The moduli space of three qutrit states

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We consider pure entangled three-partite qutrit states and their normal forms under local filtering operations (SLOCC). We describe the orbit space of the SLOCC group $SL(3,\mathbb{C})^{\times 3}$ both in its affine and projective versions in terms of a very symmetric normal form parameterized by three complex numbers. The parameters of the possible normal forms of a given state are roots of an algebraic equation, which is proved to be solvable by radicals. The structure of the sets of equivalent normal forms is related to the geometry of certain regular complex polytopes.

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I. INTRODUCTION

Since the advent of quantum computation and quantum cryptography, entanglement has been promoted to a resource that allows quantum physics to perform tasks that are classically impossible. Quantum cryptography [1, 2] proved that this gap even exists with small systems of two entangled qubits. Furthermore, it is expected that the study of higher dimensional systems and of multipartite (e.g. 3-partite) states would lead to more applications. A seminal example is the so-called 3-qutrit Aharonov-state, which "*is so elegant it had to be useful*"[3]: Fitzi, Gisin and Maurer [3] found out that the classically impossible Byzantine agreement problem [4] can be solved using 3-partite qutrit states. From a more fundamental point a view, the Aharonov state led to nontrivial counterexamples of the conjectures on additivity of the relative entropy of entanglement [5] and of the output purity of quantum channels [6]. Obviously, these results provide a strong motivation for studying 3-partite qutrit states. Furthermore, interesting families of higher-dimensional states are perfectly suited to address questions concerning local realism and Bell inequalities (see e.g. [7] for a study of three-qutrit correlations).

Motivated by this, the goal of this paper was to classify all three-qutrit states. A possible direction is to look for classes of equivalent states, in the sense that they are equivalent up to local unitary transformations [8, 9, 10] or local filtering operations (also called SLOCC operations) [10, 11, 12, 13, 14]. In the case of three qubits, especially the last classification proved to yield a lot of insights (the classification up to local unitaries has too much parameters left); the reason for that is that in the closure of each semi-stable orbit induced by SLOCC operations, there is a unique state with maximal entanglement [10, 12].

A numerical method introduced in [10] can be used to compute this maximally entangled state. Applied to the three-qutrit case, this yields the normal form

$$\begin{aligned}|\psi\rangle \simeq\ & \alpha\,(|000\rangle + |111\rangle + |222\rangle) + \\ & \beta\,(|012\rangle + |120\rangle + |201\rangle) + \\ & \gamma\,(|021\rangle + |102\rangle + |210\rangle)\end{aligned}$$

Here α, β, γ are complex parameters, and the local density operators of this state are all proportional to the identity.

The existence of this normal form was apparently first proved in 1975 by Vinberg [15, 16], who deduced it from his notion of Weyl group of a graded Lie algebra, applied to a $\mathbb{Z}_3$-grading of the exceptional Lie algebra E_6. In this case, the Weyl group is the complex reflection group G_{25} of the Shephard-Todd classification [17]. A different normal form was known by 1939 from the work of Chanler [18]. Her method is more elementary, in the sense that it relies only on simple properties of plane cubics. However, there is no hint in Chanler's normal form of the role played by the interesting group G_{25}.

Our aim in the following will be to give some applications of the explicit formulas for the basic invariants discovered by Chanler. After some comments on the structure of the orbit spaces (Section II), we will give various expressions of the basic invariants (Section IV). Next, we will show that a short geometric argument suffice to prove that Chanler's polynomials freely generate the algebra of invariants and that Vinberg's normal form is generic (Section V). After a brief discussion of the fundamental cubic covariants (Section VI), we shall observe that specialization of these invariants to Vinberg's normal form yields precisely the same invariants of the finite group G_{25} which were determined by Maschke [19] in 1889 (Section VII). This will allow a geometrical understanding of Chanler's invariants, and a solution by

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radicals of the algebraic equation giving the parameters of the normal form of a generic state from the value of its invariants. The possible degeneracies of this equation are then shown to be related to the geometry of the regular complex polytopes [20], arrangements of hyperplanes [21] and complex reflection groups [17] (Section VIII).

II. ORBIT SPACES

Let $V = \mathbb{C}^3$ and $\mathcal{H} = V \otimes V \otimes V$, regarded as a representation of the group $G = SL(3,\mathbb{C})^{\times 3}$. We are interested in the classification of the orbits of G in $\mathcal{H}$ and in the projective space $\mathbb{P}(\mathcal{H})$.

A geometric classification of the orbits has been obtained at the end of the years 1930's, in a series of papers by Thrall [22], Thrall and Chanler [23], and Chanler [18]. A different and more complete discussion, including the description of the inclusions of orbit closures, can be found in [24, 25].

