Free Quasi-Symmetric Functions, Product Actions and Quantum Field Theory of Partitions.

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Abstract: We examine two associative products over the ring of symmetric functions related to intransitive and cartesian product of permutation groups. As an application, we give an enumeration of some Feynman type diagrams arising in quantum field theory of partitions in dimension 0. To end with, we study some possiblities to construct non commutative analogous.

Résumé: Nous étudions deux lois produits associatives sur les fonctions symétriques correspondant au produits intransitif et cartesien des groupes de permutations. Nous donnons comme application l'énumeration de certain diagrammes de Feynman apparaissant dans la théorie quantique des champs des partitions en dimension 0. Enfin, nous donnons quelques pistes possibles pour construire des analogues non-commutatifs.

1 Introduction

In a relatively recent paper, Bender, Brody and Meister, introduce a special Field Theory described by the formula

$$\left(e^{(\sum_{n\geq 1} L_n \frac{z^n}{n!}\frac{\partial}{\partial z})}\right)\left(e^{(\sum_{m\geq 1} V_m \frac{z^m}{m!})}\right)\Big|_{z=0} \tag{1}$$

in the purpose of proving that any sequence of numbers $\{a_n\}$ could always be described by a suitable set of rules applied to some type of Feyman graphs [1, 2]. These graphs are in fact bipartite finite graphs with no isolated point and edges weighted with integers. It is still an open problem to exhibit properties of any generating series of these graphs (uni- or mutivariate, ordinary or exponential).

Developing one factor of (1), one can witness suprising links between: normal ordering problem (for bosons), parametric Stieltjes moment problem and kernel, substitution matrices (as generalised Stirling matrices) and one-parameter groups of analytic substitutions [6, 7, 12].

The aim of this paper is to explicit the multifaceted set of links between noncommutative symmetric functions (here **MQSym**, **FQSym** [5]) and the Feynman-type diagrams arising in the expansion of formula (1) used in combinatorial physics [12].

The structure of the contribution is the following ...

2 Actions of a direct product of permutation groups

2.1 Direct product actions

The actions of the direct product of two permutation groups (in particular, the structure of the cycles) rise interesting properties related to enumeration of non labeled objets [11]. We start this section with the definition of two actions (say intransitive and cartesian), for more details about these constructions (or for constructions involving the wreath product) the reader could refer to [3].
Consider two pairs (G_1, X_1) and (G_2, X_2) where each G_i is a permutation group acting on X_i. The *intransitive action* of $G_1 \times G_2$ on $X_1 \sqcup X_2$ (here $\sqcup$ means that we suppose the union disjoint, in the other case we replace the elements of X_2 occurring in X_1 in such a way that the union becomes disjoint) is defined by the rule

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases} . \tag{2}$$

We will denote $(G_1, X_1) \nrightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.
The *cartesian action* of $G_1 \times G_2$ on $X_1 \times X_2$ is defined by

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2). \tag{3}$$

We will denote $(G_1, X_1) \mathbin{\nearrow\!\!\!\!\!\!\searrow} (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$. Note that the two laws defined here are not commutative. A natural question asks if such a structure admits some algebraic properties. For example: Is the law $\mathbin{\nearrow\!\!\!\!\!\!\searrow}$ distributive w.r.t. $\nrightarrow$? Examine the signification of

$$(G_1, X_1) \mathbin{\nearrow\!\!\!\!\!\!\searrow} ((G_2, X_2) \nrightarrow (G_3, X_3)) = (G_1 \times G_2 \times G_3, X_1 \times (X_2 \sqcup X_3))$$

and

$$((G_1, X_1) \mathbin{\nearrow\!\!\!\!\!\!\searrow} (G_2, X_2)) \nrightarrow ((G_1, X_1) \mathbin{\nearrow\!\!\!\!\!\!\searrow} (G_3, X_3)) = (G_1 \times G_2 \times G_1 \times G_3, (X_1 \times X_2) \sqcup (X_1 \times X_3)).$$

The groups $G_1 \times G_2 \times G_1 \times G_3$ and $G_1 \times G_2 \times G_3$ being not isomorphic, the distributivity does not occur in spite of the fact that the set theoretical cartesian product is distributive. But an examination of the structure of the cycles (see [3] for general construction or the section 2.2 for a particular case) shows that the cycles are the same. More precisely, a cycle can appear with different multiplicities for the action of the two groups , but if we consider only the set of the cycles, the two structures are similar.
Now, let us give a construction which considers such a phenomenon.

