formulas* in the physics literature. These can be stated in various ways, and come in two flavors: bosonic and fermionic.

The bosonic version amounts to a familar fact about moments of Gaussian measures. Let $d\mu(x) = C\, e^{-\frac{1}{2}{}^t xAx}dx$ be a Gaussian probability mesure on $\mathbb{R}^N$. We set $\langle f\rangle = \int_{\mathbb{R}^N} f(x)d\mu(x)$. Let $f_1, \dots, f_i \dots$ be linear forms. Then, the bosonic Wick formula reads

$$\langle f_1 \cdots f_n\rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathrm{Hf}\,(\langle f_i f_j\rangle)_{1\le i,j\le n} & \text{if } n \text{ is even}, \end{cases} \tag{6}$$

the *hafnian* of a symmetric matrix of even order $n = 2r$ being defined by

$$\mathrm{Hf}\,(A) = \sum_{I,J} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_rj_r} \tag{7}$$

where the sum runs over all decompositions of $\{1, 2, \dots, 2r\}$ into disjoints subsets $I = \{i_k\}$, $J = \{j_k\}$, such that $i_1 < \cdots < i_r$, $j_1 < \cdots < j_r$ and for each k, $i_k < j_k$. Replacing A by an antisymmetric matrix and multiplying each term of the sum by the signature of the permutation $i_1j_1 \cdots i_rj_r$, one would obtain the Pfaffian of A. Actually, hafnians were introduced by Caianiello [5] in order to emphasize the similarities between the bosonic and fermionic versions of the Wick formula.

These similarities become particularly obvious when the fermionic Wick formula is stated in terms of *Berezin integrals* (integration over Grassmann variables [2]).

The Berezin integral is essentially a convenient notation for computing in Grassmann algebras. Let $\eta = \{\eta_i\}_{1\le i\le N}$ be a finite set of anticommuting symbols (*i.e.* $\eta_i\eta_j + \eta_j\eta_i = 0$). We denote by $\bigwedge_K(\eta)$ the K-algebra generated by the η_i. The Grassmann integral of an anticommutative polynomial f in the η_i is defined by

$$\int d\eta_n \cdots d\eta_1 f(\eta_1, \dots, \eta_n) = \frac{\partial}{\partial\eta_n}\cdots\frac{\partial}{\partial\eta_1} f(\eta_1, \cdots, \eta_n) \tag{8}$$

where $\frac{\partial}{\partial\eta_i}$ acts on the Grassmann algebra as a left derivation (we push η_i to the left, with a sign, and erase it). Therefore, in the fermionic calculus, integration and derivation are one and the same operation.

Let $d\mu(\eta) = Ce^{\frac{1}{2}{}^t\eta Q\eta}$ be a "fermionic Gaussian probability mesure" (Q being an antisymetric and non degenerate matrix, and $\int d\mu(\eta) = 1$) and let

$f_1, \ldots, f_n$ be linear forms in the η_i. We define

$$\langle f \rangle = \int d\mu(\eta) f(\eta) \tag{9}$$

The fermionic Wick formula reads [2]

$$\langle f_1 \cdots f_n \rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathrm{Pf}\ (\langle f_i f_j \rangle)_{1 \le i,j \le n} & \text{if } n \text{ is even}. \end{cases} \tag{10}$$

As we shall see, all these formulas are closely related, and reflect simple combinatorial identities in the shuffle algebra.

3 Pfaffians and Hafnians in shuffle algebras

3.1 Wick formulas in shuffle algebras

The Wick formula (10) is equivalent to the equality

$$\sum_{n \ge 0} \sum_{i_1 < \cdots < i_n} \langle f_{i_1} \cdots f_{i_n} \rangle \eta_{i_1} \cdots \eta_{i_n} = \prod_i^{\rightarrow} \left(1 + \sum_{i<j} \langle f_i f_j \rangle \eta_i \eta_j \right) \tag{11}$$

More generally, to an antisymmetric matrix Q of order N over a commutative ring K, we can associate the following element of the Grassmann algebra $\bigwedge_K(\eta)$ over K

$$T = \prod_i^{\rightarrow} \left(1 + \sum_{i<j} Q_{ij} \eta_i \eta_j \right) \tag{12}$$

and this product being commutative we can express it as an exponential

$$T = \exp\left\{ \frac{1}{2} {}^t\eta Q \eta \right\} = \sum_{n \ge 0} \frac{1}{n!} \left\{ \frac{1}{2} {}^t\eta Q \eta \right\}^n \tag{13}$$

where η denotes here the column vector (η_i). The coefficient $\langle \eta_I | T \rangle$ of $\eta_I = \eta_{i_1} \cdots \eta_{i_{2n}}$ in T being $\int d\eta_{i_{2n}} \cdots d\eta_{i_1} T$, the fermionic Wick formula gives

$$\langle \eta_I | T \rangle = \mathrm{Pf}\ (Q_{i_k i_l})_{1 \le k,l \le 2n} \tag{14}$$

Let us suppose now that $K = R\langle A \rangle_{\sqcup\!\sqcup}$ is the shuffle algebra over some commutative ring R. We will denote by $\mathrm{Pf}_{\sqcup\!\sqcup}(M)$ (resp. $\mathrm{Hf}_{\sqcup\!\sqcup}(M)$) the Pfaffian (resp. hafnian) of a matrix $M \in R\langle A \rangle_{\sqcup\!\sqcup}$.

Let (a_i) and (b_i) be two sequences of letters of A, and set $Q_{ij} = a_ib_j - a_jb_i$. Remarking that

$$\sum_{j_1,\cdots,j_{k+1}} \left(a_{i_1}b_{j_1}\cdots a_{i_k}b_{j_k} ⧢ a_{i_{k+1}}b_{j_{k+1}}\right)\eta_{i_1}\eta_{j_1}\cdots\eta_{i_{k+1}}\eta_{j_{k+1}}$$
$$= \sum_{j_1,\cdots,j_{k+1}} \sum_{l} a_{i_1}b_{j_1}\cdots a_{i_{l-1}}b_{j_{l-1}}a_{i_{k+1}}b_{j_{k+1}}a_{i_l}b_{j_l}\cdots a_{i_k}b_{j_k}\eta_{i_1}\eta_{j_1}\cdots\eta_{i_{k+1}}\eta_{j_{k+1}} \tag{15}$$

since the left-hand side is equal to the right-hand side, plus a sum of terms which are symmetric in one pair of indices (j_r, j_s) and are therefore zero, we find

$$\sum_{i_1<\cdots<i_{2k}} \sum_{\sigma\in\mathfrak{S}_{2k}} \epsilon(\sigma)a_{i_{\sigma(1)}}b_{j_{\sigma(2)}}\cdots a_{i_{\sigma(2k-1)}}b_{i_{\sigma(2k)}}\eta_{i_1}\cdots\eta_{i_{2k}}$$
$$= \overrightarrow{\prod_{i>0}}\left(1+\sum_{i<j}(a_ib_j - a_jb_i)\,\eta_i\eta_j\right) \tag{16}$$

where $\epsilon(\sigma)$ is the signature of the permutation σ, and formula (14) leads to

$$\sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)a_{\sigma(1)}b_{\sigma(2)}\cdots a_{\sigma(2n-1)}b_{\sigma(2n)} = \mathrm{Pf}_{\,⧢}\,(a_ib_j - a_jb_i)_{1\le i,j\le 2n} \tag{17}$$

Example 3.1 With $N = 4$ we obtain

$$\begin{aligned}
&a_1b_2a_3b_4 - a_1b_2a_4b_3 - a_1b_3a_2b_4 + a_1b_4a_4b_2 + a_1b_4a_2b_3 - a_1b_4a_3b_2\\
&-a_2b_1a_3b_4 + a_2a_4a_2b_3 + a_2b_3a_1b_4 - a_2b_3a_4b_1 + -a_2b_4a_1b_4 + a_2b_4a_3b_1\\
&-a_3b_2a_1b_4 + a_3b_2a_4b_1 + a_3b_1a_2b_4 - a_3b_1a_4b_2 - a_3b_4a_1b_2 + a_3b_4a_2b_1\\
&-a_4b_2a_3b_1 + a_4b_2a_1b_3 + a_4b_3a_2b_1 - a_4b_3a_1b_2 - a_4b_1a_2b_2 + a_4b_1a_2b_3\\
&= (a_1b_2 - a_2b_1) ⧢ (a_3b_3 - a_4b_4) - (a_1b_3 - a_3b_1) ⧢ (a_2b_4 - a_4b_2)\\
&+(a_1b_4 - a_4b_1) ⧢ (a_2b_3 - a_3b_2)
\end{aligned}$$

Similarly, if we consider the alphabet $A = \{a_{ij}\}$, we get

$$\sum_{i_1<\cdots<i_{2n}} \sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)a_{i_{\sigma(1)}i_{\sigma(2)}}\cdots a_{i_{\sigma(2n-1)}i_{\sigma(2n)}}\eta_{i_1}\cdots\eta_{i_{2n}}$$
$$= \overrightarrow{\prod_{i}}\left(1+\sum_{i<j}(a_{ij} - a_{ji})\,\eta_i\eta_j\right) \tag{18}$$

and the Wick formula gives

$$\sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(2n-1)\sigma(2n)} = \mathrm{Pf}_{\,⧢}\,(a_{kl} - a_{lk})_{1\le k,l\le 2n} \tag{19}$$

Remark that although (17) looks like the specialization $a_{ij} = a_ib_j$ of (19), this is not the case, since this specialization is not a shuffle homomorphism.