What will be of interest to us here is the description of the orbit spaces, in their affine and projective versions.

It is well known that it in general, the orbits of a group action on an algebraic variety cannot be regarded as the points of an algebraic variety. To remedy this situation, one has to discard certain degenerate orbits. It is then possible to construct a *categorical quotient* and a *moduli space*, which describe the geometry of sufficiently generic orbits, respectively in the affine and projective situation.

The *categorical quotient* $Y = \mathcal{H}//G$ is defined as the affine variety whose affine coordinate ring is the ring of polynomial invariants $R = \mathbb{C}[\mathcal{H}]^G$. The moduli space is the projective variety $\mathcal{M} = \mathrm{Proj}\,(R)$ of which R is the homogeneous coordinate ring. It is the quotient of the set $\mathbb{P}(\mathcal{H})^{\rm ss}$ of semi-stable points by the action of G (see [26]).

We shall see that the categorical quotient is a three dimensional affine space, and that the moduli space is isomorphic to the del Pezzo surface F^6, a sextic surface in $\mathbb{P}^6$.

III. THE HILBERT SERIES

Let $\mathcal{I} = \mathbb{C}[\mathcal{H}]^G$ be the graded algebra of G-invariant polynomial functions of $\mathcal{H}$. It is interesting to enquire whether there is any simple way to compute this series.

It is easily seen that its graded component $\mathcal{I}_d$ of degree d is zero if d is not multiple of 3, and that the dimension of $\mathcal{I}_{3k}$ is equal to the multiplicity of the trivial character of the symmetric group $\mathfrak{S}_{3k}$ in the cube $(\chi^{(k,k,k)})^3$ of the irreducible character labelled by the partition (k,k,k).

This formula allows the numerical calculation of the Hilbert series of invariants (we checked it up to degree 108) and this indicates that it should be

$$h(t) = \sum_{d\geq 0} \dim \mathcal{I}_d t^d = \frac{1}{(1-t^6)(1-t^9)(1-t^{12})}\,. \quad (1)$$

In principle, this can be proved directly by a residue calculation. However, this calculation seems to be out of reach at present.

That (1) is indeed correct has in fact already been proved by Vinberg [15, 16].

As already mentioned, independent invariants I_6, I_9, I_{12} of the appropriate degrees are known [18]. We shall see in the sequel that it is possible to prove that they form a system of free generators by adding a simple geometric argument to Chanler's original discussion.

Since the algebra of invariants is a polynomial algebra, the categorical quotient is just the affine space $\mathbb{C}^3$.

The moduli space is more interesting. The projective variety whose homogeneous coordinate ring is a polynomial algebra over generators of respective degrees $d_1,\dots,d_m$ is called a *weighted projective space* $\mathbb{P}(d_1,\dots,d_m)$. Hence, by definition, our moduli space is the weighted projective space $\mathbb{P}(6,9,12) \simeq \mathbb{P}(2,3,4)$. It is known that this space is isomorphic to $\mathbb{P}(1,2,3)$ [27], which in turn can be embedded as a sextic surface in $\mathbb{P}^6$, the so-called del Pezzo surface F^6 (see Ref. [28]).

IV. THE FUNDAMENTAL INVARIANTS

We shall identify a three qutrit state $|\psi\rangle \in \mathcal{H}$ with a trilinear form

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{1\leq i,j,k\leq 3} A_{ijk} x_i y_j z_k \quad (2)$$

in three ternary variables. To construct its fundamental concomitants, we shall need the notion of a transvectant, which is defined by means of Cayley's Omega process (see, e.g., Ref. [29]).

Let f_1, f_2 and f_3 be three forms in a ternary variable $\mathbf{x} = (x_1,x_2,x_3)$. Their tensor product $f_1\otimes f_2\otimes f_3$ is identified with the polynomial $f_1(\mathbf{x}^{(1)})f_2(\mathbf{x}^{(2)})f_3(\mathbf{x}^{(3)})$ in the three independent ternary variables $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$. We use the "trace" notation of [30] to denote the multiplication map $f_1\otimes f_2\otimes f_3 \to f_1f_2f_3$, that is,

$$\mathrm{tr} f_1(\mathbf{x}^{(1)})f_2(\mathbf{x}^{(2)})f_3(\mathbf{x}^{(3)}) = f_1(\mathbf{x})f_2(\mathbf{x})f_3(\mathbf{x}) \quad (3)$$