2.2 Explicit realization

Let $\mathfrak{S}_n$ and $\mathfrak{S}_m$ be two symmetric groups, we will denote by $\circ_N$ its natural action of $\mathfrak{S}_n$ on $\{0, \dots, n-1\}$, by $\circ_I$ the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \cdots, n+m-1\}$ and by $\circ_C$ the cartesian action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \dots, nm-1\}$. More precisely,

$$(\sigma_1, \sigma_2) \circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \le i \le n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \le i \le n+m-1 \end{cases} . \tag{4}$$

and

$$(\sigma_1, \sigma_2) \circ_C (j + nk) = (\sigma_1 \circ_N j) + n(\sigma_2 \circ_N k) \tag{5}$$

for $0 \le i \le n+m-1$, $0 \le j \le n-1$ and $0 \le k \le m-1$.
The *intransitive product* is the map $\nrightarrow : \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ defined by

$$\sigma_1 \nrightarrow \sigma_2 = \sigma_1 \sigma_2[n] \tag{6}$$

where $\sigma_2[n]$ denotes the shifted substitution $i \to i+n$ (here the permutations are considered as words and $\nrightarrow$ is nothing else than the shifted concatenation).

Example 2.1 *Let* $\sigma_1 = 1320 \in \mathfrak{S}_4$ *and* $\sigma_2 = 534120 \in \mathfrak{S}_6$. *Here, we denote a permutation of* $\mathfrak{S}_n$ *by the word whose ith letter is the image of* i *under its natural action on* $\{0, \ldots, n-1\}$*). With this notation, we obtain*

$$\sigma_1 \nrightarrow \sigma_2 = 1320978564$$

and

$$\sigma_2 \nrightarrow \sigma_1 = 5341207986$$

Clearly, it appears that $\nrightarrow$ *is not commutative.*

The natural action of $\mathfrak{S}_{n+m}$ is compatible with the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$.

Proposition 2.2 $(\sigma_1 \nrightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i$.

□

Let us introduce a similar construction for the cartesian action by defining the map $\times$: $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ by

$$\sigma_1 \times \sigma_2 = \prod_{i,j} c_i \times c'_j \tag{7}$$

where $\sigma_1 = c_1 \cdots c_k$ and $\sigma_2 = c'_1 \cdots c'_{k'}$ are the decompositions of σ_1 and σ_2 in a product of cycles and

$$c \times c' = \prod_{s=0}^{l \wedge l' - 1} (\phi(s, 0), \phi(s+1, 1) \cdots, \phi(s + l \vee l' - 1, l \vee l' - 1)), \tag{8}$$

where $\wedge$ denotes the gcd, $\vee$ denotes the lcm, $c = (i_0, \cdots, i_{l-1})$, $c' = (j_0, \cdots, j_{l'-1})$ are two cycles and $\phi(k, k') = i_{k \bmod l} + nj_{k' \bmod l'}$. As the intransitive action, the cartesian action is compatible with the natural action.

Proposition 2.3 $(\sigma_1 \times \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i$.

Proof — From (7), it suffices to prove the property only when $\sigma_1 = c$ and $\sigma_2 = c'$ are two cycles. But as (8) is equivalent to

$$\begin{aligned} c \times c' = & \prod_{s=0}^{l \wedge l' - 1} (i_s + nj_0, (c, c') \circ_C (i_s + nj_0), \ldots, (c^{l \vee l' - 1}, c'^{l \vee l' - 1}) \circ_C (i_s + nj_0)) \\ = & \prod_{s=0}^{l \wedge l' - 1} (i_s + nj_0, c \circ_C i_s + nc' \circ_N j_0, \ldots, c^{l \vee l' - 1} \circ_N i_s + nc'^{l \vee l' - 1} \circ_N j_0)), \end{aligned}$$

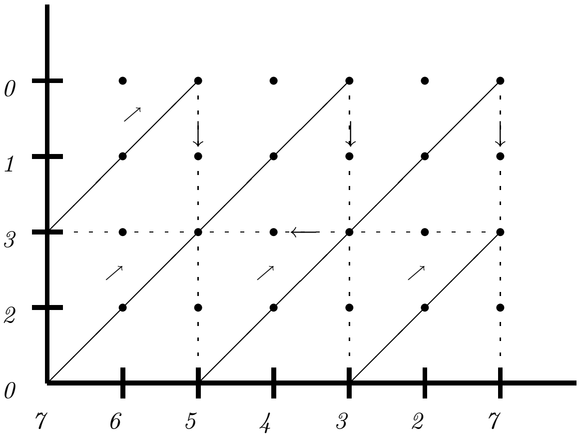
this ends the proof.