Example 3.2 with $1 \le i, j \le 4$, the formula reads

$$\begin{aligned}(a_{12}-a_{21})\sqcup\!\sqcup(a_{34}-a_{43})-(a_{13}-a_{31})\sqcup\!\sqcup(a_{24}-a_{42})+(a_{14}-a_{41})\sqcup\!\sqcup(a_{23}-a_{32})\\ = a_{12}a_{34}-a_{12}a_{43}+a_{34}a_{12}-a_{43}a_{12}-a_{21}a_{34}+a_{21}a_{43}-a_{34}a_{21}+a_{43}a_{21}\\ -a_{13}a_{24}+a_{13}a_{42}-a_{24}a_{13}+a_{42}a_{13}+a_{31}a_{24}-a_{31}a_{42}+a_{24}a_{31}-a_{42}a_{31}\\ +a_{14}a_{32}-a_{14}a_{23}+a_{32}a_{14}-a_{23}a_{14}-a_{41}a_{32}+a_{41}a_{23}-a_{32}a_{41}+a_{23}a_{41}.\end{aligned}$$

Similar identities can be obtained by expanding a bosonic version of the series (12). Let $\{\xi_i\}$ be a set of commuting symbols verifying $\xi_i^2 = 0$, and let Q be a symmetric matrix with zero diagonal. The series

$$T = \prod_i \left(1 + \sum_{i<j} Q_{ij}\xi_i\xi_j\right) \tag{20}$$

has as coefficients

$$\langle \xi_I | T \rangle = \mathrm{Hf}\ (Q_{i_k i_l}) \tag{21}$$

where $I = (i_1, \cdots, i_{2n})$. Specializing this to the shuffle algebra over variables a_{ij} and taking $Q_{kl} = a_{kl} + a_{lk}$ if $k \neq l$ and $Q_{kk} = 0$, we obtain

$$\sum_{\sigma\in\mathfrak{S}_{2n}} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)} = \mathrm{Hf}_{\,\sqcup\!\sqcup}\,(Q_{kl})_{1\le k,l\le 2n}\ . \tag{22}$$

Example 3.3 Let $A = \{a_{ij}\}_{1\le i,j\le 4}$. We have

$$\begin{aligned}(a_{12}+a_{21})\sqcup\!\sqcup(a_{34}+a_{43})+(a_{13}+a_{31})\sqcup\!\sqcup(a_{24}+a_{42})+(a_{14}+a_{41})\sqcup\!\sqcup(a_{23}+a_{32})\\ = a_{12}a_{34}+a_{12}a_{43}+a_{34}a_{12}+a_{43}a_{12}+a_{21}a_{34}+a_{21}a_{43}+a_{34}a_{21}+a_{43}a_{21}\\ +a_{13}a_{24}+a_{13}a_{42}+a_{24}a_{13}+a_{42}a_{13}+a_{31}a_{24}+a_{31}a_{42}+a_{24}a_{31}+a_{42}a_{31}\\ +a_{14}a_{32}+a_{14}a_{23}+a_{32}a_{14}+a_{23}a_{14}+a_{41}a_{32}+a_{41}a_{23}+a_{32}a_{41}+a_{23}a_{41}.\end{aligned}$$

3.2 Other Hafnian and Pfaffian identities

The simplest one in the series is

$$\sum_{\sigma\in\mathfrak{S}_{2n}} a_{\sigma(1)} \cdots a_{\sigma(2n)} = \frac{1}{(2n)!!}\mathrm{Hf}_{\,\sqcup\!\sqcup}\,(Q_{kl})_{1\le k,l\le 2n} \tag{23}$$

where $Q_{kl} = a_k a_l + a_l a_k$ if $k \neq l$ and $Q_{kk} = 0$. It amounts to the associativity and commutativity of the shuffle product.

Example 3.4 With $A = \{a_1, a_2, a_3, a_4\}$, we have

$$\frac{1}{3}\,[(a_1a_2 + a_2a_1)\sqcup\!\sqcup(a_3a_4 + a_4a_3) + (a_1a_3 + a_3a_1)\sqcup\!\sqcup(a_2a_4 + a_4a_2)$$
$$+(a_1a_4 + a_4a_1)\sqcup\!\sqcup(a_2a_3 + a_3a_2)] = a_1a_2a_3a_4 + a_1a_2a_4a_3 + a_1a_3a_2a_4 + a_1a_3a_4a_2 +$$
$$a_1a_4a_2a_3 + a_1a_4a_3a_2 + a_2a_1a_3a_4 + a_2a_1a_4a_3 + a_2a_3a_1a_4 + a_2a_3a_4a_1 + a_2a_4a_1a_3 +$$
$$a_2a_4a_3a_1 + a_3a_2a_1a_4 + a_3a_2a_4a_1 + a_3a_1a_2a_4 + a_3a_1a_4a_2 + a_3a_4a_1a_2 + a_3a_4a_2a_1 +$$
$$a_4a_2a_3a_1 + a_4a_2a_1a_3 + a_4a_3a_2a_1 + a_4a_3a_1a_2 + a_4a_1a_3a_2 + a_4a_1a_2a_3.$$

Indeed, the left hand side is just $\frac{1}{3}(3\,a_1\sqcup\!\sqcup a_2\sqcup\!\sqcup a_3\sqcup\!\sqcup a_4)$.

We can find other identities by considering the following generating series in $\bigwedge_{K[x]}(\eta)$, where $x = \{x_1, x_2, \cdots\}$ is a family of commuting variables:

$$T(x,\eta) = \sum_{I\uparrow} t_I\eta_I = \overrightarrow{\prod_i}\,(1 + x_i\eta_i) \tag{24}$$

where $I\uparrow$ means that I is an increasing sequence. Since $\eta_i^2 = 0$, one can write

$$T(x,\eta) = e^H = \overrightarrow{\prod_i}\, e^{x_i\eta_i} \tag{25}$$

where

$$H = \sum_i x_i\eta_i + \frac{1}{2}\sum_{i<j}[x_i\eta_i, x_j\eta_j] = \sum_i x_i\eta_i + \sum_{i<j} x_ix_j\eta_i\eta_j \tag{26}$$

is the Hausdorff series, here terminating in degree 2. On another hand, one has

$$t_I = \int d\eta_{\overline{I}}T(x,\eta) \tag{27}$$

where $\overline{I} = (i_n, i_{n-1}, \cdots, i_1)$ if $I = (i_1, \cdots, i_n)$. If $l(I)$ is even ($l(I) = 2n$), we have as well

$$t_I = \int d\eta_{\overline{I}}\ \exp\left\{\sum_{i<j} x_ix_j\eta_i\eta_j\right\}\,, \tag{28}$$

and since

$$\exp\left\{\sum_{i<j} x_ix_j\eta_i\eta_j\right\} = \sum_n \sum_{\substack{i_1<j_1,\dots,i_n<j_n\\ i_1<\cdots<i_n}} x_{i_1}x_{j_1}\cdots x_{i_n}x_{j_n}\eta_{i_1}\eta_{j_1}\cdots\eta_{i_n}\eta_{j_n}\,, \tag{29}$$

we have finally

$$t_I = \mathrm{Pf}\ (Q_{kl})_{1\le k,l\le 2n} \tag{30}$$

where

$$Q_{kl} = \begin{cases} x_{i_k}x_{i_l} & \text{if } k<l \\ -x_{i_k}x_{i_l} & \text{if } l<k \\ 0 & \text{if } l=k. \end{cases} \tag{31}$$

If $l(I) = 2n+1$ is odd, (27) gives

$$t_I = \int d\eta_{\overline{I}} \exp\left\{\sum_i x_i\eta_i\right\} \exp\left\{\sum_{i<j} x_ix_j\eta_i\eta_j\right\}. \tag{32}$$

But

$$e^{\sum_i x_i\eta_i} = 1 + \sum_i x_i\eta_i \tag{33}$$

and then

$$t_I = \sum_k (-1)^{k+1} x_{i_k} \mathrm{Pf}\,(Q_{ij})_{\substack{1\le i,j\le 2n+1 \\ i,j\neq k}}. \tag{34}$$

Let us now specialize this formula to the shuffle algebra. In $\bigwedge_{K\langle A\rangle_{\sqcup\!\sqcup}}(\eta)$, the generating series

$$\sum_{I\uparrow}\sum_{\sigma\in\mathfrak{S}_n} a_{i_{\sigma(1)}}\cdots a_{i_{\sigma(n)}}\eta_I = \prod_{i>0}^{\rightarrow}(1+a_i\eta_i) \tag{35}$$

gives

$$\sum_{\sigma\in\mathfrak{S}_n} a_{\sigma(1)}\cdots a_{\sigma(n)} = \begin{cases} \mathrm{Pf}_{\sqcup\!\sqcup}\,(Q_{kl})_{1\le k,l\le n} & \text{if } n \text{ is even,} \\ \sum_p (-1)^{p+1} a_p \sqcup\!\sqcup \mathrm{Pf}_{\sqcup\!\sqcup}\,(Q_{kl})_{\substack{1\le k,l\le n \\ k,l\neq p}} & \text{if } n \text{ is odd,} \end{cases} \tag{36}$$

where

$$Q_{kl} = \begin{cases} a_k \sqcup\!\sqcup a_l & \text{if } k<l \\ -a_k \sqcup\!\sqcup a_l & \text{if } l<k \\ 0 & \text{if } k=l \end{cases} \tag{37}$$

