The polynomial invariants of four qubits

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(Dated: December 14, 2002)

We describe explicitly the algebra of polynomial functions on the Hilbert space of four qubit states which are invariant under the SLOCC group $SL(2,\mathbb{C})^4$. From this description, we obtain a closed formula for the hyperdeterminant in terms of low degree invariants.

PACS numbers: O3.67.Hk, 03.65.Ud, 03.65.Fd

arXiv:quant-ph/0212069 v5 2 Feb 2003

I. INTRODUCTION

Various classifications of states with up to four qubits have been recently proposed, with the aim of understanding the different ways in which multipartite systems can be entangled [1, 2, 3, 4, 5]. However, one cannot expect that such classifications will be worked out for an arbitrary number k of qubits, and there is a need for a coarser classification scheme which would be computable for general k. In [6], Klyachko proposed to assimilate entanglement with the notion of *semi-stability* of geometric invariant theory. In this context, a semi-stable state is one which can be separated from 0 by a polynomial invariant of $SL(2,\mathbb{C})^k$, the point in the geometric approach being that explicit knowledge of the invariants is in principle not necessary to check this property.

In this paper, we construct a complete set of algebraic invariants of 4-qubit states. This allows us to identify the semi-stable states in the classification of Verstraete et al. [4], and to obtain a simple closed form for the hyperdeterminant.

Let $V=\mathbb{C}^2$ be the local Hilbert space of a spin $\frac{1}{2}$ particle, and $\mathcal{H}=V^{\otimes 4}$ be the state space of four particles, regarded as the natural representation of the group $G=SL(2,\mathbb{C})^4$, known in the context of quantum information theory (QIT) as the group of reversible stochastic local quantum operations assisted by classical communication (SLOCC) [1, 7].

If $|j\rangle$, $j=0,1$ is any basis of V, a state $|\Psi\rangle$ can be written as

$$|\Psi\rangle=\sum_{i,j,k,l=0}^{1} A_{ijkl}|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle \qquad (1)$$

and the question of which normal form can be achieved for $|\Psi\rangle$ by varying independently the bases of the four copies of V has been solved only recently [4] , although the case of three-qubit states is classical and relatively simple [21].

In the following, we give a complete [22] description of the polynomial functions $f(A_{ijkl})$ which are invariant under the SLOCC group $SL(2,\mathbb{C})^4$. This amounts to the construction of a moduli space for four qubit states. Our strategy is to find first the Hilbert series of the algebra of invariants $\mathcal{J}$. Next, we construct by classical methods four invariants of the required degrees. The knowledge of the Hilbert series reduces then the proof of algebraic independence and completeness to simple verifications. The values of the invariants on the orbits of [4] are tabulated in the Appendix.

II. THE HILBERT SERIES

Let $\mathcal{J}_d$ be the space of G-invariant homogeneous polynomial functions of degree d in the variables A_{ijkl}. Using some elementary representation theory, it is not difficult to show that $\mathcal{J}_d$ is zero for d odd, and that for $d=2m$ even, the dimension of $\mathcal{J}_d$ is equal to the multiplicity of the trivial character of the symmetric group $\mathfrak{S}_{2m}$ in the fourth power $(\chi^{mm})^4$ of its irreducible character corresponding to the partition $[m,m]$. This is the same as the scalar product $\langle(\chi^{mm})^2|(\chi^{mm})^2\rangle$, which can be evaluated by means of the formulas of [8, 9] giving the decomposition into irreducible characters of any product $\chi^\lambda\chi^\mu$ when λ and μ have at most two parts. This yields the Hilbert series of $\mathcal{J}=\bigoplus_d \mathcal{J}_d$ in the form [23]

$$\sum_{d\geq 0}\dim\mathcal{J}_d\, t^d=\frac{1}{(1-t^2)(1-t^4)^2(1-t^6)}\,. \qquad (2)$$

This formula shows that Conjecture 2.6.5.3 of [6] cannot be correct, since it predicts that the hyperdeterminant, which is of degree 24, should be one of the generators. Actually, the algebra of invariants is free on generators of degrees $2,4,4,6$, as suggested by the Hilbert series.