□

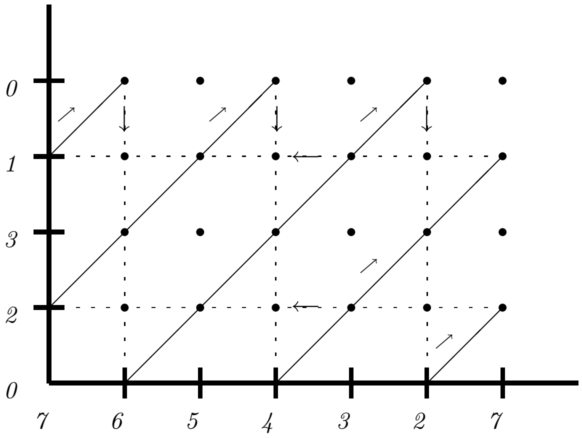
Example 2.4 *Consider the two permutations* $\sigma_1 = 2031 \in \mathfrak{S}_4$ *and* $\sigma_2 = 01723456 \in \mathfrak{S}_8$. *The permutation* σ_1 *consists in a unique cycle* $c_1 = (0, 2, 3, 1)$ *and* $\sigma_2 = c'_1 c'_2 c'_3$ *is the product of the three cycles* $c'_1 = (0)$, $c'_2 = (1)$ *and* $c'_3 = (7, 6, 5, 4, 3, 2)$. *Hence, the permutation* $\sigma_1 \times \sigma_2$ *is the product of four cycles given by*

1. $c_1 \times c'_1 = (0, 2, 3, 1)$
2. $c_1 \times c'_2 = (4, 6, 7, 5)$
3. $c_1 \times c'_3 = (28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$.

To illustrate proposition 2.3, it suffices to draw the cycles in the cartesian product $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ *whose elements are re- labeled* $(i, j) \to i + nj$. *For example, the two cycles appearing in* $c_1 \times c'_3$ *give the following partition of* $\{0, 1, 2, 3\} \times \{2, 3, 4, 5, 6, 7\}$.



(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)



(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)

On the other hand, the permutation $\sigma_2 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_1$ is the product of the four cycles

1. $c'_1 \mathbin{\diagup\!\!\!\!\!\searrow} c_1 = (0, 16, 24, 8)$
2. $c'_2 \mathbin{\diagup\!\!\!\!\!\searrow} c_1 = (1, 17, 25, 9)$
3. $c'_3 \mathbin{\diagup\!\!\!\!\!\searrow} c_1 = (7, 22, 29, 12, 3, 18, 31, 14, 5, 20, 27, 10)(6, 21, 28, 11, 2, 23, 30, 13, 4, 19, 26, 15)$

Clearly, $\sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_2 \neq \sigma_2 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_1$: the law $\mathbin{\diagup\!\!\!\!\!\searrow}$ is not commutative.

2.3 Algebraic structure

The gain of the new structures in comparison with those defined in section 2.1 consists in the omission of the operations over the groups. Hence, algebraic properties appear naturally.
First, the two laws are associative.

Proposition 2.5 *associativity*
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$
2. $\sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} (\sigma_2 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_3) = (\sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_2) \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_3$

Proof — We use the fact that two permutations of a symmetric group $\mathfrak{S}_n$ are equal if and only if their natural action on $\{0, \ldots, n-1\}$ are the same.

1. Set $\eta_1 = \sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3)$ and $\eta_2 = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$. One has

$$\eta_1 \circ_N i = \eta_2 \circ_N i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \leq i \leq n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \leq i \leq m+n-1 \\ \sigma_3 \circ_N (i-n-m) + n + m & \text{if } n+m \leq i \leq n+m+p-1 \end{cases}$$

for each $0 \leq i \leq n+m-1$. It follows $\eta_1 = \eta_2$.

2. The strategy is the same. First, we set $\eta_1 = \sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} (\sigma_2 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_3)$ and $\eta_2 = (\sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_2) \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_3$.
The action of η_1 could be computed as follows

$$\eta_1 \circ_N (i + ni') = \sigma_1 \circ_N i + n(\sigma_2 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_3) \circ_N i' = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k$$

where $0 \leq i \leq n-1$, $0 \leq i' \leq nm-1$, $0 \leq j \leq m-1$ and $0 \leq k \leq p-1$.
On the other hand, the action of η_2 is

$$\eta_2 \circ_N (k' + nmk) = (\sigma_1 \mathbin{\diagup\!\!\!\!\!\searrow} \sigma_2) \circ_N k' + nm\sigma_3 \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j + nm\sigma_3 \circ_N k$$

where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$, $0 \leq k \leq p-1$ and $0 \leq k' \leq nm-1$. Hence, $\eta_1 \circ_N i = \eta_2 \circ_N i$ for $0 \leq i \leq nmp-1$ and $\eta_1 = \eta_2$.

$\square$

From example 2.1 and 2.4, neither $\nrightarrow$ nor $\times\!\!\!\!\searrow$ is commutative. But, one has the property of left distributivity.