Example 3.5 For $A = \{a_1, a_2, a_3, a_4\}$, we have

$$(a_1a_2 + a_2a_1)\sqcup\!\sqcup(a_3a_4 + a_4a_3) - (a_1a_3 + a_3a_1)\sqcup\!\sqcup(a_2a_4 + a_4a_2) + (a_1a_4 + a_4a_1)\sqcup\!\sqcup(a_2a_3 + a_3a_2) = a_1a_2a_3a_4 + a_1a_2a_4a_3 + a_1a_3a_2a_4 + a_1a_3a_4a_2 + a_1a_4a_2a_3 + a_1a_4a_3a_2 + a_2a_1a_3a_4 + a_2a_1a_4a_3 + a_2a_3a_1a_4 + a_2a_3a_4a_1 + a_2a_4a_1a_3 + a_2a_4a_3a_1 + a_3a_2a_1a_4 + a_3a_2a_4a_1 + a_3a_1a_2a_4 + a_3a_1a_4a_2 + a_3a_4a_1a_2 + a_3a_4a_2a_1 + a_4a_2a_3a_1 + a_4a_2a_1a_3 + a_4a_3a_2a_1 + a_4a_3a_1a_2 + a_4a_1a_3a_2 + a_4a_1a_2a_3$$

Clearly, the left hand side reduces to $a_1\sqcup\!\sqcup a_2\sqcup\!\sqcup a_3\sqcup\!\sqcup a_4$, and the formula does not bring any new information.

For completeness, let us remark that there is a bosonic version of (24) in $K\langle A\rangle_{\sqcap\!\sqcap}[\xi]$, where $\sqcap\!\sqcap$ denotes the antishuffle product on $K\langle A\rangle$ (i.e. $\sqcap\!\sqcap = \sqcup\!\sqcup_q$ for $q = -1$, where the q-shuffle is defined by the recursive formula $au\sqcup\!\sqcup_q bv = a(u\sqcup\!\sqcup_q bv) + q^{|au|}b(au\sqcup\!\sqcup_q v)$)

$$\prod_i^{\rightarrow}(1 + x_i\xi_i) = \sum_{I\uparrow}\sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)a_{i_{\sigma(1)}}\cdots a_{i_{\sigma(n)}}\xi_I \tag{38}$$

leads to the following equality in the antishuffle algebra $K_{\sqcap\!\sqcap}\langle A\rangle$

$$\sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)a_{\sigma(1)}\cdots a_{\sigma(n)} = \begin{cases}\mathrm{Hf}_{\sqcap\!\sqcap}(Q_{kl})_{1\le k,l\le n} & \text{if } n \text{ is even}\\ \sum_p a_p \sqcap\!\sqcap \mathrm{Hf}_{\sqcap\!\sqcap}(Q_{kl})_{\substack{1\le k,l\le n\\ k,l\neq p}} & \text{if } n \text{ is odd}\end{cases} \tag{39}$$

where $Q_{kl} = Q_{lk} = a_ka_l - a_la_k$ if $k \le l$.

Expanding the hafnians, we see that this identity amounts to the associativity and anticommutatitity of the antishuffle.

Example 3.6 Let A=$\{a_1, a_2, a_3, a_4\}$. We have

$$(a_1a_2 - a_2a_1)\sqcap\!\sqcap(a_3a_4 - a_4a_3) + (a_1a_3 - a_3a_1)\sqcap\!\sqcap(a_2a_4 - a_4a_2) + (a_1a_4 - a_4a_1)\sqcap\!\sqcap(a_2a_3 - a_3a_2) = a_1a_2a_3a_4 - a_1a_2a_4a_3 - a_1a_3a_2a_4 + a_1a_3a_4a_2 + a_1a_4a_2a_3 + - a_1a_4a_3a_2 - a_2a_1a_3a_4 + a_2a_1a_4a_3 + a_2a_3a_1a_4 + -a_2a_3a_4a_1 - a_2a_4a_1a_3 + a_2a_4a_3a_1 + - a_3a_2a_1a_4 + a_3a_2a_4a_1 + a_3a_1a_2a_4 - a_3a_1a_4a_2 - a_3a_4a_1a_2 + a_3a_4a_2a_1 - a_4a_2a_3a_1 + a_4a_2a_1a_3 + a_4a_3a_2a_1 + -a_4a_3a_1a_2 - a_4a_1a_3a_2 + a_4a_1a_2a_3.$$

Indeed, the left hand side reduces to $a_1\sqcap\!\sqcap a_2\sqcap\!\sqcap a_3\sqcap\!\sqcap a_4$.

3.3 Applications to iterated integrals

Setting $a_i = b_i$ in (17), we find

$$\sum_{\sigma\in\mathfrak{S}_{2n}}\epsilon(\sigma)a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(2n-1)}a_{\sigma(2n)} = \mathrm{Pf}_{\sqcup\!\sqcup}(a_ka_l - a_la_k)_{1\le k,l\le 2n} \tag{40}$$

Using Chen's lemma (3), we recover de Bruijn's formula (4). Indeed, taking as alphabet $A = \{1, \dots, 2n\}$ the subscripts of our functions ϕ_i, and applying the linear form $\langle\ \rangle$ to both sides of (40), we obtain

$$\sum_{\sigma \in \mathfrak{S}_{2n}} \epsilon(\sigma)\langle\sigma\rangle = \langle \mathrm{Pf}_{\,\sqcup\!\sqcup}\,(ij - ji)\rangle = \mathrm{Pf}\,(\langle ij - ji\rangle) \tag{41}$$

which is exactly (4).

Formula (17) can also be interpreted in terms of iterated integrals, and the corresponding identity seems to be new. Taking $4n$ functions $\phi_1, \dots, \phi_{2n}$, $\psi_1, \dots, \psi_{2n}$ of $2n$ variables, we obtain

$$\int \cdots \int_{a \le x_1 < \cdots < x_{2n} \le b} \det\left(\phi_i(x_j)|\psi_i(x_j)\right) dx_1 \cdots dx_{2n} = \mathrm{Pf}\ (Q_{ij})_{1 \le i,j \le 2n} \tag{42}$$

where $(\phi_i(x_j)|\psi_i(x_j))$ denotes the $2n \times 2n$ matrix whose ith row is

$$[\phi_i(x_1), \psi_i(x_2), \dots, \phi_i(x_{2n-1}), \psi_i(x_{2n})],$$

and

$$Q_{ij} = \int\int_{a \le x < y \le b} [\phi_i(x)\psi_j(y) - \phi_j(x)\psi_i(y)] dx dy\,. \tag{43}$$

Example 3.7 Let us consider two families of complex numbers $(x_i)_{i \ge 1}$ and $(y_i)_{i \ge 1}$ (each x_i being non zero). We construct for each i two functions

$$f_i(t) = \frac{1}{x_i^2} t^{\frac{1}{x_i} - 1}$$

and

$$g_i(t) = \frac{y_i}{x_i} t^{\frac{1}{x_i} - 1}.$$

The iterated integral

$$I_n = \int \cdots \int_{0 \le t_1 < t_2 < \cdots < t_{2n} \le 1} \det(f_i(t_j)|g_i(t_j)) dt_1 \cdots dt_n$$

can be written as the Pfaffian

$$I_n = \mathrm{Pf}\left(\frac{y_j - y_i}{x_i + x_j}\right),$$

The calculation of this Pfaffian, due to Sundquist [17], is discussed in section 4.

Application of Chen's lemma to (19) gives the second de Bruijn formula (5), and we can deduce from (23) a "de Bruijn like" equality for permanents

$$\int_{a\le t_1<\cdots<t_{2n}\le b}\cdots\int \operatorname{per}\ (\phi_i(t_j))\,dt_1\cdots dt_{2n} = \frac{1}{(2n)!!}\operatorname{Hf}\left(\int\int_{a\le x<y\le b}[\phi_i(x)\phi_j(y)+\phi_j(x)\phi_i(y)]dxdy\right). \tag{44}$$

which is not very informative, as both sides are clearly equal to the product of the integrals $\int_a^b \phi_i(x)dx$. The corresponding identities for determinants and permanents of odd order can be obtained in the same way.

Example 3.8 We can write down a permanental version of Wigner's integral [13], also discussed in section 4

$$I_n = \int_{a\le t_1<t_2<\cdots<t_{2n}\le b}\cdots\int \operatorname{per}\left(e^{x_it_j}\right)dt_1\cdots dt_{2n}$$

which evaluates to

$$I_n = \frac{1}{(2n)!!}\operatorname{Hf}\left(\frac{(b_i-a_i)(b_j-a_j)}{x_ix_j}\right) = \prod_i \frac{b_i-a_i}{x_i}$$

with $a_i = e^{ax_i}$ and $b_i = e^{bx_i}$.