Cayley's operator $\Omega_{\mathbf{x}}$ is the differential operator

$$\Omega_{\mathbf{x}} = \begin{vmatrix} \frac{\partial}{\partial x_1^{(1)}} & \frac{\partial}{\partial x_1^{(2)}} & \frac{\partial}{\partial x_1^{(3)}} \\ \frac{\partial}{\partial x_2^{(1)}} & \frac{\partial}{\partial x_2^{(2)}} & \frac{\partial}{\partial x_2^{(3)}} \\ \frac{\partial}{\partial x_3^{(1)}} & \frac{\partial}{\partial x_3^{(2)}} & \frac{\partial}{\partial x_3^{(3)}} \end{vmatrix} \quad (4)$$

Now, we consider three independent ternary variables $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ together with the associated dual (contravariant) variables $\xi = (\xi_1,\xi_2,\xi_3)$, $\eta = (\eta_1,\eta_2,\eta_3)$, $\zeta = (\zeta_1,\zeta_2,\zeta_3)$ (that is, ξ_i is the linear form on the $\mathbf{x}$ space such that $\xi_i(x_j) = \delta_{ij}$).

A concomitant of f is a polynomial F in the A_{ijk}, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ξ, η, ζ, such that if $g = (g_1,g_2,g_3) \in SL(3,\mathbb{C})^3$, then

$$F(g\cdot A; g_1^{-1}\cdot\mathbf{x}, g_2^{-1}\cdot\mathbf{y}, g_3^{-1}\cdot\mathbf{z}; g_1\cdot\xi, g_2\cdot\eta, g_3\cdot\zeta) = \det(g_1)^{\mu_1}\det(g_2)^{\mu_2}\det(g_3)^{\mu_3}F(A;\mathbf{x},\mathbf{y},\mathbf{z};\xi,\eta,\zeta). \quad (5)$$

The algebra of concomitants admits only one generator of degree 1 in the A_{ijk}, which is the form f itself. Other concomitants can be deduced from f and the three mixed concomitants $P_x = \sum \xi_i x_i$, $P_y = \sum \eta_j y_j$ and $P_z = \sum \zeta_k z_k$, using transvectants. If F_1, F_2 and F_3 are three 6-uple forms in the independent ternary variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ξ, η and ζ, one defines for any $(n_1,n_2,n_3)\times(m_1,m_2,m_3)\in\mathbb{N}^3\times\mathbb{N}^3$ the multiple transvectant of F_1, F_2 and F_3 by

$$(F_1,F_2,F_3)^{n_1n_2n_3}_{m_1m_2m_3} = \mathrm{tr}\quad \Omega_{\mathbf{x}}^{n_1}\Omega_{\mathbf{y}}^{n_2}\Omega_{\mathbf{z}}^{n_3}\Omega_{\xi}^{m_1}\Omega_{\eta}^{m_2}\Omega_{\zeta}^{m_3} \prod_{i=1}^{3} F_i(\mathbf{x}^{(i)},\mathbf{y}^{(i)},\mathbf{z}^{(i)};\xi^{(i)},\eta^{(i)},\zeta^{(i)}). \quad (6)$$

For convenience, we will set $(F_1,F_2,F_3)^{n_1n_2n_3} = (F_1,F_2,F_3)^{n_1n_2n_3}_{000}$. The concomitants of degree 2 given by Chanler [18] can be obtained using these operations:

$$Q_\alpha = \frac{1}{2}(f,f,P_yP_z)^{011} \quad (7)$$
$$Q_\beta = \frac{1}{2}(f,f,P_xP_z)^{101} \quad (8)$$
$$Q_\gamma = \frac{1}{2}(f,f,P_xP_z)^{110} \quad (9)$$

The invariant I_6 is then

$$I_6 = \frac{1}{12}(Q_x,Q_x,Q_x)^{200}_{011}. \quad (10)$$

There is an alternative expression using only the ground form,

$$I_6 = \frac{1}{1152}(f^2,f^2,f^2)^{222}. \quad (11)$$

Now, in degree 3 the covariants B_α, B_β and B_γ of [18] are

$$B_\alpha = \frac{1}{6}(f,f,f)^{011} \quad (12)$$
$$B_\beta = \frac{1}{6}(f,f,f)^{101} \quad (13)$$
$$B_\gamma = \frac{1}{6}(f,f,f)^{110}. \quad (14)$$

Other concomitants in the list of Chanler can be written in a similar way. For example, we have

$$\begin{aligned} H &= \frac{1}{2}(f^2,f,P_xP_yP_z)^{111} && (15)\\ E_\alpha &= (Q_x,f,P_x)^{100} && (16)\\ E_\beta &= (Q_y,f,P_y)^{010} && (17)\\ E_\gamma &= (Q_z,f,P_z)^{001} && (18)\end{aligned}$$