Proposition 2.6 *Semi-distributivity*
Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

$$\sigma_1 \times\!\!\!\!\searrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \times\!\!\!\!\searrow \sigma_2) \nrightarrow (\sigma_1 \times\!\!\!\!\searrow \sigma_3)$$

Proof — We use the same method than in the proof of proposition 2.5. First, let us define $\eta_1 = \sigma_1 \times\!\!\!\!\searrow (\sigma_2 \nrightarrow \sigma_2)$ and $\eta_2 = (\sigma_1 \times\!\!\!\!\searrow \sigma_2) \nrightarrow (\sigma_1 \times\!\!\!\!\searrow \sigma_3)$. The action of η_1 is

$$\eta_1 \circ_N (i+nj) = \eta_1 \circ_N i + n(\sigma_2 \nrightarrow \sigma_3) \circ_N j = \begin{cases} \sigma_1 \circ_N i + n\sigma_2 \circ_N j & \text{if } 0 \leq j \leq m-1 \\ \sigma_1 \circ_N i + n\sigma_2 \circ_N (j-m) + m & \text{if } m \leq j \leq p+m-1 \end{cases} \tag{9}$$

where $0 \leq i \leq n-1$ and $0 \leq j \leq m+p-1$.
On the other hand, one has

$$\eta_2 \circ_N k = \begin{cases} (\sigma_1 \times\!\!\!\!\searrow \sigma_2) \circ_N k & \text{if } 0 \leq k \leq nm-1 \\ (\sigma_1 \times\!\!\!\!\searrow \sigma_3) \circ_N (k-nm) + nm & \text{if } nm \leq k \leq n(m+p)-1 \end{cases}. \tag{10}$$

If $0 \leq k \leq mn-1$, we set $k = i + nj$ where $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. Hence,

$$(\sigma_1 \times\!\!\!\!\searrow \sigma_2) \circ_N k = \sigma_1 \circ_N i + n\sigma_2 \circ_N j. \tag{11}$$

Similarly, if $nm \leq k \leq n(m+p)-1$, we set $(k-nm) = i + nj$ where $0 \leq i \leq n-1$ and $0 \leq j \leq p-1$. Hence,

$$(\sigma_1 \times\!\!\!\!\searrow \sigma_3) \circ_N (k-nm) + nm = \sigma_1 \circ_N i + n(\sigma_3 \circ_N (j-m) + m). \tag{12}$$

Replacing (11) and (12) in (10), one recovers the right hand side of (9). It follows immediately that $\eta_1 = \eta_2$.

$\square$

The two laws can be extent by linearity to the graded vector space $\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n]$ and confer to this space a structure of 2-associative algebra. In the next section, we construct a product $\star$ in *Sym* defined on the power sums and appearing when one examines the cycle index polynomial of a cartesian product. This product is the image of $\times\!\!\!\!\searrow$ under a particular morphism. We will prove that this last property implies the associativity and the distributivity of $\star$ over $\times$ (the natural product in *Sym*) and $+$.

3 Cycle index algebra

3.1 Cartesian product in *Sym*

We first construct a 2-associative morphism $\bigoplus_{n\geq 0} \mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ (a 2-associative algebra is just a vector space equipped with 2 associative laws [8]).
The arrow maps a permutation $\sigma \in \mathfrak{S}_n$ to a product of power sums. Let, for $j \leq 1$, $c_j(\sigma)$ be the number of cycles of σ of length j and set

$$\mathfrak{Z}(\sigma) = \prod_{j=0}^{\infty} \psi_j^{c_j(\sigma)} \tag{13}$$

where ψ_i denotes a power sum symmetric function. We claim that $\mathfrak{Z}$ is a morphism for $\nrightarrow$ and that $\times\!\!\!\!\searrow$ is compatible with $\mathfrak{Z}$ is the sense that it exists an associative law on Sym such that $\mathfrak{Z}$ is also a morphism for it. This second law is given on the power sums basis by

$$\prod_{1\leq i\leq\infty} \psi_i^{\alpha_i} \star \prod_{1\leq j\leq\infty} \psi_j^{\beta_j} = \prod_{1\leq i,j\leq\infty} \psi_{i\vee j}^{\alpha_i\beta_j(i\wedge j)} \tag{14}$$

(the sequences $(\alpha_i)_{i\geq 1}$, $(\beta_j)_{j\geq 1}$ are with finite support). It is straightforward to check that

Proposition 3.1 *i) The mapping $\mathfrak{Z} : \bigoplus_{n\geq 0}\mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ is a morphism sending the two laws $\nrightarrow$; $\times\!\!\!\!\searrow$ respectively to $\times$; $\star$ (where, here, $\times$ is for the usual product of Sym).*
More precisely, for $\sigma,\ \tau \in \sqcup_{n\geq 0}\mathfrak{S}_n$ one has

$$\mathfrak{Z}(\sigma \nrightarrow \tau) = \mathfrak{Z}(\sigma)\mathfrak{Z}(\tau)\ ;\ \mathfrak{Z}(\sigma \times\!\!\!\!\searrow \tau) = \mathfrak{Z}(\sigma)\star\mathfrak{Z}(\tau) \tag{15}$$

ii) The law $\star$ is commutative and distributive over $\times$.