III. A FUNDAMENTAL SET OF INVARIANTS

Indeed, it is possible to construct invariants of the required degrees and to check that they are algebraically independent. To reduce the size of the expressions, we shall write the components of $|\Psi\rangle$ as

$$A_{ijkl}=a_r\,,\quad r=0,\ldots,15\,, \qquad (3)$$

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where r is the integer whose binary expression is $ijkl$, that is, $r = 8i + 4j + 2k + l$. We shall consider them as the coefficients of a quadrilinear form

$$A(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) = \sum_{i,j,k,l=0}^{1} A_{ijkl} x_i y_j z_k t_l \qquad (4)$$

on $V \times V \times V \times V$. Such a form is known to have an invariant H of degree 2, which is also one of the hyperdeterminants introduced by Cayley [24]. It is given by

$$\begin{aligned} H = \;& a_0a_{15} - a_1a_{14} - a_2a_{13} + a_3a_{12} \\ & -a_4a_{11} + a_5a_{10} + a_6a_9 - a_7a_8 \,, \end{aligned} \qquad (5)$$

and the two independent invariants of degree 4 are any two of the 3 determinants which can be formed by interpreting A as a linear map $\mathbb{C}^4 \to \mathbb{C}^4$ (see [10])

$$L = \begin{vmatrix} a_0 & a_4 & a_8 & a_{12} \\ a_1 & a_5 & a_9 & a_{13} \\ a_2 & a_6 & a_{10} & a_{14} \\ a_3 & a_7 & a_{11} & a_{15} \end{vmatrix} \qquad (6)$$

$$M = \begin{vmatrix} a_0 & a_8 & a_2 & a_{10} \\ a_1 & a_9 & a_3 & a_{11} \\ a_4 & a_{12} & a_6 & a_{14} \\ a_5 & a_{13} & a_7 & a_{15} \end{vmatrix} \qquad (7)$$

$$N = \begin{vmatrix} a_0 & a_1 & a_8 & a_9 \\ a_2 & a_3 & a_{10} & a_{11} \\ a_4 & a_5 & a_{12} & a_{13} \\ a_6 & a_7 & a_{14} & a_{15} \end{vmatrix} \qquad (8)$$

$$\qquad (9)$$

One has the relation

$$L + M + N = 0\,, \qquad (10)$$

but it is easily checked that any two of them are linearly independent, and also that H^2 cannot be expressed as a linear combination of them.

To construct a sextic invariant algebraically independent from the previous ones, we shall apply the methods of classical invariant theory, and first find some covariants, that is, homogeneous G-invariant polynomials in the form coefficients A_{ijkl} and in the original variables (see, e.g., [11] for a modern presentation). The dimension of the space $\mathcal{C}_{d,k_1,k_2,k_3,k_4}$ of covariants which are of degree d in A, k_1 in $\mathbf{x}$, and so on, is equal to the multiplicity of the trivial character of $\mathfrak{S}_d$ in the product

$$\chi^{l_1+k_1,l_1}\chi^{l_2+k_2,l_2}\chi^{l_3+k_3,l_3}\chi^{l_4+k_4,l_4} \qquad (11)$$

where $d = 2l_i + k_i$ for all i. This can still be evaluated from the knowledge of the products $\chi^\lambda\chi^\mu$ of two-part characters, and one can see that A has six covariants of degree 2, which are biquadratic forms in all possible pairs of variables. Such covariants are easily constructed, these are the determinants of order 2 of the partial derivatives of A with respect to the complementary pair of variables, e.g.,

$$b_{xy}(\mathbf{x},\mathbf{y}) = \det\left(\frac{\partial^2 A}{\partial z_i \partial t_j}\right)\,. \qquad (12)$$

Each of these biquadratic forms can be interpreted as a bilinear form on the three-dimensional space $S^2(\mathbb{C}^2)$, and one can define 3×3 matrices by

$$b_{xy}(\mathbf{x},\mathbf{y}) = [x_0^2, x_0x_1, x_1^2] B_{xy} \begin{bmatrix} y_0^2 \\ y_0y_1 \\ y_1^2 \end{bmatrix} \qquad (13)$$

and similarly for the other pairs. Let, for any pair of vector variables $(\mathbf{u},\mathbf{v})$,

$$D_{uv} = \det(B_{uv})\,. \qquad (14)$$

These determinants are sextic invariants of A. Since the space of sextic invariants is four-dimensional, they must be linearly dependent. In fact,