Example 3.9 The equality

$$\sum_{\sigma\in\mathfrak{S}_m}\frac{1}{x_{\sigma(1)}}\frac{1}{x_{\sigma(1)}+x_{\sigma(2)}}\cdots\frac{1}{x_{\sigma(1)}+\cdots+x_{\sigma(m)}} = \frac{1}{x_1\cdots x_m} \tag{45}$$

can be easily shown by induction on m, remarking that

$$\sum_k \frac{1}{x_1\cdots x_{k-1}x_{k+1}\cdots x_m} = \frac{x_1+\cdots+x_m}{x_1\cdots x_m}. \tag{46}$$

But we can also see it as an integral of type (44). Indeed, direct calculation of

$$I(x_1,\cdots,x_m) = \int_{0\le t_1<\cdots<t_m\le 1}\cdots\int \operatorname{per}\left(t_i^{x_j-1}\right)dt_1\cdots dt_m$$

gives

$$I(x_1,\cdots,x_m) = \sum_\sigma \frac{1}{x_{\sigma(1)}}\frac{1}{x_{\sigma(1)}+x_{\sigma(2)}}\cdots\frac{1}{x_{\sigma(1)}+\cdots+x_{\sigma(m)}}.$$

On another hand,

$$I(x_1,\cdots,x_m) = \lambda_1\cdots\lambda_m$$

with $\lambda_i = \int_0^1 t^{x_i-1}dt = \frac{1}{x_i}$.

Finally, applying Chen's lemma to (22) we obtain

$$\int\cdots\int_{a\le t_1<\cdots<t_n\le b} \mathrm{per}\,(\phi_i(t_j)|\psi_i(t_j))\,dt_1\cdots dt_n$$
$$= \mathrm{Hf}\left(\int_a^b [\phi_i(x)\psi_j(x)+\phi_j(x)\psi_i(x)]\right) \quad (47)$$

Example 3.10 The hafnian analog of example 3.7 is

$$I(x,y) = \int\cdots\int_{0\le t_1<\cdots<t_n\le 1} \mathrm{per}\left(y_i t_j^{x_i-1}\,\left|t_j^{x_i}\right.\right) dt_1\cdots dt_n = \mathrm{Hf}\left(\frac{y_i+y_j}{x_i+x_j}\right).$$

If we compute $I(x_1,\cdots,x_{2n},y_1,\cdots,y_{2n})$ directly, we find the symmetrization identity

$$\sum_{\sigma\in\mathfrak{S}_{2n}} \frac{y_{\sigma(1)}y_{\sigma(3)}\cdots y_{\sigma(2n-1)}}{(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(1)}+x_{\sigma(2)}+x_{\sigma(3)}+x_{\sigma(4)})\cdots(x_{\sigma(1)}+\cdots+x_{\sigma(2n)})} = \mathrm{Hf}\left(\frac{y_i+y_j}{x_i+x_j}\right).$$

4 Examples

4.1 Pfaffian identities

The classical identity of Schur [15] (see [12] for an illuminating discussion of this identity and of similar ones)

$$\mathrm{Pf}\left(\frac{x_k-x_l}{x_k+x_l}\right) = \prod_{k<l}\left(\frac{x_k-x_l}{x_k+x_l}\right) \quad (48)$$

(where $k,l=1,\ldots,2m$) has been recently generalized by Sundquist [17], giving evaluations of

$$\mathrm{Pf}\left(\frac{y_k-y_l}{x_k+x_l}\right) \quad \text{and} \quad \mathrm{Pf}\left(\frac{y_k-y_l}{1+x_kx_l}\right)$$

and also by Ishikawa and Wakayama [10], giving the interpolation

$$\mathrm{Pf}\left(\frac{y_k-y_l}{a+b(x_k+x_l)+cx_kx_l}\right).$$

Actually, all three identities are equivalent. Indeed, one can obtain $\mathrm{Pf}\left(\frac{y_k-y_l}{1+x_kx_l}\right)$ from $\mathrm{Pf}\left(\frac{y_k-y_l}{z_k+z_l}\right)$ by setting $z_k=\frac{1}{2}\left(\frac{1-ix_k}{1+ix_k}\right)$ and then setting $x_k=\alpha+\beta t_k$

one obtains Pf $\left(\frac{y_k-y_l}{(1+\alpha^2)+\alpha\beta(t_k+t_l)+\beta^2 t_k t_l}\right)$. Therefore, to relate these identities to the previous considerations, it will be sufficient to consider one of them, say, the one for Pf $\left(\frac{y_k-y_l}{x_k+x_l}\right)$. To this aim, we consider the de Bruijn integral

$$I_{2m} = \int\cdots\int_{a\le t_1<\cdots<t_{2m}\le b} \det\left[e^{x_k t_l}\right] dt_1\cdots dt_{2m} = \mathrm{Pf}\,(Q) \qquad (49)$$

where $Q=(Q_{kl})$ is given in terms of $a_k = e^{ax_k}$ and $b_l = e^{bx_l}$ by

$$Q_{kl} = \frac{1}{x_k x_l(x_k+x_l)}\begin{vmatrix} u_k & v_k \\ u_l & v_l \end{vmatrix} \qquad (50)$$

and $u_k = b_k - a_k$, $v_k = x_k(a_k+b_k)$. The integral (49), which is of interest in random matrix theory, can be evaluated directly, without the help of de Bruijn's formula. This is a rather tedious calculation, originally due to E. Wigner (unpublished), which can be found in Mehta's book [13] (see p. 455, A.23). This calculation results into an expression different from the previous one ([13], Theorem 10.9.1), which in the present notation reads

$$I_{2m} = \prod_{k=1}^{2m}\frac{1}{x_k}\prod_{k<l}\frac{1}{x_k+x_l}\det(x_i^{j-1}(b_i+(-1)^j a_i))\,. \qquad (51)$$

Comparing the right hand sides of (51) and (49), and substituting $b_k - a_k = u_k$, $b_k + a_k = x_k^{-1}v_k$, one finds

$$\begin{vmatrix} u_1 & v_1 & x_1^2u_1 & x_1^2v_1 & x_1^4u_1 & \cdots & x_1^{2m-2}v_1 \\ u_2 & v_2 & x_2^2u_2 & x_2^2v_2 & x_2^4u_2 & \cdots & x_2^{2m-2}v_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{2m} & v_{2m} & x_{2m}^2u_{2m} & x_{2m}^2v_{2m} & x_{2m}^4u_{2m} & \cdots & x_{2m}^{2m-2}v_{2m} \end{vmatrix} = \prod_{i<j}(x_i+x_j)\mathrm{Pf}\left(\frac{u_iv_j-u_jv_i}{x_i+x_j}\right) \qquad (52)$$

which is one of Sundquist's formulas (when one sets $u_i = 1$ and $v_i = y_i$).

In Section 3, we have interpreted de Bruijn's formula as a combinatorial property of shuffles. In the following, we will observe that (at least) parts of Wigner's argument can be interpreted similarly.

To establish (51), Wigner starts with the simplest integral

$$J_n(x_1,\ldots,x_n) = \int\cdots\int_{a\le t_1<\cdots<t_{2m}\le b} e^{t_1x_1+\cdots+t_nx_n}dt_1\cdots dt_n\,, \qquad (53)$$

which yields I_n by antisymmetrization:

$$I_n(x_1,\dots,x_n) = \mathcal{A}J_n(x_1,\dots,x_n) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) J_n(x_{\sigma(1)},\dots,x_{\sigma(n)})\,. \tag{54}$$

If one computes the first integrals $J_1, J_2, \dots$, an induction pattern emerges, and one arrives at the expression

$$J_n(x_1,\dots,x_n) = \\ \sum_{k=0}^{n} (-1)^k a_k \cdots a_2 a_1 \cdot b_{k+1}\cdots b_n R(x_k, x_{k-1},\dots,x_1) R(x_{k+1},x_{k+2},\dots,x_n) \tag{55}$$

where we have set

$$R(z_1,\dots,z_r) = \frac{1}{z_1(z_1+z_2)\cdots(z_1+z_2+\cdots z_r)}\,. \tag{56}$$

The rest of the calculation relies upon the two identities

$$\sum_{k=0}^{n} (-1)^k R(x_k, x_{k-1},\dots,x_1) R(x_{k+1},x_{k+2},\dots,x_n) = 0 \tag{57}$$

and

$$\mathcal{A}R(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{x_i} \prod_{i<j} \frac{x_j - x_i}{x_j + x_i}\,. \tag{58}$$

Remark that if we set $\phi_z(t) = t^{z-1}$, we can write

$$R(z_1,\dots,z_r) = \langle z_1\cdots z_r\rangle = \int\cdots\int_{0\le t_1<\cdots<t_r\le 1} \phi_{z_1}(t_1)\cdots\phi_{z_r}(t_r)dt_1\cdots dt_r \tag{59}$$

so that for $u = x_k\cdots x_1$, $v = x_{k+1}\cdots x_n$ (regarded as words), $R(u)R(v) = \langle u\rangle\langle v\rangle = \langle u ⧢ v\rangle$.