Proof — i) For (15) left, one has just to remark that $c_j(\sigma \nrightarrow \tau) = c_j(\sigma) + c_j(\tau)$. For the right relation, one observes that the cartesian product of a i-cycle and a j-cycle produces $i\wedge j$ cycles of length $i\vee j$. Thus, one has $c_r(\sigma \times\!\!\!\!\searrow \tau) = \sum_{p\vee q=r} c_p(\sigma)c_q(\tau)$ whence (15).
ii) When $\sigma \in \mathfrak{S}_n$ is a cycle of maximum length, one has $\mathfrak{Z}(\sigma) = \psi_n$, hence the image of $\mathfrak{Z}$ contains also all the products of power sums and we get $Im(\mathfrak{Z}) = Sym$. Then, by proposition 3.1(i), $\star$ is distributive on the left w.r.t. $\times$. Complete distributivity follows from commutativity which straightforwardly follows from the definition.

□

The following proposition gives more details on the structure of the distributivity of $\star$ over $\times$.

Proposition 3.2 *Let P be the set of products of power sums (i.e. $P = \{\prod_{i=1}^{\infty}\psi_i^{\alpha_i}\}_{(\alpha_i)_{i\geq 1}\in\mathbb{N}^{(\mathbb{N}^*)}}$). Then P is closed by $\times$ and $\star$ and more precisely $(P,\times,\star)$ is isomorphic to a subsemiring of the semiring $\mathbb{N}[(\mathbb{N}^{(\mathfrak{p})}, sup)]$ (where $\mathfrak{p}$ stands for the set of prime numbers).*

Proof — The fact that P is closed by $\times$ and $\star$ is straightforward from the definition and (14). Now P contains the two neutrals (1 and ψ_1) and, hence (due to the established properties of the laws $\times,\ \star$) is a semiring. For every $p\in\mathfrak{p}$ and $n\in\mathbb{N}^*$, let $\nu_p(n)$ be the exponent of p in the decomposition of n in prime factors ($n = \prod_{p\in\mathfrak{p}} p^{\nu_p(n)}$). Define an arrow $\phi : P \mapsto \mathbb{N}[(\mathbb{N}^{(\mathfrak{p})}]$ by

$$\phi(\prod_{1\leq i\leq\infty}\psi_i^{\alpha_i}) = \sum_{1\leq i\leq\infty} i\alpha_i(p\mapsto\nu_p(i)). \tag{16}$$

As $\phi(m_1m_2) = \phi(m_1)+\phi(m_2)$ by construction (16), it suffices to prove that
$\phi(\psi_i\star\psi_j) = \phi(\psi_i)\times_s\phi(\psi_j)$ where $\times_s$ stands for the product in $\mathbb{N}[(\mathbb{N}^{(\mathfrak{p})}, sup)]$. But $\phi(\psi_i\star\psi_j) = \phi(\psi_{i\vee j}^{i\wedge j}) = (i\wedge j)\phi(\psi_{i\vee j})$. Now

$$\begin{aligned}(i\wedge j)\phi(\psi_{i\vee j}) = (i\wedge j)(i\vee j)(p\mapsto\nu_p(i\vee j)) &= (i\wedge j)(i\vee j)(p\mapsto sup(\nu_p(i),\nu_p(j))) = \\ ij(p\mapsto sup(\nu_p(i),\nu_p(j))) &= \phi(\psi_i)\times_s\phi(\psi_j).\end{aligned}$$

The arrow is clearly into.

□

3.2 Cycle index polynomial

Let $\mathfrak{S} = \bigsqcup_{n\geq 0} \mathfrak{S}_n$ be the disjoint union of all the symmetric groups and $\mathfrak{S}_{sg} = \bigsqcup_{n\geq 0} (\mathfrak{S}_n)_{sg}$ be the set of all the subgroups of any symmetric groups (*i.e.* the set of all the finite permutation groups). For simplicity, we identify a permutation group $G \in (\mathfrak{S}_n)_{sg}$ with its action $(G, \{0, \dots, n-1)$ (see section 2.1). The law $\nleftrightarrow$ and $\times\!\!\!\!\!\searrow$ can be defined over $\mathfrak{S}_{sg}$ by

$$G_1 \nleftrightarrow G_2 := (G_1 \times G_2, \{0, \dots, n+m-1\}) \tag{17}$$

where G_1 acts on $\{0, \dots, n-1\}$ and G_2 acts on $\{n, \dots, n+m-1\}$ and

$$G_1 \times\!\!\!\!\!\searrow G_2 := (G_1 \times G_2, \{0, \dots, nm-1\}) \tag{18}$$