$$D_{xy} = D_{zt}\,,\ D_{xz} = D_{yt} \text{ and } D_{xt} = D_{yz} \qquad (15)$$

but D_{xy}, D_{xz} and D_{xt} are linearly independent. One can check that

$$\begin{aligned} HL &= D_{xz} - D_{xt} && (16)\\ HM &= D_{xt} - D_{xy} && (17)\\ HN &= D_{xy} - D_{xz} && (18) \end{aligned}$$

and that H^3 is not in the subspace spanned by the D_{uv}. The above results, together with the knowledge of the Hilbert series, allows now to prove that the algebra of invariants is free, and that any of the D_{uv}'s can be taken as the generator of degree 6. Indeed, it is sufficient to check that the Jacobian matrix of the choosen generators has rank 4 (this is easily done using the specialization G_{abcd} of the Appendix).

We will use in the sequel

$$\mathcal{J} = \mathbb{C}[H, L, M, D_{xt}]\,. \qquad (19)$$

IV. THE HYPERDETERMINANT IN TERMS OF THE FUNDAMENTAL INVARIANTS

Here, according to the general formula of [12], the Cayley hyperdeterminant (in the sense of [5, 12]) is of degree 24. It must therefore admit an expression in terms of the fundamental invariants, whose explicitation is an interesting question. To answer it, we shall need again the covariants b_{uv}. Let us use, for example, b_{xt}. We can consider A as a trilinear form T in $\mathbf{x},\mathbf{y},\mathbf{z}$, the fourth variable $\mathbf{t}$ being treated as a parameter. The Cayley hyperdeterminant $\mathbf{Det}(T)$ of this trilinear form is homogeneous of degree 4 in its coefficients, which are themselves linear forms in $\mathbf{t}$. Hence, $R(\mathbf{t}) = \mathbf{Det}(T)$ is a binary quartic in t_0, t_1, and we can form its discriminant Δ, which will be an invariant of A. According to Schläfli [13] (see also [5, 12]), in this case, Δ is equal to $\mathbf{Det}(A)$.

It follows from the well-understood invariant theory of binary trilinear forms [14, 15] that $R(\mathbf{t})$ is equal to the discriminant of the quadratic form in $\mathbf{x}$

$$Q_t(\mathbf{x}) = b_{xt}(\mathbf{x}, \mathbf{t}) \,, \tag{20}$$

that is

$$R(\mathbf{t}) = \det\left(\frac{\partial^2 b_{xt}}{\partial x_i \partial x_j}\right) . \tag{21}$$

Let

$$R(\mathbf{t}) = c_0 t_0^4 + 4c_1 t_0^3 t_1 + 6c_2 t_0^2 t_1^2 + 4t_0 t_1^3 + c_4 t_1^4 \,. \tag{22}$$

It is well-known that the algebra of invariants of the binary quartic is free over the two generators

$$S = c_0 c_4 - 4c_1 c_3 + 3c_2^2 \,, \tag{23}$$
$$T = c_0 c_2 c_4 - c_0 c_3^2 + 2c_1 c_2 c_3 - c_1^2 c_4 - c_2^3 \,, \tag{24}$$

and that its discriminant is given by

$$\Delta = S^3 - 27\,T^2 \,. \tag{25}$$

In the classical language, S is the apolar of R with itself, and T is its catalecticant (see [11]).

The invariants S and T of R being obviously invariants of A, the problem of expressing $\mathbf{Det}(A)$ is terms of the fundamental invariants of A is reduced to the one of finding the expressions of S and T [25].

With the help of a computer algebra system, we obtain the values

$$\begin{aligned} S &= \frac{1}{12}H^4 - \frac{2}{3}H^2 L + \frac{2}{3}H^2 M - 2HD_{xt} \\ &\quad + \frac{4}{3}(L^2 + LM + M^2) \end{aligned} \tag{26}$$

and

$$\begin{aligned} T &= \frac{1}{216}H^6 - \frac{1}{18}H^4(L-M) - \frac{1}{6}H^3 D_{xt} \\ &\quad + \frac{1}{9}H^2(2L^2 - LM + 2M^2) + \frac{2}{3}H(L-M)D_{xt} \\ &\quad - \frac{8}{27}(L^3 - M^3) - \frac{4}{9}LM(L-M) + D_{xt}^2 \,. \end{aligned} \tag{27}$$

