Hyperdeterminantal aspects of the Selberg's integral

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Abstract

We show that the original proof of the Selberg's integral is nothing but the computation of a hyperdeterminant. We give a similar proof of the Aomoto's integral.

1 Introduction

Selberg's integral and its generalizations are multiple integrals encountered in many domains connected with mathematical statistics and Random matrices and can be considered as moment integrals. It appears naturally in the computation of normalization constant for some ensembles (see [32]). For example the normalization of the eigenvalue probability density function for Gaussian, circular, Laguerre, Jacobi and Cauchy ensembles are limit cases of Selberg integrals (see [35, 17]) (`vérifier la dernière ref.`). More complicated generalizations appear in the study of the partition functions $Z_n(\beta)$ of log-potential Coulomb system `(trouver ref)`. An other interest of multiple moment integrals consists in their links with the expansion of even power of the Vandermonde determinant. The expansion of such a polynomial on the Schur functions is still an open problem (see [46]) which is a key to understand some properties related to the fractional quantum Hall effect (see [38, 47]) as described by Laughlin's wavefunction [30]

$$\Psi^m_{\text{Laughlin}}(z_1,\dots,z_n)=\prod_{i<j}^{N}(z_i-z_j)^{2m+1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right\} \tag{1}$$

and to compute a lot of multiple integrals (see [31, 32]). Selberg's integral is a generalization of Euler's Beta integral introduced and computed by himself in 1944 (see [40]) in the aim to prove a generalization of a Gelfond's theorem which initially reads: *if $f(z)$ is an entire function whose exponential type is less than $k\log(1+\exp\{\frac{(1-k)}{k}\})$ and such that $f(n)$, $f'(n), \dots,$ $f^{(k-1)}(n)$ are all integers for $n \in \mathbb{N}$ then f is a polynomial.* Selberg found that his integral can be written as a product of quotients of gamma integrals

$$\begin{aligned}\mathcal{S}_n(a,b;\gamma) &= \int_0^1 \cdots \int_0^1 |\Delta(x)|^{2\gamma} \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}dx_i \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(a+j\gamma)\Gamma(b+j\gamma)\Gamma((j+1)\gamma+1)}{\Gamma(a+b+(n+j-1)\gamma)\Gamma(\gamma+1)}. \qquad (2)\end{aligned}$$

This equality was proved in different ways by other authors such as Anderson [1], Aomoto [3] and Dotsenko and Fateev [13]. These proofs are all interesting, because they involve generalization of the Selberg's integral or technics which are useful in many contexts. However, Selberg's original proof has the specificity that it is almost elementary, in the sense that he computed the integral in the case where $\gamma = k \in \mathbb{N}$ using only arguments on the degrees of the polynomials appearing after integration of the terms obtained by expansion of $\Delta(x)^{2k}$, and a very simple change of variables. Then, he concluded by involving Carlson's theorem to generalize it for $\gamma \in \mathbb{C}$ (see [35] or [43]). The elementary part of this proof appears naturally if we consider this object for $\gamma = k \in \mathbb{N}$ not as an integral but as a special hyperdeterminant defined in [41, 42]. The hyperdeterminant is a polynomial invariant for tensors of even order (in the case of matrices, its definition gives back those of the classical determinant). We have explained in [32] that Selberg's integral could be rewritten as a hyperdeterminant of Hankel type (*i.e.* whose entries depend only of the sum of the indices) built from the moments for the beta distribution. In this paper, we translate Selberg's proof in the hyperdeterminantal formalism, using only very simple tools. The hyperdeterminantal aspects of multiple integral computations look very promising : At present, we know only a few tools to compute with such a polynomial, but this is already sufficient to deal with some interesting cases. To illustrate this point, we provide a hyperdeterminantal proof of Aomoto's integral

$$\begin{aligned}\mathcal{A}_n^{a,b;k}(y) &= \int_0^n \cdots \int_0^n \Delta(x)^{2k} \prod_{i=1}^n (y-x_i)x_i^{a-1}(1-x_i)^{b-1} \qquad (3) \\ &= (-2)^{-n} S_n(a,b;k) P_n^{\frac{a}{k}-1,\frac{b}{k}-1}(1-2y) \qquad (4)\end{aligned}$$

where $P_n^{a,b}$ denotes the unitary Jacobi polynomial. Aomoto has proved this identity in [3] by means of differential equations. The main interest of Aomoto's original proof is the connection with the Calogero model (`à verifier`). Note that Barsky and Carpentier gave in [4] a alternative proof, based on Anderson's method .

2 Elementary hyperdeterminantal calculus

2.1 Definition

We shall find convenient to use a set of anticommutative variables $\eta = \{\eta_1, \ldots, \eta_i, \ldots\}$.
A tensor of order k and dimension n , that is, an element of $V^{\otimes k}$, where V is an n dimensional vector space, will be represented as

$$\mathbf{M} = \sum_{1 \le i_1, \cdots, i_{2k} \le n} M_{i_1, \cdots, i_k} \eta_{i_1} \otimes \cdots \otimes \eta_{i_k}. \tag{5}$$