The knowledge of these concomitants allows one to construct the invariants I_9 and I_{12}

$$\begin{aligned} I_9 &= \frac{1}{576}(E_\alpha,E_\beta,E_\beta)^{111}_{111} && (19)\\ I_{12} &= \frac{1}{576}(B_\alpha f,B_\alpha f,B_\alpha f)^{411}. && (20)\end{aligned}$$

V. NORMAL FORM AND INVARIANTS

It will be shown that a generic state can be reduced to the normal form

$$A_{ijk} = u\delta_{ijk} + \frac{w-v}{2}\epsilon_{ijk} + \frac{w+v}{2}|\epsilon_{ijk}| \quad (21)$$

where ϵ_{ijk} is the alternating tensor, or, otherwise said, that the generic trilinear form $f(\mathbf{x},\mathbf{y},\mathbf{z})$ is equivalent to some

$$\begin{aligned} N_{uvw}(\mathbf{x},\mathbf{y},\mathbf{z}) &= u(x_1y_1z_1 + x_2y_2z_2 + x_3y_3z_3)\\ &+v(x_1y_3z_2 + x_2y_1z_3 + x_3y_2z_1) \quad (22)\\ &+w(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2) \quad (23)\end{aligned}$$

This normal form is in general not unique, and the relations between the various N_{uvw} in a given orbit is an interesting question, which will be addressed in the sequel.

The validity of our normal form can be proved in several ways. The shortest one, although not the most elementary, relies on the results of [18], and starts with computing the invariants of N_{uvw}. Let us denote by $C_k \equiv C_k(u,v,w)$ $(k = 6,9,12)$ the values of the I_k on N_{uvw}. Direct calculation gives, denoting by m_λ the monomial symmetric functions of u,v,w

$$\begin{aligned} C_6 &= m_6 - 10m_{33}\,, && (24)\\ C_9 &= (u^3-v^3)(u^3-w^3)(v^3-w^3)\,, && (25)\\ C_{12} &= m_{12} + 4m_{93} + 6m_{66} + 228m_{633}\,. && (26)\end{aligned}$$

It is easily checked by direct calculation that the Jacobian of these three functions is nonzero for generic values of (u,v,w). Actually, its zero set consists of twelve planes, whose geometric significance will be discussed below. The categorical quotient exists, since G is a reductive group acting on an affine variety ([26], section 4.4).

Let us denote by $\varphi\ :\ \mathcal{H} \xrightarrow{(I_6,I_9,I_{12})} \mathbb{C}^3$, the map sending a trilinear form to its three invariants, so that $(C_6,C_9,C_{12}) = \varphi(N_{uvw})$. Let $S = \{N_{uvw}|(u,v,w)\in\mathbb{C}^3\}$ be the three dimensional space of normal forms. The nonvanishing of the Jacobian proves that φ induces a dominant mapping from S to $\mathbb{C}^3$ (that is, the direct image of any non-empty open subset of S contains a non-empty open subset of $\mathbb{C}^3$). Note that the independence of C_6, C_9, C_{12} implies the independence of I_6, I_9, I_{12}. Now, Chanler [18] has shown that I_6, I_9, I_{12} separate the orbits in general position. This proves that the field of rational invariants of G is freely generated by I_6, I_9, I_{12} ([26], Lemma 2.1). As a consequence, φ is a *rational quotient* ([26], section 2.4) for the action of G on $\mathcal{H}$ (actually, this also implies that φ is a categorical quotient, by [26], Proposition 2.5 and Theorem 4.12, using that $\varphi_{|S}$ is surjective, whence also φ).

There exists a nonempty open subset Y_0 of $\mathbb{C}^3$ such that the fiber of φ over each of its points is the closure of an orbit ([26], Proposition 2.5). Let then $U_0 = \varphi^{-1}(Y_0)$. This set cuts S since $\varphi_{|S}$ is dominant. Let U_1 be the union of all orbits having maximal dimension (a nonempty open set, the function *dimension of the orbit* being lower semi-continuous). It is easy to see that U_1 intersects S (for instance at $u = 1, v = 1, w = -1$, whose orbit has dimension $24 = \dim G$, as may be checked by direct calculation). Let $S_0 = U_1 \cap S$, a dense open subset of S. The set $\varphi^{-1}\varphi(S_0)$ thus contains a dense open subset U_2 of $\mathcal{H}$. One then checks that $U_0 \cap U_1 \cap U_2$ (a dense open subset, as an intersection of dense open subsets of an irreducible space) is contained in GS. This proves $\overline{GS} = \mathcal{H}$, that is, the normal form N_{uvw} is generic.

Let us remark that the above discussion proves, thanks to Igusa's theorem ([26], Theorem 4.12) that $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[I_6, I_9, I_{12}]$, that is, the algebra of invariants is freely generated by Chanler's invariants.