With this at hand, it is clear that (57) amounts to the fact that the map $S(w) = (-1)^{|w|}\bar{w}$, where $\bar{w}$ is the mirror image of the word w, is the antipode of the shuffle Hopf algebra. This means that in the convolution algebra of $K_{⧢}\langle A\rangle$, i.e., $\mathrm{End}^{\mathrm{gr}}(K_{⧢}\langle A\rangle)$ endowed with the product $f*g = \mathrm{sh}\circ(f\otimes g)\circ C$, where $\mathrm{sh}(u\otimes v) = u ⧢ v$ and $C(w) = \sum_{uv=w} u\otimes v$, one has

$$S*I = I*S = \iota\circ\epsilon$$

where ι is the unit and ϵ the counit, so that $S * I$ of a polynomial is equal to its constant term. Hence, for any non empty word w,

$$\sum_{uv=w} (-1)^{|u|} \bar{u} \sqcup\!\sqcup v = 0\,,$$

which yields (57). Equation (58) can be obtained by writing

$$\mathcal{A}R(x_1,\dots,x_n) = \int_{0\le t_1<\cdots<t_n\le 1}\!\!\cdots\int \det(\phi_{x_i}(t_j))dt_1\dots dt_n = \mathrm{Pf}\,(A)$$

where

$$A_{ij} = \int_0^1 dt_j \int_0^{t_j} dt_i (t_i^{x_i-1}t_j^{x_j-1} - t_j^{x_i-1}t_i^{x_j-1}) = \frac{x_j - x_i}{x_i x_j (x_i + x_j)}$$

and applying Schur's identity (48).

There is also an interesting interpretation of (52), involving the hyperoctahedral group. Let us introduce a family of functions labelled by the alphabet $A = \{\bar{n},\dots,\bar{2},\bar{1},1,2,\dots,n\}$

$$\phi_k(t) = b_k t^{x_k-1} \quad \text{and} \quad \phi_{\bar{k}}(t) = -a_k t^{x_k-1}\,, \tag{60}$$

so that (55) can be written as

$$J_n(x_1,\dots,x_n) = \left\langle \sum_{k=0}^{n} \bar{k}\cdots\bar{2}\bar{1} \sqcup\!\sqcup k+1\cdots n \right\rangle \tag{61}$$

and

$$\mathcal{A}J_n = \left\langle \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma)\sigma \sum_{k=0}^{n} \bar{k}\cdots\bar{2}\bar{1} \sqcup\!\sqcup k+1\cdots n \right\rangle \tag{62}$$

If the word w occurs in $\bar{k}\cdots\bar{2}\bar{1} \sqcup\!\sqcup k+1\cdots n$ with the letters $\bar{k},\dots,\bar{1}$ at positions $i_1,\dots,i_k$, the sign of the underlying permutation α, obtained by erasing the bars, is $\epsilon(\alpha) = (-1)^{||I||}$, where $||I|| = \sum_l (i_l - 1)$. We can furthermore interpret such a word w as an element of the hyperoctahedral group B_n, generated by permutations and by the involutions $\tau_i : i \mapsto \bar{i}$ and $\tau_i(j) = j$ for $j \neq i$. The double sum in (62) can now be interpreted as an element of the group algebra $\mathbb{Z}B_n$, and since $\mathcal{A}\alpha = \epsilon(\alpha)\mathcal{A}$, we have for w as above

$$\mathcal{A}w = (-1)^{||I||}\mathcal{A}\tau_{i_1}\cdots\tau_{i_k}\,. \tag{63}$$

If we extend the action of B_n to polynomials in x_i, y_i, a_i, b_i by setting $\bar{x}_i = x_i$, $\bar{y}_i = y_i$ and $\bar{b}_i = -a_i$, (62) can now be rewritten

$$\mathcal{A}J_n = \mathcal{A}\circ(1+\tau_1)\circ(1-\tau_2)\circ\cdots\circ(1-\tau_{2m})\frac{b_1b_2\cdots b_{2m}}{x_1(x_1+x_2)\cdots(x_1+\cdots+x_{2m})}, \tag{64}$$

that is,

$$\mathcal{A}J_n = \mathcal{A}(QR) \tag{65}$$

where

$$Q = \prod_{i=1}^{2m}(b_i + (-1)^i a_i) \tag{66}$$

Let $x^\rho = x_1^0 x_2^1 \cdots x_n^{n-1}$, so that $\Delta = \mathcal{A}(x^\rho)$. Then, we see that (52) is equivalent to the symmetrization identity

$$\mathcal{A}(RQ)\mathcal{A}(x^\rho) = \mathcal{A}(R)\mathcal{A}(Qx^\rho). \tag{67}$$

4.2 Anticommutative symmetric functions and alterning quasi-symmetric functions

The fermionic Wick formula admits an interpretation in the language of noncommutative symmetric functions [8]. Let us recall that the starting point of this theory is the observation that the classical algebra of symmetric functions is a polynomial algebra $Sym = \mathbb{C}[h_1, h_2\cdots]$ in an infinite number of indeterminates (here, the h_i are the complete homogeneous symmetric functions). One defines the algebra **Sym** of noncommutative symmetric functions as the free associative algebra $\mathbb{C}\langle S_1, S_2, \cdots\rangle$ over an infinite sequence of letters, together with a "commutative image" homomorphism $S_n \to h_n$. In other words, **Sym** is the tensor algebra of the vector space $V = \bigoplus \mathbb{C}S_n$ and Sym is its symmetric algebra.

Here, we propose to have a look at its exterior algebra, which we will call $ASym$ (anticommutative symmetric functions). It is the image of **Sym** under the specialization $S_n \to S_n(A) = \eta_n$ (that is, the letter A is used here as a label for the image $F(A)$ of any noncommutative symmetric function F by the ring homomorphism $S_n \mapsto \eta_n$).

The dimension of the homogeneous component of degree n of $ASym$ is equal to the number of partitions of n into distinct parts. As an algebra, $ASym$ is rather trivial, but the dual Hopf algebra looks more interesting. To study it, we can follow the same strategy as in [8], and start with the Cauchy kernel. In the notation of [8]

$$\begin{aligned}
\sigma_1(XA) &= \overrightarrow{\prod_{k\geq 1}} \sigma_{x_k}(A) = \overrightarrow{\prod_{k\geq 1}} \sum_{i_k\geq 0} x_k^{i_k}\eta_{i_k} \\
&= \sum_r \sum_{i_1<\cdots<i_r} \left(\sum_{\sigma\in\mathfrak{S}_r} \epsilon(\sigma) M_{I\sigma}(X) \right) \eta_{i_1}\cdots\eta_{i_r} \qquad (68) \\
&= \sum_r \sum_{i_1<\cdots<i_r} \left(\sum_{\sigma\in\mathfrak{S}_r} \epsilon(\sigma) M_{I\sigma}(X) \right) S^I(A)
\end{aligned}$$

where the $M_I(X)$ are the quasimonomial quasi-symmetric functions, and $I\sigma$ means $(i_{\sigma(1)}, \cdots, i_{\sigma(r)})$. The coefficients

$$V_I(X) = \sum_\sigma \epsilon(\sigma) M_{I\sigma}(X) = \int d\eta_{\bar{I}} \sigma_1(XA) \qquad (69)$$

can be evaluated by means of the fermionic Wick formula. Introducing the operators

$$\phi_k = \frac{\partial}{\partial \eta_k} = \int d\eta_k \quad \text{ and } \phi_k^*(f) = \eta_k f$$

one has

$$V_I = \langle 0|\phi_{i_r}\cdots\phi_{i_1}|\sigma(XA)\rangle \qquad (70)$$

where $|0\rangle = 1$ and $\sigma_1(XA) = \overrightarrow{\prod_{i\geq 1}} e^{\Phi(x_i)}|0\rangle$ if $\sigma_x(A) = e^{\Phi(x)}$ (here the generating series $\Phi(x)$ of the second kind power sums reduces to $x\eta_1 + x^2\eta_2 + \cdots$). The Hausdorff series for the product of exponentials is now

$$\begin{aligned}
H &= H(\Phi(x_1), \Phi(x_2), \cdots) \\
&= \sum_{k\geq 1} \psi(x_k) + \frac{1}{2}\sum_{i<j} [\Phi(x_i), \Phi(x_j)] \\
&= \sum_{n\geq 1} M_n(X)\phi_n^* + \frac{1}{2}\sum_{k\neq l} \left(M_{kl}(X) - M_{lk}(X)\right)\phi_k^*\phi_l^* \\
&= \frac{1}{2}\sum_{k\neq l} Q_{kl}\phi_k^*\phi_l^* + \sum_{n\geq 1} M_n\phi_n^* \qquad (71)
\end{aligned}$$

where $Q_{kl} = M_{kl} - M_{lk}$. Hence,

$$V_I(X) = \int d\eta_{\bar{I}} e^H = \langle 0|e^{\sum_{n\geq 0} M_n\phi_n^*} e^{\frac{1}{2}{}^t\phi^* Q\phi^*}|0\rangle \qquad (72)$$

since the quadratic terms are central, so, expanding the exponential $e^{\sum_n M_n\phi_n^*}$, we obtain the following expression for the "alternating quasisymmetric functions".

$$V_I(X) = \sum_\sigma \epsilon(\sigma) M_{I\sigma}(X) \;=\; \langle \phi_{i_r}\cdots\phi_{i_1}\rangle$$
$$= \begin{cases} \mathrm{Pf}\,(Q_{i_k i_l})_{1\le k,l\le 2m} & \text{if } r=2m \text{ is even} \\ \sum_{k=1}^{2m+1}(-1)^{k-1}M_{i_k}\langle\phi_{i_{2m+1}}\cdots\widehat{\phi_{i_k}}\cdots\phi_{i_1}\rangle & \text{if } r=2m+1 \text{ is odd.}\end{cases} \tag{73}$$

This formula is similar to Schur's expression of his symmetric functions Q_λ as Pfaffians of the $Q_{(i,j)}$. Remark that these symmetric functions are also labelled by strict partitions, and that Schur's Pfaffian can also be derived from the fermionic Wick formula [19].