where the action over $\{0, \dots, nm-1\}$ is given by $(\sigma_1, \sigma_2)k = \phi^{-1}((\sigma_1, \sigma_2)\phi(k))$, the map ϕ being the bijection $\phi : \{0, \dots, nm-1\} \to \{0, \dots, n-1\} \times \{0, \dots, m-1\}$ defined by $\phi(i + nj) = (i, j)$ if $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ and $(\sigma_1, \sigma_2)(i, j) = (\sigma_1 i, \sigma_2 j)$. Note that both $\nleftrightarrow$ and $\times\!\!\!\!\!\searrow$ are associative but $\times\!\!\!\!\!\searrow$ is not distributive over $\nleftrightarrow$.
The *Polyà cycle index polynomial* is the image of G by $Z : \mathfrak{S}_{sg} \to Sym$ defined by

$$Z(G) = \mathfrak{Z}\left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\right). \tag{19}$$

Example 3.3 *1. The cycle index of the symmetric group $\mathfrak{S}_n$ is $Z(\mathfrak{S}_n) = h_n$.*

2. The cycle index of the alternating group A_n is $Z(A_n) = h_n + e_n$.
Here h_n (resp. e_n) denotes a complete (resp. elementary) symmetric function. These examples are given as exercices in [9] ex.9 p 20-21.

As $\mathfrak{Z}$ is a morphism of 2-associative algebra, one recovers the classical relations (see [3])

$$Z(G_1 \nleftrightarrow G_2) = Z(G_1)Z(G_2) \tag{20}$$

$$Z(G_1 \times\!\!\!\!\!\searrow G_2) = Z(G_1) \star Z(G_2) \tag{21}$$

Example 3.4 *1. The cycle index polynomial of the intransitive product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is*

$$Z(\mathfrak{S}_n \nleftrightarrow \mathfrak{S}_m) = h_n h_m.$$

2. The cycle index polynomial of the cartesian product of two symmetric groups $\mathfrak{S}_n$ and $\mathfrak{S}_m$ is

$$Z(\mathfrak{S}_n \times\!\!\!\!\!\searrow \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} m_\lambda \star m_\rho = \sum_{\substack{|\lambda|=n,\\ |\rho|=m}} \frac{1}{z_\lambda z_\rho} \prod_{i,j} \psi_{\lambda_i \vee \rho_j}^{\lambda_i \wedge \rho_j},$$

where m_λ denotes a monomial symmetric function and $z_\lambda = \prod i^{n_i} n_i!$ if n_i is the number of parts of λ equal to i.

3.3 Enumeration of a type of Feynman diagrams related to quantum field theory of partitions

The cycle index polynomials are classic tools used in the topic of enumeration of non labeled objects via Polyà's theorem. Let us recall the general process. Consider a permutation group G acting on a finite set $X = \{x_1, \cdots, x_n\}$. Let $L = \{l_0, \dots, l_p, \dots\}$ (may be infinite) be an other set. Let $f : X \to L$, its *type* $t(f)$ is the vector $(i_0, \dots, i_p, \dots)$ where i_k is the number of elements of X whose image by f is l_k. The *shape* $s(f)$ of f is the partition obtained by sorting in the decreasing order $t(f)$ and erasing the 0. For example, a function f having the type $t(f) = (0, 1, 0, 9, 1, 2, 0, \dots, 0, \dots)$ has the shape $s(f) = (9, 2, 1, 1)$. The number $d^s_\lambda(G, L)$ of G-classes on L^X with the shape λ is the coefficient of m_λ in the expansion of $Z(G)$ in the basis of monomial symmetric functions:

$$Z(G) = \sum_\lambda d^s_\lambda(G, L) m_\lambda. \tag{22}$$

Now, let us apply this method to enumerate Feynman diagram arising in the expansion of formula (1). These diagrams are bipartite finite graphs with no isolated point and edges weighted with integers. First, we enumerate all bipartite finite graphs with edges weighted with integers. Let n and m be the numbers of vertices in each of the two parts. We consider the vertices as a function v from $\{0,\dots,n-1\}\times\{0,\dots,m-1\}$ to $\mathbb{N}$. The type (resp. the shape)of a graph is the type (resp. the shape) of its vertices (*i.e.* $t(v)$ -resp. $s(v)$). The number $d_\lambda(n,m)$ of graphs with type λ is equal to the number of $\mathfrak{S}_n \times \mathfrak{S}_m$-classes on $\mathbb{N}^{\{0,\dots,n-1\}\times\{0,\dots,m-1\}}$ with type λ. Hence, the generating function of the shape is

$$g(n,m) := \sum_\lambda d^s_\lambda(n,m) m_\lambda = Z(\mathfrak{S}_n) \star Z(\mathfrak{S}_m) \tag{23}$$