Setting $D = D_{xt}$, $U = H^2 - 4(L-M)$ and $V = 12(HD - 2LM)$, these expressions can be recast into the more elegant form

$$12S = U^2 - 2V \,, \tag{28}$$
$$216T = U^3 - 3UV + 216D^2 \,. \tag{29}$$

This suggests that U and V might have a geometric meaning. Actually, similar expressions occur in the course of Schläfli's calculations [13]. He does not mention their invariant theoretic meaning, however, and prefers to end up with an expression of Δ as a polynomial in H, $W = D_{xy} + D_{xz} + D_{xt}$, $\Sigma = L^2 + M^2 + N^2$ and $\Pi = (L-M)(M-N)(N-L)$, which are invariant under permutations of the indices $ijkl$.

V. CONCLUSION

A fundamental issue in QIT is the understanding of entanglement. However, as pointed out in [6], there is no universal agreement on the precise definition of entanglement and on what should be its proper measure. It is apparently this lacune which motivated recent attempts to obtain a complete classification of k-qubit states under the SLOCC group G [1, 5, 7].

Some familiarity with classical invariant theory leaves little hope that such a classification can be achieved in general. If we compare with the somewhat easier classical problem of binary forms, which, in physical terms, amounts to the classification of single spin s states under $SL(2,\mathbb{C})$, a complete solution is known only up to spin $s = 4$ (with still some unsolved questions in the case $s = 7/2$), and most experts agree that the other cases will remain out of reach.

So, it is unlikely that the complete SLOCC classification of k-qubit states will ever be found for $k \geq 8$, and it is probable that formidable computational difficulties will arise well before this value [26]. Actually, the orbit structure is still completely unknown for $k > 4$.

Now, if we adopt the definition of entanglement proposed in [6], that is, to identify entangled states with the semi-stable vectors of geometric invariant theory, the main result of the present paper can be interpreted as a numerical criterion of entanglement for 4-qubit states. Indeed, a semi-stable state is by definition a state which can be separated from 0 by some invariant polynomial. Thus, according to [6], an entangled 4-qubit state would be one for which at least one of the four polynomials H, L, M, D takes a nonzero value. As we shall see below, this definition needs to be improved. However, it is plausible that refined entanglement measures for four-qubit states might be built from the absolute values of these invariants. These would be natural generalizations of the concurrence C and the 3-tangle τ in the two and three qubit cases, which are proportional repectively to the absolute values of the determinant and of the hyperdeterminant [5], the only polynomial invariants in these cases.

From a geometric point of view, our results show that the moduli space of entangled states is the weighted projective space $\mathbb{P}(1,2,2,3)$, which can be embedded as a rational threefold in 13-dimensional projective space. Of course, the approach to semi-stability and moduli spaces by explicit construction of the polynomial invariants has its limits,and it is unlikely that this can be done for more that 5 qubits.

Note also that the notion of semi-stability can be used only to characterize some generic kind of entanglement. Indeed, even in the three-qubit cases, the so-called W-state $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ is not semi-stable (its hyperdeterminant is zero), although it should certainly be considered as entangled (even in a strong sense, according to [1]). The natural candidates for constructing further measures of entanglement appropriate to such states are the covariants of classical invariant theory, which are completely known in the three-qubit case [15].

We expect to be able to describe in a forthcoming paper the algebra of covariants in the 4-qubit case, which would not only reproduce the complete classification of the orbits, but also to give the equations of their closures, which are algebraic varieties, and provide new insights about entanglement measures for unstable states. Another case whichs seems to be readily tractable is the case of three spin 1 states. The geometric classification of the orbits was known by 1938 [16], and numerical calculation of the Hilbert series of invariants up to degree 108 indicates that it should be

$$h(t) = \frac{1}{(1-t^6)(1-t^9)(1-t^{12})}\,. \tag{30}$$

Since independent invariants I_6, I_9, I_{12} of the appropriate degrees are known [17], one can consider that the SLOCC classification and the semi-stability problems are essentially solved in this case.

Other cases of less practical importance, such as those including two spin $\frac{1}{2}$ particles and one particle of spin $s \geq 1$, are easily solved. For $s = 1$, there is only one invariant of degree 6, the hyperdeterminant. For $s = 3/2$, the hyperdeterminant is identically zero, but there is still one invariant of degree 4, which is the only determinant that can be formed by displaying the components of $|\Psi\rangle$ in a 4×4 matrix. Finally, for $s \geq 2$, there are no invariant polynomials.