5 Generalizations

5.1 Hyperpfaffians

The Pfaffian of a skew symmetric matrix of even order can be expanded as

$$\mathrm{Pf}\;(M_{ij})_{1\le i,j\le 2n} = \sum_{\sigma\in\mathfrak{E}_{2n,2}} \epsilon(\sigma) M_{\sigma(1)\sigma(2)}\cdots M_{\sigma(2n-1)\sigma(2n)} \tag{74}$$

where the sum is over

$$\mathfrak{E}_{2n,2} = \{\sigma\in\mathfrak{S}_{2n} | \sigma(2i+1)<\sigma(2i+2), \sigma(2p-1)<\sigma(2p+1), \\ 0\le i\le n-1, 1\le p\le n-1\} \tag{75}$$

We can extend this definition by setting

$$\begin{aligned}\mathfrak{E}_{kn,k} \;&=\; \{\sigma\in\mathfrak{S}_{kn}|\sigma(ki+1)<\cdots<\sigma(k(i+1)), \\ &\quad\ \sigma(k(p-1)+1)<\sigma(kp+1), 0\le i\le n-1, 1\le p\le n-1\}.\end{aligned}$$

If we consider variables $M_{i_1\cdots i_k}$ verifying $M_{i_{\sigma(1)}\cdots i_{\sigma(k)}} = \epsilon(\sigma)M_{i_1\cdots i_k}$, the polynomial

$$\mathrm{Pf}^{[k]}\,(M_{i_1\cdots i_k})_{1\le i_1,\dots,i_k\le nk} = \sum_{\sigma\in\mathfrak{E}_{kn,k}} \epsilon(\sigma) M_{\sigma(1)\dots\sigma(k)}\cdots M_{\sigma((n-1)k+1)\dots\sigma(nk)} \tag{76}$$

appears as a natural generalization of the Pfaffian. This polynomial has been called the *hyperpfaffian* of the alternating tensor M by Barvinok [1].

Similarly, higher order Hafnians, or *hyperhafnians*, can be defined by

$$\mathrm{Hf}^{[k]}(M_{i_1\cdots i_k})_{1\le i_1,\cdots,i_k\le nk} = \sum_{\sigma\in\mathfrak{C}_{kn,k}} M_{\sigma(1)\ldots\sigma(k)}\cdots M_{\sigma((n-1)k+1)\ldots\sigma(nk)}. \tag{77}$$

Alternatively, these definitions can be presented as follows. Given an alternating tensor M of order k as above, consider the element

$$\Omega_M = \sum_{1\le i_1<\ldots<i_k\le nk} M_{i_1\cdots i_k}\eta_{i_1}\cdots\eta_{i_k} \tag{78}$$

of the Grassmann algebra of rank nk. Then, in this algebra

$$\Omega_M^n = n!\mathrm{Pf}^{[k]}(M)\eta_1\eta_2\cdots\eta_{kn}\,. \tag{79}$$

When $k=2$, this is one of the classical definitions of the Pfaffian.

Similarly, when M is a symmetric tensor of order k, again over an nk dimensional space, consider the element

$$G_M = \sum_{1\le i_1<\ldots<i_k\le nk} M_{i_1\cdots i_k}\xi_{i_1}\cdots\xi_{i_k} \tag{80}$$

wher $\xi_1,\ldots,\xi_{kn}$ are commuting symbols satisfying $\xi_i^2=0$. Then

$$G_M^n = n!\mathrm{Hf}^{[k]}(M)\xi_1\xi_2\cdots\xi_{kn}\,. \tag{81}$$

5.2 Composition of hyperpfaffians

The aim of this subsection is to compute the hyperfaffian

$$\mathrm{Pf}^{[2m]}\left(\mathrm{Pf}\,(a_{i,j})_{i,j\in\{k_1,\cdots,k_{2m}\}}\right)_{k_1\cdots,k_{2m}\in\{1,\cdots,2mn\}}$$

where $A=(a_{i,j})_{1\le i,j\le 2mn}$ is a skew symmetric matrix and $m,n>0$. In order to express this as a classical Pfaffian, we consider the 2−form

$$\omega = \sum_{1\le i,j\le 2mn} a_{ij}\eta_i\eta_j\,.$$

The mnth-power of ω is by definition of Pfaffians

$$\omega^{nm} = (mn)!\mathrm{Pf}\,(a_{i,j})_{1\le i,j\le 2mn}\,.$$

But, on another hand

$$\omega^{mn} = (m!\Omega)^n$$

where

$$\Omega = \sum_{1\le k_1<k_2<\cdots<k_{2m}} \mathrm{Pf}\,(a_{ij})_{i,j\in\{k_1,\cdots,k_{2m}\}}\eta_{k_1}\cdots\eta_{k_{2m}}\,.$$

Remarking that Ω^n is given by a hyperpfaffian

$$\Omega^n = n!\mathrm{Pf}^{\,[2m]}\left(\mathrm{Pf}\,(a_{ij})_{i,j\in\{k_1,\cdots,k_{2m}\}}\right)_{1\le k_1\cdots,k_{2m}\le 2mn}\eta_1\cdots\eta_{2nm}\,,$$

by identification, we find

$$\mathrm{Pf}^{\,[2m]}\left(\mathrm{Pf}\,(a_{i,j})_{i,j\in\{k_1,\cdots,k_{2m}\}}\right)_{1\le k_1\cdots,k_{2m}\le 2mn} = \frac{(mn!)}{(m!)^n n!}\mathrm{Pf}\,(a_{ij})_{1\le i,j\le 2mn}\,. \tag{82}$$

Example 5.1 If we set $m=2$ and $a_{i,j}=\frac{x_i-x_j}{x_i+x_j}$, we can generalize Schur's formula to the 4-hyperpfaffian

$$\mathrm{Pf}^{\,[4]}\left(\prod_{1\le s,t\le 4}\frac{x_{k_s}-x_{k_t}}{x_{k_s}+x_{k_t}}\right)_{1\le k_1,k_2,k_3,k_4\le 4n} = (2n)!!\prod_{1\le i,j\le 4n}\frac{x_i-x_j}{x_i+x_j} \tag{83}$$

5.3 Hyperpfaffian of a sum

In [16], J. Stembridge gives a formula allowing to compute the Pfaffian of a sum of $2s\times 2s$ skew symmetric matrices

$$\mathrm{Pf}\,(A+B) = \sum_{t=0}^{s}\sum_{I=(i_1,\cdots,i_{2t})\uparrow}(-1)^{|I|-t}\mathrm{Pf}\,(A_I)\mathrm{Pf}\,(B_{I^c}) \tag{84}$$

where $|I| = i_1+\cdots i_{2t}$ and $I^c=\{1\cdots 2m\}-I$.
This identity appears to be a special case of a more general one for hyperpfaffians

$$\mathrm{Pf}^{\,[2m]}(A+B) = \sum_{k=0}^{n}\sum_{I=\{i_1<\cdots<i_{2km}\}}(-1)^{|I|-km}\mathrm{Pf}^{\,[2m]}(A_I)\mathrm{Pf}^{\,[2m]}(B_{I^c}) \tag{85}$$

where $A=(a_{i_1,\cdots,i_{2n}})$ and $B=(b_{i_1,\cdots,i_{2n}})$ are two skew symmetric tensors of order $2m$ over a space of dimension $2mn$.
We set

$$\omega = \sum_{1\le i_1<\cdots<i_{2m}\le 2mn}(a_{i_1\cdots i_{2m}}+b_{i_1\cdots i_{2m}})\eta_{i_1}\cdots\eta_{i_{2m}}\,.$$

The nth-power of ω is

$$\omega^n = n!\mathrm{Pf}^{\,[2m]}(A+B)\eta_1\cdots\eta_{2mn}\,.$$

On the other hand, we have

$$\omega^n = \sum_{k=0}^{n} \binom{n}{k} \omega_A^k \omega_B^{n-k},$$

where $\omega_A = \sum a_{i_1\cdots i_{2m}} \eta_{i_1}\cdots\eta_{i_{2m}}$ and $\omega_B = \sum b_{i_1\cdots i_{2m}} \eta_{i_1}\cdots\eta_{i_{2m}}$.
Since

$$\omega_A^k = k! \sum_{i_1<\cdots<i_{2km}} \mathrm{Pf}^{[2m]}(a_{i_1\cdots i_{2m}})\eta_{i_1}\cdots\eta_{i_{2km}},$$

we find

$$\omega^n = n! \sum_k \sum_{I=\{i_1<\cdots<i_{2km}\}} \mathrm{Pf}^{[2m]}(A_I)\mathrm{Pf}^{[2m]}(B_{I^c})\eta_I\eta_{I^c},$$

where $I^c = \{1,\cdots,2mn\} - I$. Finally,

$$\omega^n = n! \sum_k \sum_{\substack{I\uparrow \\ I\subset\{1\cdots 2mn\}}} (-1)^{|I|-km}\mathrm{Pf}^{[2m]}(A_I)\mathrm{Pf}^{[2m]}(B_{I^c})\eta_I$$

which by identification leads to (85).