As $g(n,m) \in Sym$, it is also the generating function counting the graph by type. It suffices to evaluate and expand the monomial functions over the alphabet $\{y_0,\dots,y_m,\dots\}$. Hence, if $d^t_I(n,m)$ denote the number of graph with type I, one has

$$g(n,m) = \sum_{I=(i_0,\dots,i_p,\dots)} d^t_I(n,m) \prod_{k=0}^{\infty} y_k^{i_k}. \tag{24}$$

Note that one can enumerate graphs having edges weighted with integers less or equal to p by specializing to the finite alphabet $\{y_0,\dots,y_p\}$.
Let us define the generating series of the type of our Feynman diagrams

$$F(n,m) := \sum_{I=(i_0,\dots,i_p,\dots)} f^t_I(n,m) \prod_{k=0}^{\infty} y_k^{i_k}, \tag{25}$$

where $f^t_I(n,m)$ denotes the number of Feynman diagrams of type I. Remark that $F(n,m)$ is a symmetric function over the alphabet $\{y_1,\dots,y_p,\dots\}$ but not over $\{y_0,\dots,y_p,\dots\}$.

Example 3.5 *Let us give the first examples of generating series for weight in* $\{0,1,2\}$.

1. $F(1,1) = y_1 + y_2$
2. $F(2,1) = F(1,2) = y_1^2 + y_1y_2 + y_2^2$
3. $F(2,2) = y_0^2y_1^2 + y_0^2y_2^2 + y_0^2y_1y_2 + y_0y_1^3 + 3y_0y_1^2y_2 + 3y_0y_1y_2^2 + y_0y_2^3 + y_1^4 + y_1^3y_2 + 3y_1^2y_2^2 + y_1y_2^3 + y_2^4$

One can remark that under this specialization,

$$F(2,2) + F(2,1)y_0^2 + F(1,2)y_0^2 + F(1,1)y_0^3 + y_0^4 = 3m_{22} + m_4 + 3m_{211} + m_{31} = g(2,2).$$

This last equality could be stated in a more general setting.

Theorem 3.6 *One has the following decomposition of the cycle index polynomial.*

$$Z(\mathfrak{S}_n \times \mathfrak{S}_m) = \sum_{(1,1)\le_{lex}(k,p)\le_{lex}(n,m)} F(k,p) y_0^{nm+kp-np-mk} + y_0^{nm}. \tag{26}$$

Proof — It suffices to remark that a bipartite graph is either a graph without isolated point either some isolated points with a smaller graph.

□

This yields a nice induction formula for the $F(n,m)$'s.

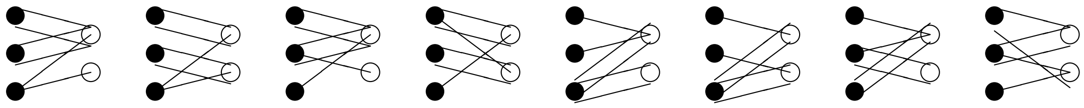
Example 3.7 *¿From theorem 3.6, one has*

$$F(2,3) = Z(\mathfrak{S}_3 \times \mathfrak{S}_2) - F(1,3)y_0^3 - F(2,2)y_0^2 - F(2,1)y_0^4 - F(1,2)y_0^4 - F(1,1)y_0^5 - y_0^6.$$

¿From example 3.5, to enumerate Feynman diagrams whose edges are weighted by 0, 1 *or* 2, *it suffices to compute* $F(1,3) = y_1^3 + y_2^3$. *After simplification, one obtains*

$$\begin{aligned} F(2,3) \quad = \quad & y_2^6 + y_2^5 y_1 + 3y_2^4 y_1 + 3y_2^4 y_1 y_0 + 2y_2^4 y_0^2 + 3y_2^3 y_1^3 + 6y_2^3 y_1^2 y_0 + 5y_2^3 y_1 y_0^2 \\ & + y_2^3 y_0^3 + 3y_2^2 y_1^4 + 3y_2^2 y_1^3 y_0 + 8y_2^2 y_1^2 y_0^2 + 3y_2^2 y_1 y_0^3 + y_2 y_1^5 + 3y_2 y_1^4 y_0 + 5y_2 y_1^3 y_0^2 \\ & + 3y_2 y_1^2 y_0^3 + y_1^6 + y_1^5 y_0 + y_1^3 y_0^3 + 2y_1^2 y_0^4. \end{aligned}$$

For example, there is 8 $(2,2,2)$- *Feynman diagrams:*



4 Non commutative realizations

4.1 Free quasi-symmetric cycle index algebra

Let $(A, <)$ be an ordered alphabet. The standardized of a word $w \in A^*$ of length n, denoted $Std(w)$, is a permutation $\sigma \in \mathfrak{S}_n$ defined recursively by

$$\sigma(i) = (\text{Number of letters} \leq w[i] \text{ in } w[1..i] + \text{number of letters} < w[i] \text{ in } w[i+1..n]) \qquad (27)$$