Is is also possible to give a direct proof using the same technique as in [18]. Chanler's method rely on the geometry of plane cubics, which will play a prominent role in the sequel.

VI. THE FUNDAMENTAL CUBICS

The trilinear form $f(\mathbf{x}, \mathbf{y}, \mathbf{z})$ can be encoded in three ways by a 3×3 matrix of linear forms $M_x(\mathbf{x})$, $M_y(\mathbf{y})$ and $M_z(\mathbf{z})$, defined by

$$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = {}^t\mathbf{y} M_x(\mathbf{x})\mathbf{z} = {}^t\mathbf{x} M_y(\mathbf{y})\mathbf{z} = {}^t\mathbf{x} M_z(\mathbf{z})\mathbf{y} \quad (27)$$

and the classification of trilinear forms amounts to the classification of one of these matrices, say $M_x(\mathbf{x})$ up to left and right multiplication by elements of $SL(3, \mathbb{C})$ and action of $SL(3, \mathbb{C})$ on the variable $\mathbf{x}$.

The most immediate covariants of f are the determinants of these matrices

$$\begin{aligned} X(\mathbf{x}) &= \det M_x(\mathbf{x})\,, & (28)\\ Y(\mathbf{y}) &= \det M_y(\mathbf{y}) & (29)\\ Z(\mathbf{z}) &= \det M_z(\mathbf{z})\,. & (30)\end{aligned}$$

These are ternary cubic forms, and for generic f the equations $X(\mathbf{x}) = 0$, etc. will define non singular cubics (elliptic curves) in $\mathbb{P}^2$. It is shown in [23] that whenever one of these curves is elliptic, so are the other two ones, and moreover, all three are projectively equivalent. Actually, one can check by direct calculation that they have the same invariants. When $f = N_{uvw}$, these three cubics have even the same equation and are in the Hesse canonical form [31]

$$X(\mathbf{x}) = -\phi(x_1^3 + x_2^3 + x_3^3) + \psi\, x_1 x_2 x_3 = Y(\mathbf{x}) = Z(\mathbf{x}) \quad (31)$$

where we introduced, following the notation of [19],

$$\phi = uvw\,, \quad \psi = u^3 + v^3 + w^3\,. \quad (32)$$

The Aronhold invariants of the cubics (31) are given by

$$\begin{aligned} 6^4 S &= -\phi(\psi^3 + (6\phi)^3) & (33)\\ 6^6 T &= (6\phi)^6 + 20(6\phi^3)\psi^3 - 8\psi^6\,. & (34)\end{aligned}$$

These are of course invariants of f. We recognize that $6^4 S = -C_{12}$, and we introduce an invariant I_{18} such that $C_{18} = I_{18}(N_{uvw}) = 6^6 T$. The three cubics have the same discriminant $64S^3 + T^2$, known to be proportional to the hyperdeterminant of f [32, 33], which we normalize as

$$\Delta = 27(64S^3 + T^2)\,. \quad (35)$$

Then $\Delta = {C'}_{12}^3$, where C'_{12} is the product of twelve linear forms

$$\begin{aligned} C'_{12} = \ & uvw(u+v+w)(\varepsilon u+v+w)(u+\varepsilon v+w)\\ & \times(\varepsilon^2 u+\varepsilon v+w)(u+\varepsilon^2 v+w)\\ & \times(\varepsilon u+\varepsilon v+w)(\varepsilon^2 u+v+w)\\ & \times(\varepsilon u+\varepsilon^2 v+w)(\varepsilon^2 u+\varepsilon^2 v+w) \qquad (36)\end{aligned}$$

where $\varepsilon = e^{2i\pi/3}$, so that $C'_{12} = 0$ is the equation in $\mathbb{P}^2$ of the twelve lines containing 3 by 3 the nine inflection points of the pencil of cubics

$$u^3 + v^3 + w^3 + 6m\, uvw = 0 \quad (37)$$

obtained from X, Y, Z by treating the original variables as parameters. We note also that the Jacobian of C_6, C_9, C_{12} is proportional to ${C'}_{12}^2$.

VII. SYMMETRIES OF THE NORMAL FORMS

The normal form N_{uvw} is not unique. A generic f has 648 different normal forms, and the points (u, v, w) for which this number is reduced form interesting geometric configurations.

To see this, we can remark that the Hilbert series (1) is also the one of the ring of invariants of the group number 25 in the classification of irreducible complex reflection groups of Shephard and Todd [17]. This group, which we will denote by K, has order 648. It is one of the groups considered by Maschke [19] in his determination of the invariants of the symmetry group of the 27 lines of a general cubic surface in $\mathbb{P}^3$ (a group with 51840 elements, which is related to the exceptional root system E_6). To define K, we first have to introduce Maschke's group H, a group of order 1296, which is generated by the matrices of the linear transformations on $\mathbb{C}^3$ given in Table 1.