That is we identify V with the vector space spanned by $\eta_1, \ldots, \eta_n$. The array $(M_{i_1,\ldots,i_k})$ is called a hypermatrix. The hyperdeterminant $\mathrm{Det}(\mathbf{M})$ of $\mathbf{M}$ can be compactly defined as the coefficient of $(\eta_1 \cdots \eta_n)^{\otimes k}$ in $\frac{\mathbf{M}^n}{n!}$:

$$\mathbf{M}^n = n!\,\mathrm{Det}(\mathbf{M})(\eta_1 \cdots \eta_n)^{\otimes k} \tag{6}$$

where the power $\mathbf{M}^n$ is evaluated in the tensor product of k copies of the Grassmann algebra over the η_i's. For convenience, we will use the notation

$$\mathrm{Det}_k(M_{i_1, \cdots, i_k})_1^n = \mathrm{Det}\,(\mathbf{M})\,. \tag{7}$$

Note first that if $\mathbf{M}$ is a tensor of odd order, one has $\mathrm{Det}(\mathbf{M}) = 0$. If $\mathbf{M}$ is a tensor of even order $2k$ and dimension n, one has

$$\mathrm{Det}(\mathbf{M}) = \frac{1}{n!} \sum_{\sigma_1, \cdots, \sigma_{2k} \in \mathfrak{S}_n} \epsilon(\sigma_1) \cdots \epsilon(\sigma_{2k}) \prod_{i=1}^{n} M_{\sigma_1(i), \ldots, \sigma_{2k}(i)}. \tag{8}$$

Setting $k = 1$, we recover the classical definition of the determinant for matrices.

2.2 The invariance property

Definition (6) is very useful for proving properties of hyperdeterminant. Let us consider the natural action of $GL(V)^{2k}$ on $V^{\otimes 2k}$

$$(g^{(1)} \otimes \cdots \otimes g^{(2k)}) \cdot \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_k = (g^{(1)}\mathbf{v}_1) \otimes \cdots \otimes (g^{(2k)}\mathbf{v}_k) \tag{9}$$

Then, for a general $\mathbf{M} \in V^{\otimes 2k}$,

$$(g^{(1)} \otimes \cdots \otimes g^{(2k)}) \cdot \mathbf{M} = \sum_{1 \le j_1, \cdots, j_{2k} \le n} M_{j_1, \cdots, j_{2k}} \times \\ \times \sum_{i_1=1}^{n} g^{(1)}_{i_1, j_1} \eta_{j_1} \otimes \cdots \otimes \sum_{i_{2k}=1}^{n} g^{(2k)}_{i_{2k}, j_{2k}} \eta_{j_{2k}}. \tag{10}$$

Hence,

$$\mathrm{Det}\left((g^{(1)} \otimes \cdots \otimes g^{(2k)}) \cdot \mathbf{M}\right) = \det\left(g^{(1)}\right) \cdots \det\left(g^{(2k)}\right) \mathrm{Det}(\mathbf{M}). \tag{11}$$

2.3 Minor summation formula

Consider two tensors

$$\mathbf{M} = \sum_{1 \le i_1, \ldots, i_{2k} \le n} M_{i_1, \ldots, i_{2k}} \eta_{i_1} \otimes \cdots \otimes \eta_{i_{2k}} \tag{12}$$

and

$$\mathbf{N} = \sum_{1 \le i_1, \ldots, i_{2k} \le n} N_{i_1, \ldots, i_{2k}} \eta_{i_1} \otimes \cdots \otimes \eta_{i_{2k}}. \tag{13}$$

Remarking that $\mathbf{MN} - \mathbf{NM} = 0$, one has

$$(\mathbf{M} + \mathbf{N})^n = \sum_{i=0}^{n} \binom{n}{i} \mathbf{M}^i \mathbf{N}^{n-i}. \tag{14}$$

If $I = (I_1, \cdots, I_{2k})$ is a $2k$-uple of subsets of $\{1, \ldots, n\}$, we will denote by $\mathbf{M}[I]$ the tensor

$$\mathbf{M}[I] = \sum_{i_1 \in I_1, \ldots, i_{2k} \in I_{2k}} M_{i_1, \cdots, i_{2k}} \eta_{i_1} \otimes \cdots \otimes \eta_{i_{2k}} \tag{15}$$

and by η_I the product

$$\eta_I = \overrightarrow{\prod_{i \in I_1}} \eta_i \otimes \cdots \otimes \overrightarrow{\prod_{i \in I_{2k}}} \eta_i \tag{16}$$

where the symbol $\overrightarrow{\prod_i}$ denotes the product for the elements i taken in increasing order.

Let $\mathfrak{C}^r_{n,k}$ the set of pairs (I, J) of $2k$-uples $I = (I_1, \cdots, I_{2k})$ and $J = (J_1, \cdots, J_{2k})$ such that for each $s \in \{1, \ldots, 2k\}$, (I_s, J_s) is a partition of $\{1, \ldots, n\}$ in two blocks of size r and $n - r$. For each $(I, J) \in \mathfrak{C}^r_{n,k}$ we will denote by $\epsilon(I,J)$ the product of the signs of the permutations $\sigma_s = (i_1,\dots,i_r,j_1,\dots,j_n)$ (written as a word) where $I_s = \{i_1 < \cdots < i_r\}$ and $J_s = \{j_1 < \cdots < j_{n-r}\}$.