Recall that the algebra **FQSym** is defined by one of its bases, indexed to $\mathfrak{S}$ and defined as follows

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A \rangle\rangle \qquad (28)$$

In [5], it is shown that **FQSym** is freely generated by the $\mathbf{F}_\sigma$ where σ runs over the connected permutations (see [4]) (*i.e.* permutations such that $\sigma([1,k] \neq [1,k]$ for each k). The algebra **FQSym** is spanned by a linear basis whose product implements the intransitive action $\nrightarrow$:

$$\mathbf{F}^\sigma = F_{\sigma_1} \cdots F_{\sigma_n} \qquad (29)$$

where $\sigma = \sigma_1 \nrightarrow \cdots \nrightarrow \sigma_n$ is the maximal factorisation of σ in connected permutations. As a consequence of this definition, one has

$$F^\sigma F^\tau = F^{\sigma \nrightarrow \tau}. \qquad (30)$$

This induces naturally a morphism of algebra

$$\begin{aligned} \underline{\mathfrak{Z}} : \left(\bigoplus \mathbb{Q}[\mathfrak{S}_n], \nrightarrow, +\right) \quad & \rightarrow \quad (\mathbf{FQSym}, ., +) \\ \sigma \quad & \rightarrow \quad \mathbf{F}^\sigma. \end{aligned} \qquad (31)$$

One defines the product $\star$ on **FQSym** by $F^\sigma \star F^\tau := F^{\sigma \times\!\!\!\!\nearrow \tau}$. By this way, $\underline{\mathfrak{Z}}$ becomes a morphism of 2-associative algebra. Furthermore, $\star$ is associative, distributive over the sum and semi-distributive over the shifted concatenation .

4.2 Free quasi-symmetric Polyà cycle index polynomial

Let G be a permutation group. The *free quasi-symmetric Polyà cycle index polynomial* of G is its image by $\underline{Z} : \mathfrak{S}_{sg} \rightarrow$ **FQSym** defined by

$$\underline{Z}(G) := \underline{\mathfrak{Z}}\left(\frac{1}{|G|}\sum_{\sigma \in G} \sigma\right). \qquad (32)$$

The polynomial $\underline{Z}(G)$ has similar properties than $Z(G)$, in particular regarding the laws $\nrightarrow$ and $\times\!\!\!\!\nearrow$.

Proposition 4.1 *Let $G_1, G_2 \in \mathfrak{S}_{sg}$ be two permutation groups, one has*

1. $\underline{Z}(G_1 \nrightarrow G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
2. $\underline{Z}(G_1 \bigtimes G_2) = \underline{Z}(G_1) \star \underline{Z}(G_2)$.

Consider the morphism, $z : \mathbf{FQSym} \to Sym$ defined by $z(\mathbf{F}^\sigma) = \mathfrak{z}(\sigma)$. Note that it is not a morphism of Hopf algebra as $z(F^{321}) = \psi_2\psi_1$.
The following diagram is commutative

$$\begin{array}{ccc} \mathfrak{S}_{sg} & \xrightarrow{\underline{Z}} & \mathbf{FQSym} \\ Z\downarrow & {}^{z}\swarrow & \uparrow \underline{\mathfrak{z}} \\ Sym & \xleftarrow[\mathfrak{z}]{} & \bigoplus \mathbb{Q}[\mathfrak{S}_n] \end{array} \tag{33}$$

Example 4.2

1. *The free quasi-symmetric cycle index of $\mathfrak{S}_n$ is*

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} F^\sigma.$$

One can consider it as a free quasi-symmetric analogous of the complete symmetric function h_n as $z(\mathbf{H_n}) = Z(\mathfrak{S}_n) = h_n$.

2. *One can define free quasi-symmetric analogous of elementary symmetric functions considering the cycle index polynomial of the alternative groups:*

$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n).$$

We get $z(\mathbf{E_n}) = Z(A_n) - Z(\mathfrak{S}_n) = e_n$.

3. *The knowledge of analogous of other symmetric functions should be useful to understand the combinatorics of free quasi-symmetric cycle index. In particular, it should be interesting to find free quasi-symmetric functions whose images by z are the monomial symmetric functions.*

4.3 Other realizations

`parler de MQSYM`

Other realizations in Hopf algebras are possible. For example, if we consider Hopf algebras of graphs $GQSym^{110}$ and $GTSym^{110}$ as defined in [10]. An interesting mapping from $\bigoplus \mathbb{Q}[\mathfrak{S}_N]$ to $GQSym^{110}$ or $GTSym^{110}$ can be construct sending each cycle to an equivalent loop.

5 Conclusion

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