TABLE I: The generators of H

	A	B	C	D	E
u'	v	u	u	u	$\frac{1}{i\sqrt{3}}(u+v+w)$
v'	w	w	εv	εv	$\frac{1}{i\sqrt{3}}(u+\varepsilon v+\varepsilon^2 w)$
w'	u	v	$\varepsilon^2 w$	εw	$\frac{1}{i\sqrt{3}}(u+\varepsilon^2 v+\varepsilon w)$

This group contains in particular the permutation matrices, and simultaneous multiplication by $\pm\varepsilon^k$, since $E^2 = -B$. The subgroup H is the one in which odd permutations can appear only with a minus sign. It is generated by A, C, D, E.

Then, as proved by Maschke, the algebra of invariants of K in $\mathbb{C}[u,v,w]$ is precisely $\mathbb{C}[C_6, C_9, C_{12}]$.

Hence, we can conclude that K is the symmetry group of the normal forms N_{uvw} (see Note [36]).

Taking into account the results of Section V, we see that

$$S = \{N_{uvw} \mid (u,v,w) \in \mathbb{C}^3\} \tag{38}$$

is what is usually called a Chevalley section of the action of G on $\mathcal{H}$, with Weyl group K (see Ref. [26], p. 174).

VIII. THE FORM PROBLEM

Klein (see [34]) has introduced and investigated the notion of "Formenproblem" associated to a finite group action. This is the following: given the numerical values of the invariants, compute the coordinates of a point of the corresponding orbit.

In our case, we shall see that the problem of finding the parameters (u,v,w) of the normal form of a given generic f, given the values of the invariants, can be reduced to a chain of algebraic equations of degree at most 4, hence sovable by radicals.

Let $a = I_6$, $b = I_{12}$ and $c = I_{18}$ (we start with I_{18}, because C_{18} is a symmetric function of u^3, v^3, w^3, and at the end of the calculation, select the solutions which give the correct sign for C_9, which is alternating).

What we have to do is to determine the elementary symmetric functions $e_1 = \psi, e_2 = \chi, e_3 = \phi^3$ of u^3, v^3, w^3. Let $\lambda = 216\phi^3$. Then,

$$\psi^4 + \lambda\psi - b \ = 0 \tag{39}$$

$$\psi^6 - \frac{5}{2}\lambda\psi^3 - \frac{1}{8}\lambda^2 - c \ = 0\,. \tag{40}$$

Eliminating λ from these equations, we get a quartic equation for ψ^2

$$27\psi^8 - 18b\psi^4 - 8c\psi^2 - b^2 = 0\,. \tag{41}$$

The discriminant (with respect to ψ) of this polynomial is proportional to $D = b^2(b^3-c^2)^4$. When it is non zero, we get 8 values for e_1, each of which determines univocally e_2 and e_3. Hence, we obtain eight cubic equations whose

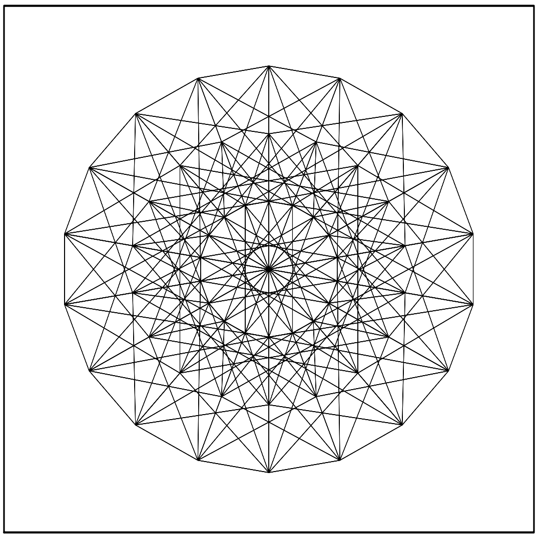

FIG. 1: The polyhedron 2{4}3{3}3

roots are the possible values of u^3, v^3, w^3. This gives 8 sets, whence $8 \times 6 = 48$ triples, each of which providing generically 27 values of (u,v,w), in all $48 \times 27 = 1296$ triples corresponding to the given values of a, b, c, among which exactly $1296/2 = 648$ give the correct sign for I_9. The common discriminant of the 8 cubics is $\delta = a^3 - 3ab + 2c$. Clearly, when $\delta \neq 0$, we will have 648 triples. If $\delta = 0$, one can check that the cubics cannot have a triple root, and that no root is zero. Hence, in this case, we obtain again 648 triples.