We can now write, for each $p \le n$

$$\mathbf{M}^p = p! \sum_I \mathrm{Det}(\mathbf{M}[I])\eta_I \tag{17}$$

where the sum is over the $2k$-uples $I = (I_1,\cdots,I_{2k})$ of subsets of $\{1,\dots,n\}$ of cardinality p. Hence, by (14),

$$\mathrm{Det}(\mathbf{M}+\mathbf{N}) = \sum_{r=0}^{n} \sum_{(I,J)\in\mathfrak{C}^r_{n,k}} \epsilon(I,J)\mathrm{Det}(\mathbf{M}[I])\mathrm{Det}(\mathbf{N}[J]) \tag{18}$$

Now, if we set

$$\mathbf{M}' = \sum_{1\le i_2,\dots,i_{2k}\le n} M_{1,i_2,\dots,i_{2k}}\eta_1\otimes\eta_{i_2}\otimes\cdots\otimes\eta_{i_{2k}} \tag{19}$$

and

$$\mathbf{M}'' = \sum_{i_1=2}^{n} \sum_{1\le i_2,\dots,i_{2k}\le n} M_{i_1,i_2,\dots,i_{2k}}\eta_{i_1}\otimes\cdots\otimes\eta_{i_{2k}}, \tag{20}$$

remarking that $\mathbf{M} = \mathbf{M}' + \mathbf{M}''$ we obtain from (18),

$$\mathrm{Det}(\mathbf{M}) = \sum_{I=(i_1=1\le i_2,i_3,\dots,i_{2k}\le n)} \mathrm{sign}(I) M_{1,i_2,\dots,i_{2k}}\mathrm{Det}(\mathbf{M}[\overline{I}]) \tag{21}$$

where $\mathrm{sign}(I) = (-1)^{i_1+i_2+\cdots+i_{2k}}$ and $\overline{I} = (\{1,\dots,n\}-i_1,\dots,\{1,\dots,n\}-i_{2k})$. Equalities (18) and (21) could be found in [5, 41, 42].

3 Hankel hypermatrices

For convenience, if $I = (i_1,\dots,i_{2k})$ we will denote $|I| = i_1+\cdots+i_{2k}$ and $\eta_I = \eta_{i_1}\otimes\cdots\otimes\eta_{i_{2k}}$.

3.1 Heine's integrals for Hankel determinant

Let μ be a measure and $c_n = \int x^n d\mu(x)$ be the moments associated to this measure. Heine's integrals are evaluated as Hankel determinants whose entries are the moments c_n (see [23])

$$\int \Delta(x)^n d\mu(x_1)\cdots\mu(x_n) = n!\det(c_{i+j})_0^{n-1}. \tag{22}$$

This can be seen as a particular case of the deBruijn equality

$$\int\cdots\int_{a\le x_1<\cdots<x_n\le b} \det\left(\phi_i(x_j)|\psi_i(x_j)\right) dx_1\cdots dx_n = \mathrm{Pf}\ (Q_{ij})_{1\le i,j\le 2n} \qquad (23)$$

where $Q_{ij}=\int_a^b[\phi_i(x)\psi_j(x)-\phi_j(x)\psi_i(x)]dx$ and $(\phi_i(x_j)|\psi_i(x_j))$ denotes the matrix whose ith row is $[\phi_i(x_1),\psi_i(x_1),\phi_i(x_2),\psi_i(x_2),\dots,\phi_i(x_n),\psi_i(x_n)]$ and $\mathrm{Pf}\,(M)$ denotes the Pfaffian of the matrix M. In [31], we have stated a generalization its equality involving $2kn$ functions on the left hand side and hyperpfaffian (see [5]) in the right hand side. In the case where the determinant is a even power of the Vandermonde determinant, we obtain a generalized Heine's theorem turning multiple integrals in Hankel hyperdeterminant. This generalization is the aim of [32] and the following subsection.

3.2 Generalization: Hankel hyperdeterminant

We consider here, the integral

$$\mathcal{I}^{\mu}_{n,k}=\int\cdots\int\Delta(x_1,\cdots,x_n)^{2k}d\mu(x_1)\cdots d\mu(x_n) \qquad (24)$$

Expanding the even power of the Vandermonde determinant, this integral can be turn in an Hankel hyperdeterminant whose entries are the moments $c_n=\int x^n d\mu(x)$:

$$\mathcal{I}^{\mu}_{n,k}=n!\mathrm{Det}\left(c_{|I|}\right)_0^{n-1}. \qquad (25)$$

Consider now the two following tensors

$$\mathbf{S}_n(a,b;k)=\sum_{0\le i_1,i_2,\dots,i_{2k}\le n-1}\mathrm{B}(a+|I|,b)\eta_I \qquad (26)$$

and

$$\mathbf{A}_n^{a,b;k}(y)=\sum_{0\le i_1,i_2,\dots,i_{2k}\le n-1}\mathrm{B}(a+|I|,b)\quad\left(y-\frac{a+|I|}{a+b+|I|}\right)\eta_I, \qquad (27)$$

where $\mathrm{B}(a,b)$ denotes the Beta integral $\mathrm{B}(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$. Remarking that the $\mathrm{B}(a+n,b)$ are the moments for the Beta distribution, one obtains after applying (25)

$$\mathcal{S}_n(a,b;k)=n!\mathrm{Det}\left(\mathbf{S}_n(a,b;k)\right) \qquad (28)$$

and in the same way

$$\mathcal{A}_n^{a,b;k}(y)=n!\mathrm{Det}\left(\mathbf{A}_n^{a,b;k}(y)\right). \qquad (29)$$