5.4 Minor summation formula

In [9], M. Ishikawa and M. Wakayama prove a minor summation formula for Pfaffians

$$\sum_{1\le k_1<\cdots<k_m\le n} \mathrm{Pf}\,(B^{k_1\cdots k_m}_{k_1\cdots k_m})\det(T^{1\cdots m}_{k_1\cdots k_m}) = \mathrm{Pf}\,(Q), \qquad (86)$$

where $A^{i_1\cdots i_m}_{j_1\cdots j_m}$ denotes the submatrix of A which consists of the rows $i_1\cdots i_m$ and the columns $j_1,\cdots,j_m$, B is a skew symmetric matrix, T is an arbitrary matrix and $Q = TB^tT$. We shall now extend this formula to hyperpfaffians.

Let $A = (a_{i_1\cdots i_{2m}})_{1\le i_1,\cdots,i_{2m}\le 2mn}$ be a skew symmetric tensor of order $2m$ over a space of dimension $2mn$ and $T = (T_{ij})_{1\le i,j\le 2mn}$ be a square matrix. We want to find a skew symmetric tensor $Q = (Q_{i_1,\cdots,i_{2m}})_{1\le i_1\cdots i_{2n}\le 2mn}$ verifying

$$\sum_{1\le k_1<\cdots<k_{2mt}\le 2mn} \mathrm{Pf}^{[2m]}(A_{k_1\cdots k_{2mt}})\det(T^{1\cdots 2mt}_{k_1\cdots k_{2mt}}) = \mathrm{Pf}^{[2m]}(Q), \qquad (87)$$

where $A_{k_1,\cdots,k_{2mt}} = (a_{i_1,\cdots,i_{2m}})_{i_1,\cdots,i_{2m}\in\{k_1,\cdots,k_{2mt}\}}$.
In order to compute such a tensor, we need first to remark that

$$\det\left(T^{1\cdots 2mn}_{k_1\cdots k_{2mn}}\right) = \sum_{\sigma\in\mathfrak{C}_{2mn,2m}} \epsilon(\sigma)\det\left(T^{\sigma(1)\cdots\sigma(2m)}_{k_1\cdots k_{2m}}\right)\cdots\det\left(T^{\sigma(2(n-1)m+1)\cdots\sigma(2mn)}_{k_{2(n-1)m+1}\cdots k_{2mn}}\right) \qquad (88)$$

which can be obtained by considering $\omega_k = \sum_{i=1}^{2nt} T_{ik}\eta_i$ and computing $\omega_{k_1}\cdots\omega_{k_t}$. Now, let us consider

$$Q_{i_1\cdots i_{2m}} = \sum_{1\le k_1<\cdots<k_{2m}\le 2mn} a_{k_1\cdots k_{2m}} \det\left(T^{i_1\cdots i_{2m}}_{k_1\cdots k_{2m}}\right)\,,$$

and the polynomial

$$\omega = \sum_{1\le i_1<\cdots<i_{2m}\le 2mt}\left(\sum_{k_1<\cdots<k_{2m}} a_{k_1\cdots k_{2m}} \det\left(T^{i_1\cdots i_{2m}}_{k_1\cdots k_{2m}}\right)\right)\eta_{i_1}\cdots\eta_{i_{2m}}\,.$$

A simple calculation gives

$$w^t = t!\mathrm{Pf}^{\,[2m]}(Q)\eta_1\cdots\eta_{2mn}$$

On another hand

$$\begin{aligned}\omega^t &= \sum_{\sigma\in\mathfrak{E}_{2m,2mt}} Q_{i_1\cdots i_{2m}}\cdots Q_{i_{2m(t-1)+1}\cdots i_{2mt}}\eta_{i_1}\cdots\eta_{i_{2nt}}\\ &= \sum_{1\le k_1<\cdots<k_{2mt}\le 2mn}\ \sum_{\sigma_1\in\mathfrak{E}_{2m,2mt}} a_{k_{\sigma(1)}\cdots k_{\sigma(2m)}}\cdots a_{k_{\sigma(2m(t-1)+1)}\cdots a_{\sigma(2mt)}}\\ &\sum_{\sigma_2\in\mathfrak{E}_{2m,2mt}}\epsilon(\sigma_2)\det\left(T^{\sigma_2(1)\cdots\sigma_2(2m)}_{k_{\sigma_1(1)}\cdots k_{\sigma_1(2m)}}\right)\cdots\det\left(T^{\sigma_2(2m(t-1)+1)\cdots\sigma_2(2mt)}_{k_{\sigma_1(2m(t-1)+1)}\cdots k_{\sigma_1(2mt)}}\right)\\ &\qquad\times\eta_1\cdots\eta_{2mt}\,. \qquad (89)\end{aligned}$$

By formula (88), we find

$$\begin{aligned}\omega^t = \sum_{\substack{1\le k_1<\cdots<k_{2mt}\le 2mn\\ \sigma\in\mathfrak{E}_{2m,2mt}}} & a_{k_{\sigma(1)}\cdots k_{\sigma(2m)}}\cdots a_{k_{\sigma(2m(t-1)+1)}\cdots a_{\sigma(2mt)}}\\ &\times\det(T^{1\cdots 2mt}_{k_{\sigma(1)}\cdots k_{\sigma(2mt)}})\eta_1\cdots\eta_{2mt} \qquad (90)\end{aligned}$$

so that

$$\omega^t = t!\sum_{1\le k_1<\cdots<k_{2mt}\le 2mn}\mathrm{Pf}^{\,[2m]}(A_{k_1,\cdots,k_{2mt}})\det(T^{1\cdots 2mt}_{k_1\cdots k_{2mt}})\eta_1\cdots\eta_{2mt}\,,$$

which leads to the identity (87).

5.5 Generalized de Bruijn formulas

We consider now a commutative ring K and for $2k$ a non zero even integer, a set of scalars $\{Q_{i_1\cdots i_{2k}}\}_{i_1<\cdots<i_{2k}}$. We shall first calculate the coefficients P_I of the $\bigwedge_K(\eta)$-series

$$F=\sum_{I\uparrow}P_I\eta_I=\prod_{i_1}\left(1+\sum_{i_1<\cdots<i_k}Q_{i_1\cdots i_{2k}}\eta_{i_1}\cdots\eta_{i_{2k}}\right). \tag{91}$$

As $\eta_i^2=0$, one has

$$F=\sum_n\sum_{\substack{i_{2k(p-1)+1}<\cdots<i_{2kp},\ 1\le p\le n\\ i_{2k(q-1)+1}<i_{2kq+1},\ 1\le q\le n-1}}Q_{i_1\ldots i_{2k}}\cdots Q_{i_{2k(n-1)+1}\cdots i_{2kn}}\eta_{i_1}\cdots\eta_{i_{2nk}} \tag{92}$$

and if $I=\{i_1,\cdots,i_{2nk}\}$ is an increasing sequence, the coefficient P_I of η_I is

$$P_I=\mathrm{Pf}^{[2k]}\left(Q_{i_{j_1}\cdots i_{j_{2k}}}\right). \tag{93}$$

Similarly, in the commutative algebra $K[\xi]$ ($\xi_i^2=0$)

$$\sum_{\substack{I\uparrow\\ l(I)=2nk}}\mathrm{Hf}^{[2k]}\left(Q_{i_{j_1}\cdots i_{j_{2k}}}\right)\xi_I=\prod_{i_1}\left(1+\sum_{i_1<\cdots<i_{2k}}Q_{i_1\cdots i_{2k}}\xi_{i_1}\cdots\xi_{i_{2k}}\right). \tag{94}$$

In the remainder of this section, the ring K is the shuffle algebra $R\langle A\rangle_{\text{Ш}}$ for the alphabet $A=\{a_{i_1\cdots i_{2k}}\}_{i_1,\cdots,i_{2k}\ge1}$.We consider here the series

$$\begin{aligned}T(A|\eta)&=\overrightarrow{\prod_{i_1}}\left(1+\sum_{i_1<\cdots<i_k}\left(\sum_{\sigma\in\mathfrak{S}_{2k}}\epsilon(\sigma)a_{i_{\sigma(1)}\cdots i_{\sigma(2k)}}\right)\eta_{i_1}\cdots\eta_{i_{2k}}\right)\\&=\sum_{\substack{I\uparrow\\ l(I)=2kn}}P_I\eta_I\end{aligned}$$

If we set $\mathbf{S}(i_1,\cdots,i_{2k})=\sum_{\sigma\in\mathfrak{S}_{2k}}\epsilon(\sigma)a_{i_{\sigma(1)}\cdots i_{\sigma(2k)}}$, we obtain for each increasing sequence $I=(i_1,\cdots,i_{2kn})$

$$\begin{aligned}P_I&=\int d\eta_{\bar I}T(A|\eta)\\&=\int d\eta_{\bar I}\overrightarrow{\prod_{i_1\in I}}(1+\sum_{\substack{i_2,\cdots,i_{2k}\in I\\ i_1<\cdots<i_{2k}}}\mathbf{S}(i_1,\cdots,i_{2k})\eta_{i_1}\cdots\eta_{i_{2k}})\\&=\sum_{\sigma\in\mathfrak{C}_{2n,2nk}}\epsilon(\sigma)\mathbf{S}(i_{\sigma(1)},\cdots,i_{\sigma(2k)})\text{Ш}\cdots\text{Ш}\mathbf{S}(i_{\sigma(2(n-1)k+1)},\cdots,i_{\sigma(2nk)})\\&=\sum_{\sigma\in\mathfrak{S}_{2nk}}\epsilon(\sigma)a_{i_{\sigma(1)}\cdots i_{\sigma(2k)}}\cdots a_{i_{\sigma(2(n-1)k+1)}\cdots i_{\sigma(2nk)}}.\end{aligned}$$