If $D = 0$, we can have $b^3 = c^2$ or $b = 0$. In the first case, setting $b = q^2, c = q^3$, the equation becomes

$$(\psi^2 - q)^3(\psi^2 + 2q)^3 = 0\,. \tag{42}$$

In this case, we get only four quartics for ψ^2. If $C_9 \neq 0$, we obtain 216 triples. If $C_9 = 0$ and $b = a^2/4$, $c = -a^3/8$ we obtain again 216 triples which form the centers of the edges of a complex polyhedron of type 2{4}3{3}3 in $\mathbb{C}^3$ (see Fig. 1), in the notation of Ref. [20]. The vertices of this polyhedron are the vertices of two reciprocal Hessian polyhedra (see Fig. 2) and its edges [37] join each vertex of one Hessian polyhedron to the 8 closest vertices of the other one. If $C_9 = 0$ and $b = a^2$, $c = a^3$, we obtain only 72 triples which are the centers of the edges of a Hessian polyhedron. In the case where $b = 0$, we have to distinguish between the cases $c \neq 0$ and $c = 0$. If $c \neq 0$, we find 648 triples, whatever the value of a. If $c = 0$, we obtain 27 triples if $a \neq 0$, and only one if $a = 0$.

Indeed, for $b = c = 0$, the ψ-equation reduces to $\psi^8 = 0$, and all the cubics collapse to $12U^3 - aU = 0$. For $a \neq 0$ we obtain precisely 27 triples (u,v,w) which form the vertices of a Hessian polyhedron in $\mathbb{C}^3$ (see Ref. [35]).

From the results of Ref. [21] about the arrangement of 12 hyperplanes formed by the mirrors of the pseudoreflections of $K = G_{25}$, we can determine the structure of the stabilizers of the normal forms. The only nontrivial cases are:

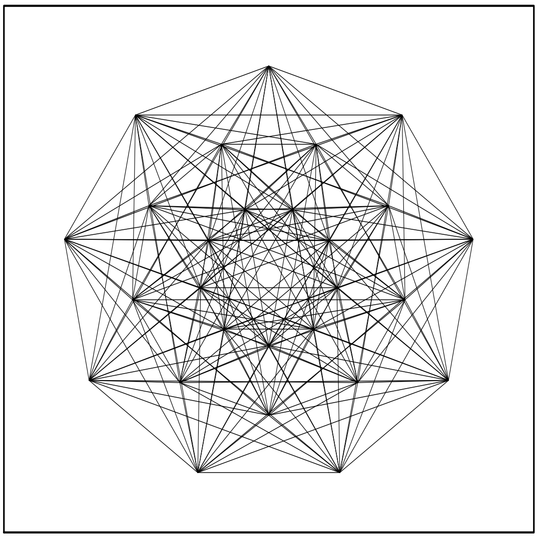

FIG. 2: The Hessian polyhedron

- the orbits with 216 elements, for which the stabilizer is the cyclic group C_3;
- the orbits with 72 elements, for which it is $C_3 \times C_3$;
- the Hessian orbits with 27 elements, for which it is the group G_4 of the Shephard-Todd classification.

IX. CONCLUSION

We described the geometry of the normal forms of semi-stable orbits of three qutrit states under the action of $SL(3, \mathbb{C})^{\times 3}$, the group of SLOCC-operations. From a physical point a view, our results yield a nice starting point for studying the richness of the entanglement of three qutrits and its differences with that of the simpler qubit systems. From a mathematical point a view, we have worked out an interesting example of a problem in invariant theory, using both classical algebraic and modern geometric methods, and found a surprising connection with the geometry of complex polytopes. Also, this example provides a good illustration of the ideas presented in Refs. [10] and [12].

[1] C.H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. Proc. of the IEEE Int. Conf. on Computers, Systems and Signal Processing, pages 175–179, 1984.

[2] A. Ekert. Quantum cryptography based on Bell's theorem. *Phys. Rev. Lett.*, 67:661, 1991.

[3] M. Fitzi, N. Gisin, and U. Maurer. Quantum solution to the byzantine agreement problem. *Phys. Rev. Lett.*, 87:217901, 2001.

[4] L. Lamport, R. Shostak, and M. Pease. *ACM Trans. Programming Languages Syst.*, 4:382, 1982.

[5] K.G.H. Vollbrecht and R.F. Werner. Entanglement measure under symmetry. *Phys. Rev. A*, 64:062307, 2001.

[6] R.F. Werner and A.S. Holevo. Counterexample to an additivity conjecture for output purity of quantum channels, 2002. quant-ph/0203003.