3.3 Expansion of Hankel hyperdeterminant

In this section we consider the hyperdeterminant of a general Hankel tensor

$$\mathcal{H}_{n,k} = \mathrm{Det}_{2k}\left(X_{i_1+\cdots+i_{2k}}\right)_0^{n-1}. \tag{30}$$

Expanding this polynomial, one finds

$$\mathcal{H}_{n,k} = \sum_{\lambda} c_{\lambda}^{n,k} X_{\lambda_1}\cdots X_{\lambda_n} \tag{31}$$

where the sum is under the n-uples $\lambda = (\lambda_1, \cdots, \lambda_n)$ verifying $\sum \lambda_i = kn(n-1)$ and for each $i \in \{1, \cdots, n\}$, $(i-1)k \leq \lambda_i \leq k(n+i-2)$.
We have obtained such an expansion in [32], through umbral calculus, considering the hyperdeterminant as the expansion of an even power of Vandermonde. Remark that, it is the way considered by Selberg in its proof (see [35, 40]).

Let $\mathfrak{d}_{n,k}$ be the coefficient of $X_{k(n-1)}^n$ in (31):

$$\mathfrak{d}_{n,k} = c_{(k(n-1))^n}. \tag{32}$$

If we set $X_i = 0$ for $i > k(n-1)$, we obtain from (31) that

$$\mathrm{Det}_{2k}\left(X_{i_1+\cdots+i_{2k}}\right)_0^{n-1} = \mathfrak{d}_{n,k} X_{k(n-1)}^n. \tag{33}$$

A closed form for $\mathfrak{d}_{n,k}$ follows from a well-known constant term identity ("the Dyson conjecture")[1]

$$\mathcal{C}_{n,k} = \mathrm{C.T.}\left\{\prod_{\substack{i,j=1\\ i\neq}}^{n}\left(1-\frac{x_i}{x_j}\right)^{a_i}\right\} = \binom{a_1+\cdots+a_n}{a_1,\cdots,a_n}. \tag{34}$$

On an other hand,

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\frac{x_i}{x_j}\right)^{k} = (-1)^k \Delta(x)^{2k}\prod_{i=1}^{n}\frac{1}{x_i^{k(n-1)}}. \tag{35}$$

Expanding the power of the Vandermonde determinant, one find

$$\begin{aligned}
\mathcal{C}_{n,k} &= (-1)^k \mathrm{C.T.}\left\{\sum_{\sigma_1,\cdots,\sigma_{2k}\in\mathfrak{S}_n}\epsilon(\sigma_1)\cdots\epsilon(\sigma_{2k})\prod_{i=1}^{n} x_i^{\sigma_1(i)+\cdots+\sigma_{2k}(i)-2k-k(n-1)}\right\}\\
&= (-1)^k n!\,\mathrm{Det}_{2k}\left(\delta_{i_1+\cdots+i_{2k},n}\right)_0^{n-1}
\end{aligned} \tag{36}$$

[1]This was conjectured by Dyson [14] in 1962, I.J. Good gave, in 1970, an elegant elementary proof involving only Lagrange interpolation [22].

where $\delta_{i,j}=1$ if $i=j$ and 0 elsewhere denotes the Kronecker symbol. Using (33), we obtain

$$\mathcal{C}_{n,k}=(-1)^k n!\mathfrak{d}_{n,k}. \tag{37}$$

Hence,

$$\mathfrak{d}_{n,k}=(-1)^k\frac{1}{n!}\left(\begin{array}{c}kn\\ k,\cdots,k\end{array}\right). \tag{38}$$

3.4 Minors of a Hankel hyperdeterminant

We consider here a hankel hyperdeterminant

$$\mathbf{M}=\sum_{I=0^{2k}}^{n^{2k}} M_{|I|}\eta_{i_1}\otimes\cdots\otimes\eta_{i_{2k}} \tag{39}$$

and a family of $2k$ subsets $J=(J^1,\dots,J^{2k})$ of a fixed size $m\leq n$ with $J^l=\{j^l_0\leq j^l_1\leq\cdots\leq j^l_m\}\subset\{0,\dots,n-1\}$. We want to expands the polynomial $M[J]^m$ where

$$\mathbf{M}[J]=\sum_{I=0^{2k}}^{m^{2k}} M_{j^1_{i_1}+\cdots+j^{2k}_{i_{2k}}}\eta_{i_1}\otimes\cdots\otimes\eta_{i_{2k}}. \tag{40}$$

One has

$$\mathrm{Det}(M[J])=\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_{2k}\in\mathfrak{S}_m}\prod_{l=1}^m M_{j^1_{\sigma_1(l)}+\cdots+j^{2k}_{\sigma_{2k}(l)}}. \tag{41}$$

But for each $2k$ uples $(\sigma_1,\dots,\sigma_{2k})$, one has

$$\prod_{l=1}^m M_{j^1_{\sigma_1(l)}+\cdots+j^{2k}_{\sigma_{2k}(l)}}=\prod_{l=1}^m M_{\sigma_1(l)+\cdots+\sigma_{2k}(l)-2k+\epsilon^1_{\sigma_1(l)}+\cdots+\epsilon^{2k}_{\sigma_{2k}(l)}} \tag{42}$$

with $0\leq\epsilon^i_l\leq n-m$. Hence, by a similar computation than for a Hankel hyperdeterminant

$$\prod_{l=1}^m M_{j^1_{\sigma_1(l)}+\cdots+j^{2k}_{\sigma_{2k}(l)}}=\prod_{l=1}^m M_{\lambda_i+\epsilon_i} \tag{43}$$

with $k(i-1)\leq\lambda_i\leq k(m+i-2)$ and $0\leq\epsilon_i\leq 2k(n-m)$. And

$$\mathrm{Det}(M[J])=\sum_{\tilde\lambda}\prod_{i=1}^m \tilde c^{n,k;J}_{\tilde\lambda}M_{\tilde\lambda_i} \tag{44}$$

where $k(i-1)\leq\tilde\lambda_i\leq k(2n-m+i-2)$.