On the other hand, we find by (93)

$$P_I = \mathrm{Pf}\,_{\sqcup\!\sqcup}^{[2k]} \left(\sum_{\sigma\in\mathfrak{S}_{2k}} \epsilon(\sigma) a_{j_{\sigma(1)}\cdots i_{\sigma(2k)}} \right)_{j_1,\cdots,j_n\in I} \tag{95}$$

By Chen's lemma, this equality leads a generalized de Bruijn formula. We consider a set of $4k^2n$ functions $\Phi = \{\phi_{ij}\}_{\substack{1\le i\le 2k\\ 1\le j\le 2kn}}$. Equation (95) gives

$$\begin{aligned} \mathfrak{W}_\Phi(a,b) &= \int\cdots\int_{a\le x_1<\cdots<x_n\le b} \det(\phi_{1i}(x_j)|\cdots|\phi_{2ki}(x_j))_{\substack{1\le i\le 2nk\\ 1\le j\le n}} dx_1\ldots dx_n \\ &= \mathrm{Pf}^{[2k]} \left(\int_a^b \sum_{\sigma\in\mathfrak{S}_{2k}} \epsilon(\sigma)\phi_{1i_{\sigma(1)}}(x)\cdots\phi_{2ki_{\sigma(2k)}}(x)dx \right)_{1\le i_1,\cdots,i_{2k}\le 2nk}. \end{aligned} \tag{96}$$

Similarly there is a permanent version of this identity

$$\begin{aligned} \mathfrak{M}_\Phi(a,b) &= \int\cdots\int_{a\le x_1<\cdots<x_n\le b} \mathrm{per}(\phi_{1i}(x_j)|\cdots|\phi_{2ki}(x_j))_{\substack{1\le i\le 2nk\\ 1\le j\le n}} dx_1\ldots dx_n \\ &= \mathrm{Hf}^{[2k]} \left(\int_a^b \sum_{\sigma\in\mathfrak{S}_{2k}} \phi_{1i_{\sigma(1)}}(x)\cdots\phi_{2ki_{\sigma(2k)}}(x)dx \right)_{1\le i_1,\cdots,i_{2k}\le 2nk}. \end{aligned} \tag{97}$$

5.6 Sums of even powers of the Vandermonde

We can use formula (96) to obtain the average of certain determinants. For example, let us consider n real numbers $x_1,\cdots,x_n$ chosen at random from N values $y_1,\cdots,y_N$ with uniform probability. The average of the polynomial $\Delta^{2m}(x_1,\cdots,x_n)$ (for $m\in N$) can be expressed as an integral

$$\mathcal{A}(n,m) = \frac{1}{N^n}\int\cdots\int \Delta^{2m}(x_1,\cdots,x_n)\prod_{i=1}^n\left(\sum_{p=1}^N \delta(x_i-y_p)\right)dx_i$$

where δ denotes the Dirac distribution. We can transform $\mathcal{A}(n,m)$ into the interated integral of single determinant

$$\mathcal{A}(n,m) = \frac{n!}{N^n}\int\cdots\int_{-\infty<x_1<\cdots<x_n<\infty} \det\left(f_i^1(x_j)|f_i^2(x_j)|\cdots|f_i^{2m}(x_j)\right)dx_1\cdots dx_n$$

where

$$f_i^1(x) = \begin{cases} x^{i-1}\sum_{p=1}^N \delta(x-y_p) & \text{if } i \le n \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_i^s(x) = \begin{cases} x^{i-(s-1)n-1} & \text{if } (s-1)n+1 \le i \le sn \\ 0 & \text{otherwise} \end{cases}$$

if $1 < s \le 2m$. Using formula (96) we find

$$\mathcal{A}(n,m) = \frac{n!}{N^n}\mathrm{Pf}^{[2m]}\left(M_{i_1\cdots i_{2m}}\right) \tag{98}$$

where if $i_1 < \cdots < i_{2m}$

$$M_{i_1,\cdots,i_{2m}} = \begin{cases} \int_{-\infty}^{\infty} x^{s(i_1,\cdots,i_{2m})}\sum_{p=1}^N \delta(x-y_p)dx & \begin{array}{l}\text{if for each } s \le 2m,\\ (s-1)n < i_s \le sn\end{array} \\ 0 & \text{otherwise} \end{cases} \tag{99}$$

with

$$s(i_1,\cdots,i_{2m}) = \sum_{k=1}^{2m} i_k - m(2n(m-1)+2)$$

and

$$M_{i_{\sigma(1)}\cdots i_{\sigma(2m)}} = \epsilon(\sigma)M_{i_1\cdots i_{2m}}\,.$$

The nonzero entries of the hyperpfaffian are therefore the power-sums

$$M_{i_1,\cdots,i_{2m}} = \sum_{p=1}^N y_p^{s(i_1,\cdots,i_{2m})}. \tag{100}$$

with $i_1 < \cdots < i_{2m}$ and $(s-1)n < i_s \le sn$ for each $s \le 2m$.

In the case where $m = 1$, we find

$$\mathcal{A}(n,1) = \frac{n!}{N^n}\det\left(\sum_{p=1}^N y_p^{i+j-2}\right). \tag{101}$$

More generally, if we consider the measure

$$d\mu = \left(\frac{k}{N\sqrt{\pi}}\right)^n \prod_{p=1}^n \left(\sum_{p=1}^N e^{-k^2(x_i-y_p)^2}\right) dx_i\,,$$

we can generalize formula (98) to evaluate

$$\mathcal{A}(k,n,m) = \int\cdots\int \Delta^{2m}(x_1,\cdots,x_n)d\mu$$
$$= \frac{n!}{N^n}\mathrm{Pf}^{\,[2m]}\left(M^{(k)}_{i_1,\cdots,i_{2m}}\right) \quad (102)$$

where

$$M^{(k)}_{i_1\dots i_{2m}} = \sum_{p=1}^{N}\sum_{t=0}^{s(i_1,\cdots,i_{2m})}\binom{s(i_1,\cdots,i_{2m})}{t} y_p^{s(i_1,\dots,i_{2m})-t}\frac{H_t(0)}{(2k)^t} \quad (103)$$

where H_t denotes a Hermite polynomial. We can verify that $\lim\limits_{k\to\infty}\mathcal{A}(k,n,m) = \mathcal{A}(n,m)$, as expected.

Finally, one can generalize formula (98) by considering a matrix of functions

$$F = (f_i^k)_{\substack{1\leq i\leq 2nm\\ 1\leq k\leq 2m}}$$

and the determinant

$$P_F(x_1,\cdots,x_n) = \det\left(f_i^1(x_j)|\cdots|f_i^{2m}(x_j)\right).$$

Its average, when the x_i run over the set $\{y_1,\cdots,y_N\}$ endowed with the uniform probability, can be computed as above:

$$\begin{aligned}\mathcal{A}(n,m) &= \frac{n!}{N^n}\int\limits_{-\infty<x_1<\cdots<x_{2m}<\infty}\cdots\int P_F(x_1,\cdots,x_n)\prod_{i=1}^{n}\left(\sum_{p=1}^{N}\delta(x_i-y_p)\right)dx_i\\ &= \frac{n!}{N^n}\int\limits_{-\infty<x_1<\cdots<x_{2m}<\infty}\cdots\int P_{F'}(x_1,\cdots,x_n)dx_1\cdots dx_n\end{aligned} \quad (104)$$

where

$$F'(x) = \begin{pmatrix} \sum_{p=1}^{N}\delta(x-y_p)f_1^1 & f_1^2(x) & \cdots & f_1^{2m}(x)\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{p=1}^{N}\delta(x-y_p)f_{2mn}^1(x) & f_{2mn}^2(x) & \cdots & f_{2mn}^{2m}(x)\end{pmatrix} \quad (105)$$

By formula (96), we find

$$A(n,m) = \frac{n!}{N^n}\mathrm{Pf}^{\,[2m]}\left(M_{i_1\cdots i_{2m}}\right) \quad (106)$$

where

$$\begin{aligned} M_{i_1\cdots i_{2m}} &= \int_{-\infty}^{\infty} \sum_{\sigma\in\mathfrak{S}_{2m}} \epsilon(\sigma) \sum_{p=1}^{N} \delta(x-y_p) f_1^{i_{\sigma(1)}}(x)\cdots f_{2m}^{i_{\sigma(2m)}}(x)dx \\ &= \sum_{\sigma\in\mathfrak{S}_{2m}} \epsilon(\sigma) \sum_{p=1}^{N} f_1^{i_{\sigma(1)}}(y_p)\cdots f_{2m}^{i_{\sigma(2m)}}(y_p) \end{aligned}$$

To conclude, let us remark that the hyperpfaffians (98) and (102) can alternatively be expressed as hyperdeterminants in the sense of [1]. We note also that in dimension nk, the hyperdeterminant of an alternating tensor A of (even) rank k satisfies $\mathrm{DET}(A) = (\mathrm{Pf}^{[k]}(A))^k$, generalizing the classical case $k = 2$. Such relations will be investigated in a forthcoming paper.

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Institut Gaspard Monge
Université de Marne-la-Vallée
5 Boulevard Descartes
Champs-sur-Marne

77454 Marne-la-Vallée cedex 2
FRANCE