[7] Dagomir Kaszlikowski, Darwin Gosal, E. J. Ling, L. C. Kwek, Marek Ukowski, and C. H. Oh. Three-qutrit correlations violate local realism more strongly than those of three qubits. *Phys. Rev. A*, 66:032103, 2002.

[8] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach. Schmidt decomposition and classification of three-quantum-bit states. *Phys. Rev. Lett.*, 85:1560, 2000.

[9] H.A. Carteret, A. Higuchi, and A. Sudbery. Multipartite generalization of the Schmidt decomposition. *J. Math. Phys.*, 41:7932, 2000.

[10] F. Verstraete, J. Dehaene, and B. De Moor. Normal forms and entanglement measures for multipartite quantum systems. quant-ph/0105090.

[11] W. Dür, G. Vidal, and J.I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62:062314, 2000.

[12] Alexander A. Klyachko. Coherent states, entanglement, and geometric invariant theory. quant-ph/0206012.

[13] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde. *Phys. Rev. A*, 65:052112, 2002.

[14] J.-G. Luque and J.-Y. Thibon. Polynomial invariants of four qubits. *Phys. Rev. A*, 67:042303, 2003.

[15] È. B. Vinberg. Linear groups that are connected with periodic automorphisms of semisimple algebraic groups. *Dokl. Akad. Nauk SSSR*, 221(4):767–770, 1975.

[16] È. B. Vinberg. The Weyl group of a graded Lie algebra. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(3):488–526, 709, 1976.

[17] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canad. J. Math.*, 6:274–304, 1954.

[18] J. H. Chanler. The invariant theory of the ternary trilinear form. *Duke Math. J.*, 5:552–566, 1939.

[19] H. Maschke. Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen Substitutionen. *Math. Ann.*, 33:317–344, 1889.

[20] H. S. M. Coxeter. *Regular complex polytopes*. Cambridge University Press, Cambridge, second edition, 1991.

[21] P. Orlik and L. Solomon. Arrangements defined by unitary reflection groups. *Math. Ann.*, 261:339–357, 1982.

[22] R. M. Thrall. Metabelian groups and trilinear forms. *Amer. J. Math.*, 40:383–415, 1938.

[23] R. M. Thrall and J.H. Chanler. Ternary trilinear forms in the field of complex numbers. *Duke Math. J.*, 4:678–690, 1938.

[24] A. G. Nurmiev. Orbits and invariants of third-order matrices. *Mat. Sb.*, 191(5):101–108, 2000.

[25] A. G. Nurmiev. Closures of nilpotent orbits of cubic matrices of order three. *Uspekhi Mat. Nauk*, 55(2(332)):143–144, 2000.

[26] E. B. Vinberg and V. L. Popov. Invariant theory. In I. R. Shafarevich, editor, *Algebraic geometry. IV*, volume 55 of

Encyclopaedia of Mathematical Sciences, pages vi+284. Springer-Verlag, Berlin, 1994.

[27] I. Dolgachev. Weighted projective varieties. In *Group actions and vector fields (Vancouver, B.C., 1981)*, volume 956 of *Lecture Notes in Math.*, pages 34–71. Springer, Berlin, 1982.

[28] J. Harris. *Algebraic geometry, a first course.* Springer-Verlag, Berlin, 1992.

[29] H. W. Turnbull. *The theory of determinants, matrices and invariants.* Dover, New York, 1960.

[30] P.J. Olver. *Classical Invariant Theory.* Cambridge University Press, Cambridge, 1999.

[31] J. L. Coolidge. *A treatise on algebraic plane curves.* Oxford University Press (Clarendon), London, 1931.

[32] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants.* Birkhäuser, Boston, 1994.

[33] A. Miyake. Classification of multipartite entangled states by multidimensional determinants. *Phys. Rev. A*, 67:012108, 2003.

[34] F. Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade.* Teubner, Leipzig, 1884.

[35] H. S. M. Coxeter. The polytope 2_{21}, whose twenty-seven vertices correspond to the lines on the general cubic surface. *Amer. J. Math.*, 62:457–486, 1940.

[36] There was another, equally natural possibility leading to the same Hilbert series. The symmetry group L of the equianharmonic cubic surface Σ : $z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0$ acting on the homogeneous coordinate ring $\mathbb{C}[\Sigma]$ has as fundamental invariants the elementary symmetric functions of the z_i^3, the first one being 0 by definition, so that the Hilbert series of $\mathbb{C}[\Sigma]^L$ coincides with (1). Moreover, L is also of order 648, but it is known that it is not isomorphic to K.

[37] The edges of the complex polyhedron, which are complex lines, are represented by real equilateral triangles, so that the figure can as well be interpreted as a 2-dimensional projection of a 6-dimensional Gosset polytope 2_{21}.