4 Selberg's integral

4.1 From the Hankel form to the symmetric form

Selberg's integral admits an other hyperdeterminantal representation which can be obtain directly (without use of the integral). It suffices to remark the following property

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\mathrm{B}(a+i,b)=\mathrm{B}(a,b+i). \tag{45}$$

Hence, we have

$$\mathbf{S}_n(a,b;k)=(\overbrace{\mathrm{I},\cdots,\mathrm{I}}^{\times k},\overbrace{B,\cdots,B}^{\times k}).\mathbf{S}_n^{Sym}(a,b;k) \tag{46}$$

where

$$\mathbf{S}_n^{Sym}(a,b;k)=\sum_{I,J\in\{0,\dots,n-1\}^k}\mathrm{B}(a+|I|,b+|J|)\eta_I\otimes\eta_J \tag{47}$$

and B is the $n\times n$ matrix $B=\sum_{1\leq i,j\leq n}(-1)^j\binom{i}{j}\eta_i\otimes\eta_j$. It follows

$$\det(B)^k\mathrm{Det}\left(\mathbf{S}_n^{Sym}(a,b;k)\right)=\mathrm{Det}\left(\mathbf{S}_n(a,b;k)\right) \tag{48}$$

and, by

$$\det(B)=(-1)^{\lfloor\frac{n}{2}\rfloor}=(-1)^{\frac{n(n-1)}{2}}, \tag{49}$$

we find

$$\mathrm{Det}\left(\mathbf{S}_n^{Sym}(a,b;k)\right)=(-1)^{\frac{kn(n-1)}{2}}\mathrm{Det}\left(\mathbf{S}_n(a,b;k)\right) \tag{50}$$

without the help of the integral representation.

4.2 Selberg's original proof

Each step of Selberg's proof can be viewed as a manipulation on a hyperdeterminant (see [35] for example). It can be divided into two parts. In the first part, Selberg proved that

$$\mathcal{S}_n(a,b;k)=c_{n,k}\prod_{i=0}^{n-1}\frac{\Gamma(a+jk)\Gamma(b+jk)}{\Gamma(a+b+(n+j-1)k)}. \tag{51}$$

where $c_{n,k}$ does not depend on a and b. In this proof, Selberg started from the expansion of the Vandermonde determinant. In term of hyperdeterminant, it is equivalent to the expansion (**??**). For our purpose equality (51) reads

$$\mathrm{Det}(\mathbf{S}_n(a,b;k)) = \alpha_{n,k}\prod_{i=0}^{n-1}\frac{\Gamma(a+jk)\Gamma(b+jk)}{\Gamma(a+b+(n+j-1)k}\tag{52}$$

where $\alpha_{n,k}=n!c_{n,k}$. To obtain it, start from formula (31)

$$\mathrm{Det}(\mathbf{S}_n(a,b;k)) = \sum_\lambda c_\lambda^{n,k}\mathrm{B}(a+\lambda_1,b)\cdots\mathrm{B}(a+\lambda_n,b)\tag{53}$$

where the sum is over all n-uples $\lambda=(\lambda_1,\cdots,\lambda_n)$ verifying $\sum\lambda_i=kn(n-1)$ and for each $i\in\{1,\cdots,n\}$, $(i-1)k\le\lambda_i\le k(n+i-2)$.
Then, extracting common factors, we arrive at an expression of the form

$$\mathrm{Det}(\mathbf{S}_n(a,b;k)) = \frac{Q(a,b)}{R(b)}\prod_{i=0}^{n-1}\frac{\Gamma(a+jk)\Gamma(b+jk)}{\Gamma(a+b+(n+j-1)k}\tag{54}$$

where $Q(a,b)$ is a polynomial of degree at most $kn(n-1)/2$ and $R(b)$ is polynomial of degree $kn(n-1)/2$. But (50) implies that

$$\mathrm{Det}(\mathbf{S}_n(a,b;k)) = \mathrm{Det}(\mathbf{S}_n(b,a;k))\tag{55}$$

and then

$$\frac{Q(a,b)}{R(b)}=\frac{Q(b,a)}{R(a)}\tag{56}$$

which implies that $\alpha_{n,k}=\frac{Q(a,b)}{R(b)}$ is independent of a and b.