Acknowledgments

We would like to thank A. Klyachko and A. Miyake for their comments on preliminary versions of this text, and in particular M. Grassl for pointing out a gap in our original argument, together with a way of filling it.

APPENDIX A: APPLICATION TO THE CLASSIFICATION OF VERSTRAETE ET AL.

To conclude, let us discuss the semi-stability of the orbits obtained in [4] (see this reference for notation). For the family G_{abcd}, the values of the fundamental invariants are [27]

$$\begin{aligned} H &= \frac{1}{2}(a^2+b^2+c^2+d^2) && \text{(A1)}\\ L &= abcd\,, && \text{(A2)}\\ M &= \left[\left(\tfrac{c-d}{2}\right)^2-\left(\tfrac{a-b}{2}\right)^2\right]\left[\left(\tfrac{a+b}{2}\right)^2-\left(\tfrac{c+d}{2}\right)^2\right] && \text{(A3)}\\ D &= -\frac{1}{4}(ad-bc)(ac-bd)(ab-cd) && \text{(A4)} \end{aligned}$$

and the hyperdeterminant is $\frac{1}{256}V(a^2,b^2,c^2,d^2)^2$, where V denotes the Vandermonde determinant. For L_{abc_2},

$$\begin{aligned} H &= \frac{1}{2}(a^2+b^2+2c^2) && \text{(A5)}\\ L &= abc^2 && \text{(A6)}\\ M &= \left[c^2-\left(\tfrac{a+b}{2}\right)^2\right]\left(\tfrac{a-b}{2}\right)^2 && \text{(A7)}\\ D &= -\frac{1}{4}c^2(a-b)^2(ab-c^2)\,, && \text{(A8)} \end{aligned}$$

and $\Delta = 0$. For $L_{a_2b_2}$,

$$\begin{aligned} H &= a^2+b^2 && \text{(A9)}\\ L &= a^2b^2 && \text{(A10)}\\ M &= 0 && \text{(A11)}\\ D &= 0 && \text{(A12)} \end{aligned}$$

and $\Delta = 0$. For L_{ab_3},

$$\begin{aligned} H &= \frac{1}{2}(3a^2+b^2) && \text{(A13)}\\ L &= a^3b && \text{(A14)}\\ M &= \left[a^2-\left(\tfrac{a+b}{2}\right)^2\right]\left(\tfrac{a-b}{2}\right)^2 && \text{(A15)}\\ D &= \frac{1}{4}a^3(a-b)^3 && \text{(A16)} \end{aligned}$$

and $\Delta = 0$. For L_{a_4},

$$\begin{aligned} H &= 2a^2 && \text{(A17)}\\ L &= a^4 && \text{(A18)}\\ M &= 0 && \text{(A19)}\\ D &= 0 && \text{(A20)} \end{aligned}$$

and $\Delta = 0$. For $L_{a_2 0_{3\oplus\bar{1}}}$, there is still one nonzero invariant

$$H = a^2\,, \tag{A21}$$

so that all the above 6 families of orbits are semi-stable, whilst the remaining three are unstable.

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[21] In the physics literature, the solution appears in [1], but an equivalent problem had been solved in classical invariant theory since at least 1881 [14, 15]. Note also that a problem equivalent to the classification of four-qubit states had been studied by Segre in 1922 [10], but he only obtained a partial classification.
[22] We note that a combinatorial method for computing invariants of fourth-rank tensors is proposed in [18]. The point in the present paper is that we can prove that our system of invariants is complete.
[23] This is a tedious method. With the help of computer algebra system, the Hilbert series is more straightforwardly obtained by a residue calculation. Similar computations (for $SU(2)$ and $U(2)$ invariants) appear in [19].
[24] Cayley actually considered several different notions under the same generic name, see [20].
[25] The values of S and T may depend on our choices of the pair of variables (x, t) and then on the choice of eliminating x, but the combination $S^3 - 27T^2$ will not depend on these choices.
[26] For 5 qubits, the generic orbits depend on 16 free parameters, and for eight qubits, the moduli space would be of dimension 231.
[27] It is interesting to observe that these polynomials can be regarded as a fundamental system of invariants for the standard action of the Weyl group D_4 on the four-dimensional vector space spanned by a, b, c, d.