The second part of Selberg's proof consists in establishing a recurrence relation between the $\alpha_{n,k}$'s:

$$\alpha_{n,k}=\frac{(nk)!}{n!}\alpha_{n-1,k}.\tag{57}$$

This can be found from the limit

$$L_{b;k,n}=\lim_{a\to0^+}a\,\mathrm{Det}(\mathbf{S}_n(a,b;k))\tag{58}$$

Hence, one has

$$L_{b;k,n}=\lim_{a\to0^+}\mathrm{Det}(\mathbf{T}_a)\tag{59}$$

where $\mathbf{T}_a$ is the tensor defined by

$$\begin{aligned}\mathbf{T}_a = &\sum_{I\in\{0\}\times\{0,\dots,n-1\}^{2k-1}} a\mathrm{B}(a+|I|,b)\eta_I + \\ &+\sum_{I\in\{1,\cdots,n-1\}\times\{0,\dots,n-1\}^{2k-1}}\mathrm{B}(a+I,b)\eta_I\end{aligned}\tag{60}$$

Remarking, that

$$\lim_{a\to 0^+} a\,\mathrm{B}(a+n,b) = \delta_{n,0} \tag{61}$$

where $\delta_{i,j}$ is the Kronecker symbol, one obtains

$$L_{b;k,n} = \mathrm{Det}(\mathbf{T}) \tag{62}$$

where $\mathbf{T}$ is the tensor

$$\mathbf{T} = \eta_0 \otimes \cdots \otimes \eta_0 + \sum_{I\in\{1,\cdots,n-1\}\times\{0,\dots,n-1\}^{2k-1}} \mathrm{B}(|I|,b)\eta_I \tag{63}$$

Hence, expanding by (21) with respect to the first subscript, one has

$$L_{b;k,n} = \mathrm{Det}(\mathbf{S}_{n-1}(1,b;k)). \tag{64}$$

Together with (52) this equality gives the recurrence relation (57) and proves Selberg's identity.

4.3 An alternative end for Selberg's proof

We give a simpler proof of Selberg's identity by reducting it to the Dyson conjecture. We start from the Hankel form

$$\mathrm{Det}\left(\mathbf{S}_n(a,b;k)\right) = \mathrm{B}(a,b)^n \mathrm{Det}_{2k}\left(\frac{(a)_{|I|}}{(a+b)_{|I|}}\right)_0^{n-1}. \tag{65}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. The result will follow if we can compute a closed form for the hyperdeterminant

$$\mathcal{D}_{n,k}(a,b) = \mathrm{Det}_{2k}\left(\frac{(a)_{|I|}}{(b)_{|I|}}\right). \tag{66}$$

We start as the previous proof, obtaining an analog of (52)

$$\mathcal{D}_{k,n}(a,b) = \alpha'_{n,k}\prod_{m=1}^{n}\frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(b)_{k(n+m-1)}} \tag{67}$$

where the constant $\alpha'_{n,k}$ is independent from a and b. Now, for every a'

$$\mathcal{D}_{k,n}(a,b) = \prod_{m=1}^{n}\frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(b)_{k(n+m-1)}(a')_{k(m-1)}}\mathrm{Det}_{2k}\left((a')_{|I|}\right)_0^{n-1}. \tag{68}$$

This means that Selberg's integral can be deduced from the knowledge of $\mathrm{Det}_{2k}((a')_{|I|})_0^{n-1}$ for any particular choice of a'. If we set $a' = -k(n-1)$, one has by (33)

$$\mathrm{Det}_{2k}\left((-k(n-1))_{|I|}\right)_0^{n-1} = (-1)^{kn(n-1)}\mathfrak{d}_{n,k}(k(n-1))!^n. \tag{69}$$

But $\mathfrak{d}_{n,k}$ has been evaluated by means of the Dyson conjecture (38) and the closed form of the Selberg integral follows.

5 Aomoto's integral

5.1 An other hyperdeterminantal representation

Observing the integral representation, we obtain easily the following identity through the substitution $x_i \to 1 - x_i$

$$\mathcal{A}_n^{a,b;k}(y) = (-1)^n \mathcal{A}_n^{b,a;k}(1-y). \tag{70}$$

There is also, a simple hyperdeterminantal proof of this assertion. We consider the tensor

$$\mathbf{A}_n^{b,a;k}(1-y) = \sum_{I\in\{0,\dots,n-1\}^{2k}} \left(1 - y - \frac{b+|I|}{a+b+|I|}\right) \mathrm{B}(b+|I|, b)\eta_I \tag{71}$$

and the action of $2k$ copies of the matrix B defined before. Using equality (45), one has

$$(B,\dots,B)\mathbf{A}_n^{b,a;k}(1-y) = -\mathbf{A}_n^{a,b;k}(y) \tag{72}$$

Hence,

$$\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y)) = (-1)^n \mathrm{Det}(\mathbf{A}_n^{b,a;k}(1-y)). \tag{73}$$

5.2 Minor of Selberg's hyperdeterminant

Before proving Aomoto's identity, we need some preliminary results on Selberg's hyperdeterminant. Let us consider a family of $2k$ subsets $I = (I^1, \dots, I^{2k})$ with $I^j = \{i_0^j \le i_1^j, \dots, \le i_{m-1}^j\} \subset \{1, \cdots, n\}$ and the associated sub-tensor of the Selberg's tensor:

$$\mathbf{S}_n(a,b;k)[I] = \sum_{0\le j_1,\dots,j_{2k}\le m-1} \mathrm{B}(a + i_{j_1}^1 + \cdots + i_{j_{2k}}^{2k}, b)\eta_{i_{j_1}^1} \otimes \cdots \otimes \eta_{i_{j_k}^{2k}}. \tag{74}$$

One can write by (44)

$$\mathrm{Det}(\mathbf{S}_n(a,b;k)[I]) = \sum_\lambda \tilde{C}_\lambda \prod_{i=1}^m \mathrm{B}(a+\lambda_i) \tag{75}$$

where $k(i-1) \le \lambda_i \le k(2n - m + i - 1)$. Hence,

$$\begin{aligned}\mathrm{Det}(\mathbf{S}_n(a,b;k)[I]) = &\prod_{i=1}^m \frac{\Gamma(a+k(i-1))\Gamma(b)}{\Gamma(a+b+k(2n-m+i-2))}\mathfrak{p}_I(a,b) \\ &\prod_{i=1}^m \frac{\Gamma(a+k(i-1))\Gamma(b)}{\Gamma(a+b+k(2n-i-1))}\mathfrak{p}_I(a,b)\end{aligned} \tag{76}$$

where $\mathfrak{p}_I(a,b)$ is a polynomial. In the same way, we will use the following identity

$$\mathrm{Det}(\mathbf{S}_n(a+1,b;k)[I]=\prod_{i=1}^{n}\frac{1}{a+b+k(2n-i-1)}\prod_{i=1}^{m}(a+k(i-1))\times \tag{77}$$
$$\times\prod_{i=m+1}^{n}(a+b+k(2n-i-1))\prod_{i=1}^{m}\frac{\Gamma(a+k(i-1))\Gamma(b)}{\Gamma(a+b+k(2n-i-1))}\mathfrak{p}_I(a+1,b).$$

which follows immediately from (76).

5.3 A proof of Aomoto's identity

We start from the hyperdeterminantal representation and we remark that

$$\mathbf{A}_n^{a,b;k}(y)=\mathbf{S}(a,b;k)y-\mathbf{S}(a+1,b;k). \tag{78}$$

By (18) we obtain

$$\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y))=\sum_{r=0}^{n}(-1)^{n-r}y^r\times$$
$$\times\sum_{(I,J)\in\mathfrak{C}_{n,k}^r}\epsilon(I,J)\mathrm{Det}(\mathbf{S}(a,b;k)[I])\mathrm{Det}(\mathbf{S}(a+1,b;k)[J]). \tag{79}$$

And by (76) and (77), one has

$$\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y))=\prod_{i=1}^{n}\frac{1}{a+b+k(2n-i-1)}\times$$
$$\times\sum_{r=0}^{n}y^r\prod_{i=1}^{n}\frac{\Gamma(a+k(i-1))\Gamma(b)}{\Gamma(a+b+k-2n-i-1)}\times \tag{80}$$
$$\times\prod_{i=1}^{n-r}(a+k(i-1))\prod_{i=n-r+1}^{n}(a+b+k(2n-i-1))\mathfrak{P}_r(a,b)$$

where $\mathfrak{P}_r(a,b)$ is a polynomial in a and b.

On the other hand, we expand the hyperdeterminantal representation of Aomoto's integral by (31) and obtain after extracting common factors as above

$$\begin{aligned}\mathrm{Det}(\mathbf{A}_n^{a,b;k}) &= \mathrm{Det}(\mathbf{S}_n(a,b;k))\sum_\lambda\frac{C_\lambda}{Q(b)}\prod_{i=1}^{n}\Big\{y(a)_{k(i-1)}^{\lambda_i-1}(a+b)_{\lambda_i}^{k(n+i-2)-1}\\ &\quad -(a)_{k(i-1)}^{\lambda_i}(a+b)_{\lambda_i+1}^{k(n+i-2)-1}\Big\}\end{aligned} \tag{81}$$

where $Q(b)$ is a polynomial of degree $kn(n-1)$ in b, $(a)_n^m$ denotes $(a+n)(a+n+1)\cdots(a+m)$ if $n\leq m$ and $(a)_n^m=1$ otherwise and the sum is over the partitions $\lambda=(\lambda_1\leq\cdots\leq\lambda_n)$ verifying $k(i-1)\leq\lambda_i\leq k(n+i-1)$ for each $i\in\{1,\cdots,n\}$. After combining the coefficients of the y^i's, one obtains

$$\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y))=\mathrm{Det}(\mathbf{S}_n(a,b;k))\sum_{r=0}^{n}y^r\frac{\mathfrak{Q}_r(a,b)}{Q(b)} \tag{82}$$

where $\mathfrak{Q}_r(a,b)$ is a polynomial. Hence, comparing the coefficients of the y^i in the expressions (80) and (82), we found

$$\begin{aligned}\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y))=\mathrm{Det}(\mathbf{S}_n(a,b;k))\prod_{i=0}^{n-1}\frac{1}{a+b+k(n+i-1)}\times\\ \times\sum_{i=0}^{n}y^i\frac{P_i(a,b)}{Q(b)}\prod_{j=1}^{n-i}(a+k(n-j))\prod_{j=n-i+1}^{n}(a+b+k(2n-j-1))\end{aligned} \tag{83}$$

where $P_i(a,b)$ is a polynomial. We remark that each $P_i(a,b)$ is of degree in b at most $kn(n-1)$. Now, we apply (73) and we identify the coefficients of the y^i's in the left and right hand sides. After simplification, one obtains for each i

$$\sum_{j=i}^{n}\binom{j}{i}P_j(b,a)F_j(b,a)Q(b)=(-1)^{n-i}P_i(a,b)F_i(a,b)Q(a)) \tag{84}$$

where

$$F_i(a,b)=\prod_{m=1}^{n-i}(a+k(n-m))\prod_{m=n-i+1}^{n}(a+b+k(2n-m-1)). \tag{85}$$

Remarking that $Q(b)$ does not divide $F_i(a,b)$, equality (84) implies

$$P_i(a,b)=R_i(a)Q(b). \tag{86}$$

Hence, setting as in the previous section, $a=-k(n-1)$ in

$$\sum_{j=i}^{n}\binom{j}{i}R_j(b)F_j(b,a)=(-1)^{n-i}R_i(a)F_i(a,b) \tag{87}$$

and remarking that

$$F_i(-k(n-1),b)=0 \tag{88}$$

and

$$F_i(b, -k(n-1)) = \prod_{m=1}^{n} (b + k(n-m)), \tag{89}$$

one obtains the recurrence relation

$$R_i(b) = -\sum_{j=i+1}^{n} \binom{j}{i} R_j(b). \tag{90}$$

The starting point of the induction is $R_n(b) = \frac{1}{n!}$, which follows from the fact that the coefficient of y^n in the expansion of the hyperdeterminant is the value of Selberg's integral. Solving the induction, one finds that each $R_i(b)$ is in fact independent of b, and that

$$R_i(b) = \frac{(-1)^{n-i}}{n!} \binom{n}{i} \tag{91}$$

Hence,

$$\mathrm{Det}(\mathbf{A}_n^{a,b;k}(y)) = \mathrm{Det}(\mathbf{S}_n(a,b;k)) \frac{1}{n!} \prod_{i=0}^{n-1} \frac{1}{a+b+k(n+i-1)} \times$$

$$\times \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \prod_{m=1}^{n-i} (a + k(n-m)) \prod_{m=n-i+1}^{n} (a+b+k(2n-m-1)) y^i \tag{92}$$

$$= (-1)^n \mathrm{Det}(\mathbf{S}_n(a,b;k)) \frac{\left(\frac{a}{k}\right)_n}{n! \left(\frac{a}{k} + \frac{b}{k} + n - 1\right)_n} \sum_{i=0}^{n} \frac{(-n)_i (\frac{a}{k} + \frac{b}{k} + n - 1)_i}{(\frac{a}{k})_i} \frac{y^i}{i!}.$$

We recognize a hypergeometric function of type ${}_2F_1$ which can be evaluated as a unitary Jacobi polynomial (see [2] for more details)

$$\begin{aligned} \mathrm{Det}(\mathbf{A}_n^{a,b;k}(y)) &= (-1)^n \mathrm{Det}(\mathbf{S}_n(a,b;k)) \frac{\left(\frac{a}{k}\right)_n}{n! \left(\frac{a}{k} + \frac{b}{k} + n - 1\right)_n} \times \\ &\quad \times {}_2F_1 \left(\left. \begin{matrix} -n, n + \frac{a}{k} + \frac{b}{k} - 1 \\ \frac{a}{k} \end{matrix} \right| y \right) \\ &= \frac{(-2)^{-n}}{n!} \mathrm{Det}(\mathbf{S}_n(a,b;k)) P_n^{\frac{a}{k}-1, \frac{b}{k}-1}(1-2y) \end{aligned} \tag{93}$$

Aomoto's equality follows.

6 Conclusion

This discussion shows that hyperdeterminantal calculus is a pertinent tools to compute some multiple integrals. Hyperdeterminant is a particular invariant of hypermatrices. In the case of the antisymmetric hypermatrices, an other example of invariant has similar properties : the hyperpfaffian [5, 31]. In the classical case, deBruijn has shown in [11] that multiple integrals of some determinants could be evaluated as Pfaffian. In [31], we have found some generalizations to hyperpfaffian of these identities. In fact, generalized Heine's theorem is a particular case of the generalized deBuijn integral where the determinant can be factorized in a product of determinants of matrices whose dimensions are equal to the number of integration variables. There exist other invariant polynomials of hypermatrices [8, 9, 10, 20] which can be, in principle, computed using Omega Cayley process (see [33, 6, 7] for example involving $2 \times 2 \times 2 \times 2$ and $3 \times 3 \times 3$ hypermatrices). A natural question asks if there exist other integral identities involving some of them.

On an other hand, other generalizations of Selberg integral are encountered in the mathematical physical litterature. For example the Dotsenko-Fateev (see [13]) and Kaneko's integral (see [24]). Kaneko's integral reads

$$K_{n,\gamma}^{a,b}(y_1, \cdots, y_m) = \int_0^1 |\Delta(x)|^{2\gamma} \prod_{i=1}^{n} \left(x_i^{a-1}(1-x_i)^{b-1} \prod_{j=1}^{n}(x_i - y_j)dx_i \right) \quad (94)$$

and is evaluated as a generalized orthogonal polynomial associated with Jack polynomials (see [27, 28, 29]). By generalized Heine's theorem, when $\gamma \in \mathbb{N}$, The Kaneko integral can be evaluated as an hyperdeterminant of moments (see [32]). The Kaneko proof of its integral is related to the Calogero-Sutherland model. Actually, we are unable to find a hyperdeterminantal interpretation of this. In the same way, Anderson's (see [1]) and Aomoto's (see [3]) proofs of Selberg's integral seems to contain other information than the hyperdeterminantal representation. `parler des autres systèmes de racines`

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